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Interval oscillation criteria for functional differential equations of fractional order

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Abstract

In this paper, we are concerned with the oscillatory behavior of a class of fractional differential equations with functional terms. The fractional derivative is defined in the sense of the modified Riemann-Liouville derivative. By using a variable transformation, a generalized Riccati transformation, Philos type kernels, and the averaging technique, we establish new interval oscillation criteria. Illustrative examples are also given.

Keywords: fractional ODE; oscillation; functional term

1 Introduction

Differential equations of fractional order have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, viscoelasticity, chemical physics, electrical networks, fluid flows, control, dynamical processes in self-similar and porous structures, *etc.* (see, for example, [1–6]). There have appeared lots of works in which fractional derivatives are used for a better description of considered material properties; mathematical modeling based on enhanced rheological models naturally leads to differential equations of fractional order and to the necessity of the formulation of initial conditions to such equations. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications. The books on the subject of fractional integrals and fractional derivatives by Diethelm [7], Miller and Ross [8], Podlubny [9] and Kilbas *et al.* [10] summarize and organize much of fractional calculus and many of theories and applications of fractional differential equations. Many papers have studied some aspects of fractional differential equations, especially the existence of solutions (or positive solutions) of nonlinear initial (or boundary) value problems for fractional differential equation (or system) by the use of techniques of nonlinear analysis (fixed point theorems, Leray-Schauder theory), the methods for explicit and numerical solutions and the stability of solutions; we refer to [11–21] and the references cited therein.

However, to the best of our knowledge, very little is known regarding the oscillation of fractional differential equations up to now. The oscillation theory as a part of the qualitative theory of differential equations has been developed rapidly in the last decades, and there has been a great deal of work on the oscillatory behavior of integer order differential equations; see the monographs [22, 23].

Recently Feng and Meng [24] have established oscillation criteria for the nonlinear fractional differential equations of the form

$$D_t^\alpha\big[r(t)\psi\big(x(t)\big)D_t^\alpha x(t)\big] + q(t)f\big(x(t)\big) = e(t), \quad t \geq t_0 > 0, 0 < \alpha < 1,$$

where $D_t^\alpha(\cdot)$ denotes the modified Riemann-Liouville derivative [25] with respect to variable t.

In this paper, we are concerned with the oscillation of functional differential equations of fractional order in the form of

$$D_t^\alpha\big[r(t)\psi\big(x(t)\big)D_t^\alpha x(t)\big] + F\big(t, x(t), x\big(\tau(t)\big)\big) = e(t), \quad t \geq t_0 > 0, 0 < \alpha < 1, \tag{1}$$

where $D_t^\alpha(\cdot)$ denotes the modified Riemann-Liouville derivative, the function $r \in C^\alpha([t_0, \infty), R^+)$, which is the set of functions with continuous derivative or fractional order α, the function $e \in C([t_0,\infty), R)$, the function ψ belongs to $C(R,R)$ with $0 < \psi(x) \leq M$ for all $x \in R$ and some $M \in R^+$, the function $F \in C([t_0,\infty) \times R^2, R)$, and the function τ belongs to $C([t_0,\infty), R^+)$ with $\lim_{t\to\infty} \tau(t) = \infty$.

A solution of (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Equation (1) is called oscillatory if its all solutions are oscillatory.

Some of the key properties of the modified Riemann-Liouville derivative are as follows:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-a)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi) - f(0))\,d\xi, & 0 < \alpha < 1, \\ (f^{(n)}(t))^{(\alpha-n)}, & 1 \leq n \leq \alpha < n+1, \end{cases}$$

$$D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha},$$

$$D_t^\alpha\big(f(t)g(t)\big) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t),$$

$$D_t^\alpha f\big(g(t)\big) = f_g'\big(g(t)\big)D_t^\alpha g(t) = D_g^\alpha f\big(g(t)\big)\big(g'(t)\big)^\alpha,$$

which do not hold for classical Riemann-Liouville and Caputo derivatives. Especially Leibniz?s rule (product rule) and Faà di Bruno?s formula (chain rule) are important tools in our proofs.

We will use a transformation technique, also used in [24, 26–30], in our proofs. For the sake of convenience, in the rest of this paper, we denote

$$\xi = \xi(t) := \frac{t^\alpha}{\Gamma(1+\alpha)}, \qquad \xi_{t_0} := \xi(t_0) = \frac{t_0^\alpha}{\Gamma(1+\alpha)},$$

and for any function f, we denote $\tilde{f} = f \circ \xi^{-1}$, *i.e.*, $\tilde{f}(\xi) = f(t)$. We immediately get the conclusion $D_t^\alpha f(t) = \tilde{f}'(\xi)$, which gives us the ability to build a connection between fractional and integer order derivatives of functions. We will transform fractional order differential inequalities to integer order differential inequalities in our proofs with this connection.

Now we introduce a functional that will be used in the proofs of some results.

Let

$$B(s_i, t_i) = \big\{u \in C^1[s_i, t_i] : u(t) \neq 0 \text{ for } t \in (s_i, t_i), u(s_i) = u(t_i) = 0\big\}$$

for $i = 1, 2$. We define the functional $A(\cdot; s_i, t_i)$ for $G \in B(s_i, t_i)$ as

$$A(g; s_i, t_i) = \int_{s_i}^{t_i} G^2(t) g(t)\, dt, \quad s_i \le t \le t_i, i = 1, 2,$$

where $g \in C([t_0, \infty), R)$. It is easily seen that the linear functional $A(\cdot; s_i, t_i)$ satisfies

$$A(g'; s_i, t_i) = -A\left(2\frac{G'}{G} g; s_i, t_i\right) \ge -A\left(2\left|\frac{G'}{G}\right| |g|; s_i, t_i\right).$$

In the proofs of some of our results, we will also use another class of averaging functions $H \in C(D, R)$ which satisfy

(i) $H(t, t) = 0$, $H(t, s) > 0$ for $t > s$;

(ii) H has partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H}{\partial t} = h_1(t, s)\sqrt{H(t, s)}, \qquad \frac{\partial H}{\partial s} = -h_2(t, s)\sqrt{H(t, s)},$$

where $D = \{(t, s) : t_0 \le s \le t < \infty\}$ and $h_1, h_2 \in L_{\text{loc}}(D, \mathbb{R}^+)$.

Before our main results, now we state a useful lemma (Young?s inequality).

Lemma 1 ([31]) *If A and B are nonnegative constants and $m, n \in R$ such that $\frac{1}{m} + \frac{1}{n} = 1$, then*

$$\frac{1}{m}A + \frac{1}{n}B \ge A^{\frac{1}{m}} B^{\frac{1}{n}}.$$

We shall make use of the following conditions in our results:

(C_1) For any $T \ge t_0$, there exist s_1, s_2, t_1, t_2 such that $T \le s_1 < t_1 \le s_2 < t_2$ and

$$e(t) \le 0 \quad \text{for } t \in [s_1, t_1],$$
$$e(t) \ge 0 \quad \text{for } t \in [s_2, t_2].$$

(C_2) There exist a function $q(t) > 0$ and a constant $\gamma \ge 1$ such that $F(t, x, u)/x \ge q(t)|x|^{\gamma-1}$ holds for $t \in [s_1, t_1] \cup [s_2, t_2]$ and $x \ne 0$, $u \in \mathbb{R}$.

2 Main results

Theorem 2 *Suppose that conditions* (C_1)-(C_2) *hold. If there exists a function $G \in B(s_i, t_i)$ such that the inequality*

$$A(\widetilde{Q}; \xi_{s_i}, \xi_{t_i}) > A\left(M\frac{(G')^2}{G^2}\tilde{r}; \xi_{s_i}, \xi_{t_i}\right) \tag{2}$$

holds for $i = 1, 2$, where $Q(t) = \gamma(\gamma - 1)^{(1-\gamma)/\gamma}[q(t)]^{1/\gamma}|e(t)|^{\frac{(\gamma-1)}{\gamma}}$ with the convention $0^0 = 1$, then Eq. (1) *is oscillatory.*

Proof On the contrary, suppose that Eq. (1) has a nonoscillatory solution $x(t)$. Then $x(t)$ eventually must have one sign, *i.e.*, $x(t) \ne 0$ on $[T_0, \infty)$ for some large $T_0 \ge t_0$. Define

$$w(t) := \frac{r(t)\psi(x(t))D_t^\alpha x(t)}{x(t)} \tag{3}$$

for $t \geq T_0$. Then we deduce

$$D_t^\alpha w(t) = \frac{e(t)}{x(t)} - \frac{F(t, x(t), x(\tau(t)))}{x(t)} - \frac{1}{r(t)\psi(x(t))} w^2(t)$$

for $t \geq T_0$. By assuming (C_2) and condition on ψ, we obtain

$$D_t^\alpha w(t) \leq \frac{e(t)}{x(t)} - q(t)\left|x(t)\right|^{\gamma-1} - \frac{1}{Mr(t)} w^2(t) \tag{4}$$

for $t \geq T_0$. By assuming (C_1), if $x(t) > 0$, then we can choose $s_1, t_1 \geq T_0$ such that $e(t) \leq 0$ for $t \in [s_1, t_1]$. Similarly if $x(t) < 0$, then we can choose $s_2, t_2 \geq T_0$ such that $e(t) \geq 0$ for $t \in [s_2, t_2]$. So $\frac{e(t)}{x(t)} \leq 0$ (*i.e.*, $-\frac{e(t)}{x(t)} = |\frac{e(t)}{x(t)}|$) for $t \in [s_i, t_i]$, $i = 1, 2$, and from (4) one can deduce

$$D_t^\alpha w(t) \leq -\left|\frac{e(t)}{x(t)}\right| - q(t)\left|x(t)\right|^{\gamma-1} - \frac{1}{Mr(t)} w^2(t).$$

For $\gamma > 1$, by setting $m = \gamma$, $n = \frac{\gamma}{\gamma-1}$, $A = \gamma q(t)|x(t)|^{\gamma-1}$, $B = \frac{\gamma}{\gamma-1}|\frac{e(t)}{x(t)}|$ and using Lemma 1, we obtain

$$q(t)\left|x(t)\right|^{\gamma-1} + \left|\frac{e(t)}{x(t)}\right| \geq Q(t). \tag{5}$$

Note that inequality (5) trivially holds for $\gamma = 1$. Hence the inequality

$$D_t^\alpha w(t) \leq -Q(t) - \frac{1}{Mr(t)} w^2(t) \tag{6}$$

holds for $t \in [s_1, t_1]$ or $t \in [s_2, t_2]$.

Now let $w(t) = \widetilde{w}(\xi)$. Then we have $D_t^\alpha w(t) = \widetilde{w}'(\xi)$. So (6) is transformed to

$$\widetilde{w}'(\xi) \leq -\widetilde{Q}(\xi) - \frac{1}{M\tilde{r}(\xi)} \widetilde{w}^2(\xi) \tag{7}$$

for $\xi \in [\xi_{s_i}, \xi_{t_i}]$, $i = 1$ or 2.

Now multiplying $G^2(\xi)$ throughout inequality (7) and integrating from ξ_{s_i} to ξ_{t_i} for $i =$ 1 or 2, we obtain

$$A(\widetilde{Q}; \xi_{s_i}, \xi_{t_i}) \leq A\left(2\frac{|G'|}{|G|}|\widetilde{w}| - \frac{1}{M\tilde{r}(\xi)} \widetilde{w}^2; \xi_{s_i}, \xi_{t_i}\right). \tag{8}$$

Setting

$$m(v) = 2\frac{|G'|}{|G|} v - \frac{1}{M\tilde{r}} v^2, \quad v > 0,$$

we have $m'(v^*) = 0$ and $m''(v^*) < 0$, where $v^* = M\frac{|G'|}{|G|}\tilde{r}$, which implies that $m(v)$ obtains its maximum at v^*. So we have

$$m(v) \leq m\left(v^*\right) = M\frac{(G')^2}{G^2} \tilde{r}. \tag{9}$$

Then, by using (9) in (8), we obtain

$$A(\widetilde{Q};\xi_{s_i},\xi_{t_i}) \le A\left(M\frac{(G')^2}{G^2}\tilde{r};\xi_{s_i},\xi_{t_i}\right),$$

which contradicts (2). The proof is complete. □

Theorem 3 *Suppose that conditions* (C_1)-(C_2) *hold. If there exist some* $\delta_i \in (s_i,t_i)$ *for* $i = 1,2$, $H \in C(D,R)$ *satisfying* (i)-(ii) *and a positive function* $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ *such that*

$$\frac{1}{H(\delta_i,\xi_{s_i})}\int_{\xi_{s_i}}^{\delta_i}\tilde{\rho}(s)\left[H(s,\xi_{s_i})\widetilde{Q}(s) - \frac{M\tilde{r}(s)}{4}H_1^2(s,\xi_{s_i})\right]ds + \frac{1}{H(\xi_{t_i},\delta_i)}\int_{\delta_i}^{\xi_{t_i}}\tilde{\rho}(s)\left[H(\xi_{t_i},s)\widetilde{Q}(s) - \frac{M\tilde{r}(s)}{4}H_2^2(s,\xi_{t_i})\right]ds > 0 \tag{10}$$

for $i = 1,2$, *where* $H_i(t,s) := h_i(t,s) - \frac{\tilde{\rho}'(t)}{\tilde{\rho}(t)}\sqrt{H(t,s)}$, *then Eq.* (1) *is oscillatory.*

Proof On the contrary, suppose that Eq. (1) has a nonoscillatory solution $x(t)$. Then $x(t)$ eventually must have one sign, *i.e.*, $x(t) \neq 0$ on $[T_0,\infty)$ for some large $T_0 \ge t_0$. Define

$$z(t) := \rho(t)\frac{r(t)\psi(x(t))D_t^\alpha x(t)}{x(t)} \tag{11}$$

for $t \ge T_0$. Then we deduce

$$D_t^\alpha z(t) = \rho(t)\frac{e(t)}{x(t)} - \rho(t)\frac{F(t,x(t),x(\tau(t)))}{x(t)} + \frac{D_t^\alpha\rho(t)}{\rho(t)}z(t) - \frac{1}{\rho(t)r(t)\psi(x(t))}z^2(t)$$

for $t \ge T_0$. By assuming (C_2) and condition on ψ, we obtain

$$D_t^\alpha z(t) \le \rho(t)\left(\frac{e(t)}{x(t)} - q(t)\left|x(t)\right|^{\gamma-1}\right) + \frac{D_t^\alpha\rho(t)}{\rho(t)}z(t) - \frac{1}{M\rho(t)r(t)}z^2(t) \tag{12}$$

for $t \ge T_0$. By assuming (C_1), if $x(t) > 0$, then we can choose $s_1, t_1 \ge T_0$ such that $e(t) \le 0$ for $t \in [s_1,t_1]$. Similarly if $x(t) < 0$, then we can choose $s_2,t_2 \ge T_0$ such that $e(t) \ge 0$ for $t \in [s_2,t_2]$. So $\frac{e(t)}{x(t)} \le 0$ (*i.e.*, $-\frac{e(t)}{x(t)} = |\frac{e(t)}{x(t)}|$) for $t \in [s_i,t_i]$, $i = 1$ or 2, and from (12) one can deduce

$$D_t^\alpha z(t) \le \rho(t)\left(-\left|\frac{e(t)}{x(t)}\right| - q(t)\left|x(t)\right|^{\gamma-1}\right) + \frac{D_t^\alpha\rho(t)}{\rho(t)}z(t) - \frac{1}{M\rho(t)r(t)}z^2(t).$$

As in the proof of previous result, using Lemma 1, we obtain

$$D_t^\alpha z(t) \le \rho(t)Q(t) + \frac{D_t^\alpha\rho(t)}{\rho(t)}z(t) - \frac{1}{M\rho(t)r(t)}z^2(t) \tag{13}$$

for $\gamma \ge 1$ and $t \in [s_i,t_i]$, $i = 1$ or 2.

Now let $z(t) = \tilde{z}(\xi)$. Then we have $D_t^\alpha z(t) = \tilde{z}'(\xi)$ and $D_t^\alpha\rho(t) = \tilde{\rho}'(\xi)$. So (13) is transformed to

$$\tilde{z}'(\xi) \le \tilde{\rho}(\xi)\widetilde{Q}(\xi) + \frac{\tilde{\rho}'(\xi)}{\tilde{\rho}(\xi)}\tilde{z}(\xi) - \frac{1}{M\tilde{\rho}(\xi)\tilde{r}(\xi)}\tilde{z}^2(\xi) \tag{14}$$

for $\xi \in [\xi_{s_i},\xi_{t_i}]$, $i = 1$ or 2.

Let δ_i be an arbitrary point in (ξ_{s_i}, ξ_{t_i}). Substituting ξ with s, multiplying (14) with $H(\xi,s)$ and integrating it over $[\delta_i,\xi)$ for $\xi \in [\delta_i, \xi_{t_i})$, $i = 1$ or 2, we obtain

$$\begin{aligned}
&\int_{\delta_i}^{\xi} H(\xi,s)\tilde{\rho}(s)\widetilde{Q}(s)\,ds \\
&\quad\le -\int_{\delta_i}^{\xi} H(\xi,s)\tilde{z}'(s)\,ds + \int_{\delta_i}^{\xi} H(\xi,s)\left[\frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)}\tilde{z}(s) - \frac{1}{M\tilde{\rho}(s)\tilde{r}(s)}\tilde{z}^2(s)\right]ds \\
&\quad= H(\xi,\delta_i)\tilde{z}(\delta_i) - \int_{\delta_i}^{\xi} \tilde{z}(s)h_2(\xi,s)\sqrt{H(\xi,s)}\,ds \\
&\qquad+ \int_{\delta_i}^{\xi} H(\xi,s)\left[\frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)}\tilde{z}(s) - \frac{1}{M\tilde{\rho}(s)\tilde{r}(s)}\tilde{z}^2(s)\right]ds \\
&\quad= H(\xi,\delta_i)\tilde{z}(\delta_i) - \int_{\delta_i}^{\xi}\left[\left(\frac{H(\xi,s)}{M\tilde{\rho}(s)\tilde{r}(s)}\right)^{1/2}\tilde{z}(s) - \frac{1}{2}\bigl(M\tilde{\rho}(s)\tilde{r}(s)\bigr)^{1/2}H_2(\xi,s)\right]^2 ds \\
&\qquad+ \int_{\delta_i}^{\xi} \frac{M\tilde{\rho}(s)\tilde{r}(s)}{4}H_2^2(\xi,s)\,ds \\
&\quad\le H(\xi,\delta_i)\tilde{z}(\delta_i) + \int_{\delta_i}^{\xi} \frac{M\tilde{\rho}(s)\tilde{r}(s)}{4}H_2^2(\xi,s)\,ds.
\end{aligned}$$

Now letting $\xi \to \xi_{t_i}^-$ and dividing it by $H(\xi_{t_i},\delta_i)$, we obtain

$$\begin{aligned}
&\frac{1}{H(\xi_{t_i},\delta_i)}\int_{\delta_i}^{\xi_{t_i}} H(\xi_{t_i},s)\tilde{\rho}(s)\widetilde{Q}(s)\,ds \\
&\quad\le \tilde{z}(\delta_i) + \frac{1}{H(\xi_{t_i},\delta_i)}\int_{\delta_i}^{\xi_{t_i}} \frac{M\tilde{\rho}(s)\tilde{r}(s)}{4}H_2^2(\xi_{t_i},s)\,ds. \qquad (15)
\end{aligned}$$

On the other hand, substituting ξ with s, multiplying (14) with $H(s,\xi)$ and integrating it over (ξ,δ_i) for $\xi \in [\xi_{s_i},\delta_i)$, $i = 1$ or 2, with similar calculations, we obtain

$$\begin{aligned}
&\int_{\xi}^{\delta_i} H(s,\xi)\tilde{\rho}(s)\widetilde{Q}(s)\,ds \\
&\quad\le -\int_{\xi}^{\delta_i} H(s,\xi)\tilde{z}'(s)\,ds + \int_{\xi}^{\delta_i} H(s,\xi)\left[\frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)}\tilde{z}(s) - \frac{1}{M\tilde{\rho}(s)\tilde{r}(s)}\tilde{z}^2(s)\right]ds \\
&\quad= -H(\delta_i,\xi)\tilde{z}(\delta_i) - \int_{\xi}^{\delta_i} \tilde{z}(s)h_1(s,\xi)\sqrt{H(s,\xi)}\,ds \\
&\qquad+ \int_{\xi}^{\delta_i} H(s,\xi)\left[\frac{\tilde{\rho}'(s)}{\tilde{\rho}(s)}\tilde{z}(s) - \frac{1}{M\tilde{\rho}(s)\tilde{r}(s)}\tilde{z}^2(s)\right]ds \\
&\quad\le -H(\delta_i,\xi)\tilde{z}(\delta_i) + \int_{\xi}^{\delta_i} \frac{M\tilde{\rho}(s)\tilde{r}(s)}{4}H_1^2(s,\xi)\,ds.
\end{aligned}$$

Now letting $\xi \to \xi_{s_i}^+$ and dividing it by $H(\delta_i,\xi_{s_i})$, we obtain

$$\begin{aligned}
&\frac{1}{H(\delta_i,\xi_{s_i})}\int_{\xi_{s_i}}^{\delta_i} H(s,\xi_{s_i})\tilde{\rho}(s)\widetilde{Q}(s)\,ds \\
&\quad\le -\tilde{z}(\delta_i) + \frac{1}{H(\delta_i,\xi_{s_i})}\int_{\xi_{s_i}}^{\delta_i} \frac{M\tilde{\rho}(s)\tilde{r}(s)}{4}H_1^2(s,\xi_{s_i})\,ds. \qquad (16)
\end{aligned}$$

By combining (15) and (16), we obtain

$$\begin{aligned}&\frac{1}{H(\delta_i,\xi_{s_i})}\int_{\xi_{s_i}}^{\delta_i} H(s,\xi_{s_i})\tilde{\rho}(s)\widetilde{Q}(s)\,ds + \frac{1}{H(\xi_{t_i},\delta_i)}\int_{\delta_i}^{\xi_{t_i}} H(\xi_{t_i},s)\tilde{\rho}(s)\widetilde{Q}(s)\,ds \\ &\quad\le \frac{1}{H(\delta_i,\xi_{s_i})}\int_{\xi_{s_i}}^{\delta_i} \frac{M\tilde{\rho}(s)\tilde{r}(s)}{4}H_1^2(s,\xi_{s_i})\,ds + \frac{1}{H(\xi_{t_i},\delta_i)}\int_{\delta_i}^{\xi_{t_i}} \frac{M\tilde{\rho}(s)\tilde{r}(s)}{4}H_2^2(\xi_{t_i},s)\,ds,\end{aligned}$$

which contradicts (10). The proof is complete. □

3 Examples

Example 1 Consider the fractional differential equation

$$D_t^\alpha\left(\sin^2\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right)e^{-x^2(t)}D_t^\alpha x(t)\right) + x(t)\left[N + x^2\left(\tau(t)\right)\right] = \sin\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right) \tag{17}$$

for $t \geq 2$, $0 < \alpha < 1$ and $N > 0$. This corresponds to (1) with $r(t) = \sin^2(\frac{t^\alpha}{\Gamma(1+\alpha)})$, $\psi(x) = e^{-x^2}$, $F(t,x,u) = x(N+u^2)$, $e(t) = \sin(\frac{t^\alpha}{\Gamma(1+\alpha)})$, and therefore $\tilde{r}(\xi) = \sin^2\xi$, $M = 1$, $\gamma = 1$, $q(t) = Q(t) = N$. Now, by choosing $G(t) = \sin^2(t)$, $\xi_{s_1} = k\pi$, $\xi_{t_1} = \xi_{s_2} = (k+1)\pi$, $\xi_{t_2} = (k+2)\pi$ for some sufficiently large k, it is easy to verify that

$$N\frac{\pi}{2} = A(\widetilde{Q};\xi_{s_i},\xi_{t_i}) > A\left(M\frac{(G')^2}{G^2}\tilde{r};\xi_{s_i},\xi_{t_i}\right) = 2\pi$$

for $i = 1,2$ and $N > 4$. Thus, according to Theorem 2, Eq. (17) is oscillatory for $N > 4$.

Example 2 Consider the fractional differential equation

$$D_t^\alpha\left(D_t^\alpha x(t)\right) + x(t)\left[1 + x^2\left(\tau(t)\right)\right] = \sin\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right) \tag{18}$$

for $t \geq 2$ and $0 < \alpha < 1$. This corresponds to (1) with $r(t) = 1$, $\psi(x) = 1$, $F(t,x,u) = x(1+u^2)$, $e(t) = \sin(\frac{t^\alpha}{\Gamma(1+\alpha)})$, and therefore $\tilde{r}(\xi) = 1$, $M = 1$, $\gamma = 1$, $q(t) = Q(t) = 1$. Now, by choosing $H(t,s) = (t-s)^2$, $\xi_{s_1} = k\pi$, $\xi_{t_1} = \xi_{s_2} = (k+2)\pi$, $\xi_{t_2} = (k+4)\pi$, $\delta_1 = (k+1)\pi$, $\delta_2 = (k+3)\pi$ for some sufficiently large k and $\rho(t) = 1$, we have $h_i(t,s) = H_i(t,s) = 2$ for $i = 1,2$. Since

$$\int_{k\pi}^{(k+1)\pi}\left[\left((k+1)\pi - s\right)^2 - 1\right]ds = \frac{\pi^3}{3} - \pi > 0,$$

inequality (10) clearly holds. Oscillation of Eq. (18) follows from Theorem 3.

Competing interests
The author declares that they have no competing interests.

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