Zhu *Boundary Value Problems* (2024) 2024:85
https://doi.org/10.1186/s13661-024-01894-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Existence of positive periodic solutions for Liénard equation with a singularity of repulsive type

Yu Zhu[1*]

[*]Correspondence:
404665116@qq.com
[1]Ma'anshan University Maanshan 243000, P.R. China

Abstract

In this paper, the existence of positive periodic solutions is studied for Liénard equation with a singularity of repulsive type,

$$x''(t) + f(x(t))x'(t) + \varphi(t)x^{\mu}(t) - \frac{1}{x^{\gamma}(t)} = e(t),$$

where $f : (0, +\infty) \to R$ is continuous, which may have a singularity at the origin, the sign of $\varphi(t)$, $e(t)$ is allowed to change, and μ, γ are positive constants. By using a continuation theorem, as well as the techniques of a priori estimates, we show that this equation has a positive T-periodic solution when $\mu \in [0, +\infty)$.

Keywords: Liénard equation; Periodic solutions; Singularity; Continuation theorem

1 Introduction

Since singular equations have a wide range of application in physics, engineering, mechanics, and other subjects (see [1–7]), the periodic problem for a certain second order differential equation has attracted much attention from many researchers. In the past years, lots of papers (see [8–14]) were concerned with the problem of periodic solutions to the second order singular equation without the first derivative term,

$$x'' + \varphi(t)x - \frac{b(t)}{x^{\mu}} = h(t), \tag{1.1}$$

where $f : [0, \infty) \to \mathbb{R}$ is continuous, $\varphi, b, h \in L^1[0, T]$, and $\mu > 0$ is a constant. Among these papers, we notice that the coefficient function $\varphi(t)$ is required to be

$$\varphi(t) \geq 0 \quad \text{for a.e. } t \in [0, T]. \tag{1.2}$$

This is because (1.2), together with other conditions, can ensure that the function $G(t, s) \geq 0$ for $(t, s) \in [0, T] \times [0, T]$, where the $G(t, s)$ is the Green function associated with the

Springer

boundary value problem for Hill's equation

$$x''(t) + \varphi(t)x(t) = h(t), \quad x(0) = x(T),\ x'(0) = x'(T).$$

The condition $G(t,s) \geq 0$ for $(t,s) \in [0,T] \times [0,T]$ is crucial for obtaining the positive periodic solutions to (1.1) by means of some fixed point theorems on cones. Beginning with the paper of Habets–Sanchez [15], many works (see [16–21]) discussed the existence of a periodic solution for Liénard equations with singularities,

$$x''(t) + f(x(t))x'(t) + \varphi(t)x(t) - \frac{1}{x^{\gamma}(t)} = e(t), \tag{1.3}$$

where $\varphi(t)$ and $e(t)$ are T-periodic with $\varphi, e \in L^1[0,T]$, while γ is a constant with $\gamma > 0$. However, in those papers, the conditions of $\varphi(t) \geq 0$ for a.e. $t \in [0,T]$, the strong singularity $\gamma \in [1,+\infty)$, and $f(x)$ being continuous on $[0,+\infty)$ are needed. To the best of our knowledge, there are fewer papers dealing with the equation where the function $f(x)$ possesses a singularity at $x = 0$. We find that Hakl, Torres, and Zamora in [22] considered the periodic problem for the singular equation of repulsive type,

$$x''(t) + f(x(t))x'(t) + \varphi(t)x^{\mu}(t) + g(x(t)) = 0, \tag{1.4}$$

where $\mu \in (0,1]$ is a constant, φ is a T-periodic function with $\varphi \in L^1([0,T],R)$, and the sign of $\varphi(t)$ can change, while $f \in C((0,+\infty),R)$ may be singular at $x = 0$ and $g \in C((0,+\infty),R)$ has a repulsive singularity at $x = 0$, i.e., $\lim_{x\to 0^+} g(x) = -\infty$. By using Schauder's fixed point theorem, some results on the existence of positive T-periodic solutions were obtained. However, the strong singularity condition $\int_0^1 g(s)ds = -\infty$ is also required. In a recent paper [23], the authors consider the periodic problem to (1.4) for the special case $g(x) = \frac{1}{x^{\gamma}}$, where $\gamma \in (0,+\infty)$. But, in [23], the function $\varphi(t)$ is required to satisfy $\varphi(t) \geq 0$ a.e. $t \in [0,T]$ for the case $\mu > 1$ (see Theorem 3.1, [23]). Motivated by this, in the present paper, we continue to study the periodic problem for the singular equation,

$$x''(t) + f(x(t))x'(t) + \varphi(t)x^{\mu}(t) - \frac{1}{x^{\gamma}} = e(t), \tag{1.5}$$

where f, φ are as same as those in (1.4); $\mu > 0$ and $\gamma > 0$ are constants, e is a T-periodic function with $e \in L^1([0,T],R)$, and $\int_0^T e(s)ds = 0$. By means of a continuation theorem of coincidence degree principle developed by Manásevich and Mawhin, as well as the techniques of a priori estimates, some new results on the existence of positive periodic solutions are obtained. The interesting point in this paper is that the function $f(x)$ has a singularity at $x = 0$, the sign of $\varphi(t)$ is allowed to change, and $\mu, \gamma \in (0,+\infty)$. Compared with [22], we allow the singular term $\frac{1}{x^{\gamma}}$ to have a weak singularity, i.e., $\gamma \in (0,1)$. Also, for the case of $\mu > 1$, the sign of $\varphi(t)$ is allowed to change, which is essentially different from the condition $\varphi(t) \geq 0$ for a.e. $t \in [0,T]$ in [23].

2 Essential definitions and lemmas

Throughout this paper, let $C_T = \{x \in C(R,R) : x(t+T) = x(t), \forall t \in R\}$ with the norm $|x|_{\infty} = \max_{t\in[0,T]} |x(t)|$. Clearly, C_T is a Banach space. For any T-periodic function $x(t)$, we denote

$\bar{x} = \frac{1}{T}\int_0^T x(s)ds$, $x_+(t) = \max\{x(t), 0\}$, and $x_- = -\min\{x(t), 0\}$. Thus, $x(t) = x_+(t) - x_-(t)$ for all $t \in R$, and $\overline{x} = \overline{x_+} - \overline{x_-}$. Furthermore, for each $u \in C_T$, let $\|u\|_p = (\int_0^T |u(s)|^p ds)^{\frac{1}{p}}$, $p \in [1, +\infty)$.

Lemma 2.1 ([24]) *Assume that there exit positive constants M_0 and M_1, with $0 < M_0 < M_1$, such that the following conditions hold*:

(1) *for each $\lambda \in (0, 1]$, each possible positive T-periodic solution u to the equation*

$$x''(t) + \lambda f(x(t))x'(t) + \lambda\varphi(t)x^{\mu}(t) - \frac{\lambda}{x^{\gamma}(t)} = 0$$

satisfies the inequality $M_0 < u(t) < M_1$ for all $t \in [0, T]$;

(2) *each possible solution $c \in (0, +\infty)$ to the equation*

$$\frac{1}{c^{\gamma}} - \overline{\varphi}c^{\mu} = 0$$

satisfies the inequality $M_0 < c < M_1$;

(3) *the inequality*

$$\left(\frac{1}{M_0^{\gamma}} - \overline{\varphi}M_0^{\mu}\right)\left(\frac{\overline{\alpha}}{M_1^{\gamma}} - \overline{\varphi}M_1^{\mu}\right) < 0$$

holds.

Then equation has at least one positive T-periodic solution $u(t)$ such that $M_0 < u(t) < M_1$ for all $t \in [0, T]$.

Lemma 2.2 ([22]) *Let $u(t) : [0, \omega] \to R$ be an arbitrary absolutely continuous function with $u(0) = u(\omega)$. Then the inequality*

$$\left(\max_{t\in[0,\omega]} u(t) - \min_{t\in[0,\omega]} u(t)\right)^2 \leq \frac{\omega}{4}\int_4^{\omega} |u'(s)|^2 ds$$

holds.

Remark 2.3 If $\overline{\varphi} > 0$, then there are constants C_1 and C_2 with $0 < C_1 < C_2$ such that

$$\frac{1}{x^{\gamma}} - \overline{\varphi}x^{\mu} > 0 \quad \forall x \in (0, C_1) \tag{2.1}$$

and

$$\frac{1}{x^{\gamma}} - \overline{\varphi}x^{\mu} < 0 \quad \forall x \in (C_2, +\infty). \tag{2.2}$$

Now, we embed equation (1.5) into the following equation family with a parameter $\lambda \in (0, 1]$:

$$x''(t) + \lambda f(x(t))x'(t) + \lambda\varphi(t)x^{\mu}(t) - \lambda\frac{1}{x^{\gamma}(t)} = \lambda e(t). \tag{2.3}$$

Let

$$D = \left\{x \in C_T^1 : x''(t) + \lambda f(x(t))x'(t) + \lambda\varphi(t)x^{\mu}(t) - \lambda\frac{1}{x^{\gamma}(t)} = \lambda e(t), \lambda \in (0, 1]\right\}, \tag{2.4}$$

and

$$F(x) = \int_1^x f(s)ds, G(x) = \int_1^x s^\gamma f(s)ds, \quad x \in (0, +\infty). \tag{2.5}$$

Lemma 2.4 *Assume $\overline{\varphi} > 0$ and $\overline{e} = 0$, then there are two constants $\tau_1, \tau_2 \in [0, T]$ for each $u \in D$, such that*

$$u(\tau_1) \le \max\left\{1, \left(\frac{2}{\overline{\varphi}}\right)^{\frac{1}{\mu}}\right\} := A_0 \tag{2.6}$$

and

$$u(\tau_2) \ge \min\left\{1, \left(\frac{1}{\overline{\varphi_+}}\right)^{\frac{1}{\gamma}}\right\} := A_1. \tag{2.7}$$

Proof Let $u \in D$, then

$$u''(t) + \lambda f(u(t))u'(t) + \lambda\varphi(t)u^\mu(t) - \frac{\lambda}{u^\gamma(t)} = \lambda e(t). \tag{2.8}$$

Dividing both sides of (2.8) by $u^\mu(t)$ and integrating over the interval $[0, T]$, we obtain

$$\int_0^T \frac{u''(t)}{u^\mu(t)}dt + \lambda T\overline{\varphi} = \lambda \int_0^T \frac{1}{u^{\mu+\gamma}(t)}dt + \lambda \int_0^T \frac{e(t)}{u^\mu(t)}dt.$$

Since the inequality $\int_0^T \frac{u''(t)}{u^\mu(t)}dt \ge 0$ holds, it is easy to see that

$$\lambda T\overline{\varphi} \le \lambda T\frac{1}{u^{\mu+\gamma}(\xi)} + \lambda T\frac{\overline{e_+}}{u^\mu(\tau_1)},$$

i.e.,

$$0 < \overline{\varphi} \le \frac{1}{u^{\mu+\gamma}(\xi)} + \frac{1}{u^\mu(\tau_1)}. \tag{2.9}$$

From this, we can verify (2.6). In fact, if (2.6) does not hold, then

$$u(t) > \max\left\{1, \left(\frac{2}{\overline{\varphi}}\right)^{\frac{1}{\mu+\gamma}}\right\}, \quad \forall t \in [0, T], \tag{2.10}$$

which together with (2.9) gives

$$0 < \overline{\varphi} \le \frac{1}{u^\mu(\tau_1)} + \frac{\overline{\varphi}}{2},$$

i.e.,

$$u(\tau_1) < \left(\frac{2}{\overline{\varphi}}\right)^{\frac{1}{\mu}}. \tag{2.11}$$

On the other hand, (2.10) implies that $u(\tau_1) > 1$. It follows from (2.11) that $(\frac{2}{\overline{\varphi}})^{\frac{1}{\mu}} > 1$, i.e., $\frac{2}{\overline{\varphi}} > 1$. By using (2.11) again, we get

$$u(\tau_1) < \left(\frac{2}{\overline{\varphi}}\right)^{\frac{1}{\mu+\gamma}}, \tag{2.12}$$

which contradicts with (2.10), verifying (2.6).

Integrating both sides of (2.8) over the interval $[0,T]$, we obtain

$$\int_0^T \varphi(t)u^\mu(t)dt - \int_0^T \frac{1}{u^\gamma(t)}dt = \int_0^T e(t)dt.$$

Since $\int_0^T e(t)dt = T\overline{e} = 0$, it follows that $\int_0^T \varphi(t)u^\mu(t)dt = \int_0^T \frac{1}{u^\gamma(t)}dt$. If

$$u(t) < 1 \quad \forall t \in [0,T], \tag{2.13}$$

then

$$\int_0^T \frac{1}{u^\gamma(t)}dt \le \int_0^T \varphi_+(t)u^\mu(t)dt \le T\overline{\varphi_+}.$$

By using the mean value theorem for integrals, we get that there is a point $\xi \in [0,T]$ such that

$$\frac{T}{u^\gamma(\xi)} \le T\overline{\varphi_+},$$

i.e.,

$$u(\xi) \ge \left(\frac{1}{\overline{\varphi_+}}\right)^{\frac{1}{\gamma}}. \tag{2.14}$$

Thus (2.7) immediately follows from (2.13) and (2.14). □

Lemma 2.5 *Assume $\overline{\varphi} > 0$ and $\overline{e} = 0$ for a.e. $t \in [0,T]$ and suppose that the following assumptions*:

$$B_0 = \inf_{[A_1,+\infty)} H(x) > -\infty \tag{2.15}$$

and

$$\lim_{s\to 0^+}\left(F(s) + \frac{T}{s^\gamma}\right) < B_0 - T\overline{e_+} \tag{2.16}$$

hold, where $H(x) = F(x) - T\overline{\varphi_-}x^\mu$. Then there is a constant $\gamma_0 > 0$ such that

$$\min_{t\in[0,T]} u(t) \ge \gamma_0, \quad \textit{uniformly for } u \in D. \tag{2.17}$$

Proof Let $u \in D$, then u satisfies

$$u''(t) + \lambda f(u(t))u'(t) + \lambda\varphi(t)u^\mu(t) - \lambda\frac{1}{u^\gamma(t)} = \lambda e(t). \tag{2.18}$$

Since $u \in D$, it is easy to see that there are two points $t_1, t_2 \in R$ such that $u(t_1) = \max_{t\in[0,T]} u(t)$, $u(t_2) = \min_{t\in[0,T]} u(t)$, and $0 < t_1 - t_2 \le T$. By integrating (2.18) over the interval $[t_2, t_1]$, we get

$$\begin{aligned} F(u(t_1)) &= F(u(t_2)) + \int_{t_2}^{t_1} \frac{1}{u^\gamma(t)} dt - \int_{t_2}^{t_1} \varphi(t) u^\mu(t) dt + \int_{t_2}^{t_1} e(t) dt \\ &\le F(u(t_2)) + \frac{T}{u^\gamma(t_2)} + T\overline{\varphi_-} u^\mu(t_1) + T\overline{e_+}, \end{aligned}$$

and then

$$\begin{aligned} F(u(t_2)) + \frac{T}{u^\gamma(t_2)} &\ge F(u(t_1)) - T\overline{\varphi_-} u^\mu(t_1) - T\overline{e_+} \\ &\ge \inf_{[A_1,+\infty)} H(x) - T\overline{e_+} \\ &= B_0 - T\overline{e_+}. \end{aligned} \tag{2.19}$$

Assumption (2.16) ensures that there is a constant $\gamma_0 > 0$ such that

$$F(s) + \frac{T}{s^\gamma} < B_0 - T\overline{e_+}, \quad \text{for } s \in (0, \gamma_0). \tag{2.20}$$

Combining (2.19) with (2.20), we get that

$$\min_{t\in[0,T]} u(t) = u(t_2) \ge \gamma_0. \tag{2.21}$$

□

Lemma 2.6 *Assume* $\overline{\varphi} > 0$ *and* $\overline{e} = 0$ *for a.e.* $t \in [0, T]$ *and suppose that the following assumptions*:

$$B_0 = \inf_{[A_1,+\infty)} H(x) > -\infty, \tag{2.22}$$

$$\lim_{s\to 0^+} \left(F(s) + \frac{T}{s^\gamma} \right) < B_0 - T\overline{e_+}, \tag{2.23}$$

and

$$\lim_{s\to +\infty} (F(s) - T\overline{\varphi_+} s^\mu) = +\infty \tag{2.24}$$

hold. Then, there exists a constant $\gamma_1 > 0$ *such that*

$$\max_{t\in[0,T]} u(t) \le \gamma_1, \quad \textit{uniformly for } u \in D. \tag{2.25}$$

Proof Since $u \in D$, the function u satisfies (2.18). Then there are two points $t_1, t_2 \in R$ such that $u(t_1) = \max_{t\in[0,T]} u(t)$, $u(t_2) = \min_{t\in[0,T]} u(t)$, and $0 < t_2 - t_1 < T$. By integrating over the interval $[t_1, t_2]$, we get

$$\begin{aligned} F(u(t_1)) &= F(u(t_2)) - \int_{t_1}^{t_2} \frac{1}{u^\gamma(t)} dt + \int_{t_1}^{t_2} \varphi(t) u^\mu(t) dt - \int_{t_1}^{t_2} e(t) dt \\ &\le F(u(t_2)) + T\overline{\varphi_+} u^\mu(t_1) + T\overline{e_-}, \end{aligned}$$

thus, by the assumptions of (2.6), (2.22), and (2.24), according to the proof of Lemma 2.4, we obtain

$$\gamma_0 \le u(t_2) = \min_{t\in[0,T]} u(t) \le A_0. \tag{2.26}$$

So, we have

$$\begin{aligned} F(u(t_1)) - T\overline{\varphi_+}u^\mu(t_1) &\le F(u(t_2)) + T\overline{e_-} \\ &\le \max_{x\in[\gamma_0,A_0]} F(x) + T\overline{e_-}. \end{aligned} \tag{2.27}$$

Assumption (2.24) now ensures that there is a constant $\gamma_1 > \gamma_0 > 0$ such that

$$F(s) - T\overline{\varphi_+}s^\mu > \max_{x\in[\gamma_0,A_0]} F(x) + T\overline{e_-} \quad \text{for all } s \in (\gamma_1, +\infty). \tag{2.28}$$

Therefore, (2.27) and (2.28) imply

$$\max_{t\in[0,T]} u(t) = u(t_1) \le \gamma_1, \quad \text{uniformly for } u \in D. \tag{2.29}$$

□

Lemma 2.7 *Assume $\overline{\varphi} > 0$ and $\overline{e} = 0$ for a.e. $t \in [0,T]$ and suppose that the following assumptions*:

$$C_0 = \sup_{[A_1,+\infty)} H_1(x) < +\infty \tag{2.30}$$

and

$$\lim_{s\to 0^+} (F(s) - \frac{T}{s^\gamma}) > C_0 + T\overline{e_+} \tag{2.31}$$

hold, where $H_1(x) = F(x) + T\overline{\varphi_-}x^\mu$. Then there is a constant $\gamma_2 > 0$ such that

$$\min_{t\in[0,T]} u(t) \ge \gamma_2, \quad \textit{uniformly for } u \in D. \tag{2.32}$$

Proof Since $u \in D$, it is easy to see that there exist two points $t_1, t_2 \in R$ such that $u(t_1) = \max_{t\in[0,T]} u(t)$, $u(t_2) = \min_{t\in[0,T]} u(t)$, and $0 < t_2 - t_1 < T$. By integrating over the interval $[t_1, t_2]$, we get

$$\begin{aligned} F(u(t_2)) &= F(u(t_1)) + \int_{t_1}^{t_2} \frac{1}{u^\gamma(t)} dt - \int_{t_1}^{t_2} \varphi(t)u^\mu(t)dt + \int_{t_1}^{t_2} e(t)dt \\ &\le F(u(t_1)) + \frac{T}{u^\gamma(t_2)} + T\overline{\varphi_-}u^\mu(t_1) + T\overline{e_+}, \end{aligned}$$

and then

$$\begin{aligned} F(u(t_2)) - \frac{T}{u^\gamma(t_2)} &\le F(u(t_1)) + T\overline{\varphi_-}u^\mu(t_1) + T\overline{e_+} \\ &\le \sup_{[A_1,+\infty)} H_1(x) + T\overline{e_+} \\ &= C_0 + T\overline{e_+}. \end{aligned} \tag{2.33}$$

Assumption (2.31) ensures that there is a constant $\gamma_2 > 0$ such that

$$F(s) - \frac{T}{s^\gamma} > C_0 + T\overline{e_+}, \quad \text{for } s \in (0, \gamma_2).$$

So, it is easy to see from (2.33) that

$$u(t_2) = \min_{t\in[0,T]} u(t) \geq \gamma_2. \tag{2.34}$$

□

Lemma 2.8 *Assume* $\overline{\varphi} > 0$ *and* $\overline{e} = 0$ *for a.e.* $t \in [0, T]$ *and suppose that the following assumptions*:

$$C_0 = \sup_{[A_1,+\infty)} H_1(x) < +\infty, \quad H_1(x) = F(x) + T\overline{\varphi_-}x^\mu, \tag{2.35}$$

as well as

$$\lim_{s\to 0^+} (F(s) - \frac{T}{s^\gamma}) > C_0 + T\overline{e_+} \tag{2.36}$$

and

$$\lim_{s\to +\infty} (F(s) + T\overline{\varphi_-}s^\mu + \frac{T}{s^\gamma}) = -\infty, \tag{2.37}$$

hold. Then, there exists a constant $\gamma_3 > 0$ *such that*

$$\max_{t\in[0,T]} u(t) \leq \gamma_3, \quad \text{uniformly for } u \in D. \tag{2.38}$$

Proof Let $u \in D$, then u satisfies (2.18). Let t_1 and t_2 be defined as in the proof of Lemma 2.6, that is, $u(t_1) = \max_{t\in[0,T]} u(t)$, $u(t_2) = \min_{t\in[0,T]} u(t)$, and $0 < t_2 - t_1 < T$. By integrating over the interval $[t_1, t_2]$, we get

$$\begin{aligned} F(u(t_1)) &= F(u(t_2) - \int_{t_1}^{t_2} \frac{1}{u^\gamma(t)} dt + \int_{t_1}^{t_2} \varphi(t)u^\mu(t)dt - \int_{t_1}^{t_2} e(t)dt \\ &\geq F(u(t_2)) - \frac{T}{u^\gamma(t_1)} - T\overline{\varphi_-}u^\mu(t_1) - T\overline{e_+}. \end{aligned} \tag{2.39}$$

Thus, by the assumptions of (2.6), (2.35), and (2.36), and according to the proof of Lemma 2.6, we have

$$\gamma_2 \leq u(t_2) = \min_{t\in[0,T]} u(t) \leq A_0, \tag{2.40}$$

which together with (2.39) yields

$$\begin{aligned} F(u(t_1)) + T\overline{\varphi_-}u^\mu(t_1) + \frac{T}{u^\gamma(t_1)} &= F(u(t_2) - T\overline{e_+} \\ &\geq \min_{x\in[\gamma_2,A_0]} F(x) - T\overline{e_+}. \end{aligned} \tag{2.41}$$

On the other hand, assumption (2.37) gives that there exits a constant $\gamma_3 > 0$ such that

$$F(s) + T\overline{\varphi_-}s^\mu + \frac{T}{s^\gamma} < \min_{x\in[\gamma_2,A_0]} F(x) - T\overline{e_+}, \quad s \in (\gamma_3, +\infty). \tag{2.42}$$

Combining (2.41) with (2.42), we get that

$$u(t_1) = \max_{t\in[0,T]} u(t) \le \gamma_3. \tag{2.43}$$

□

3 Main results

Theorem 3.1 *Assume $\overline{\varphi} > 0$ and $\overline{e} = 0$ for a.e. $t \in [0, T]$ and suppose that the assumptions of (2.15) and (2.16) in Lemma 2.4, as well as the assumption (2.24) in Lemma 2.5, hold. Then for each $\mu \in [0, +\infty)$, equation (1.5) has at least one positive T-periodic solution.*

Proof Due to assumptions of Lemma 2.4, we see that there are two constants $\gamma_0 > 0$, $\gamma_1 > 0$ such that $\min u(t) \ge \gamma_0$, $\max u(t) \le \gamma_1$.

Now, we will show that there exists a positive constant $M > 0$ such that $\max_{t\in[0,T]} |u'(t)| \le M$, uniformly for $u \in D$. If $u(t_1) = \max_{t\in[0,T]}$, $t_1 \in [0, T]$, then $u'(t_1) = 0$. Letting $t \in [0, T]$, we integrate (2.8) over the interval $[t_1, t]$ and get

$$\int_{t_1}^t u''(t)dt + \lambda \int_{t_1}^t f(u(t))u'(t)dt + \lambda \int_{t_1}^t \varphi(t)u^\mu(t)dt - \lambda \int_{t_1}^t \frac{1}{u^\gamma(t)}dt = \lambda \int_{t_1}^t e(t)dt, \tag{3.1}$$

which yields

$$u'(t) = \lambda \int_{t_1}^t (-f(u(t))u'(t) - \varphi(t)u^\mu(t) + \frac{1}{u^\gamma(t)} + e(t))dt, \tag{3.2}$$

and then we obtain

$$\begin{aligned} |u'(t)| &\le \lambda|F(u(t)) - F(u(t_1))| + \lambda \int_{t_1}^{t_1+T} |\frac{1}{u^\gamma(t)}|dt + \lambda \int_{t_1}^{t_1+T} |e(t)|dt \\ &\quad + \lambda \int_{t_1}^{t^1+T} |\varphi(t)u^\mu(t)|dt \\ &\le 2 \max_{\gamma_0\le u\le\gamma_1} |F(u(t))| + \frac{T}{\gamma_0^\gamma} + T\overline{e_+} + T\overline{|\varphi|}\gamma_1^\mu \\ &:= M, \quad \text{for all } t \in [0, T]. \end{aligned} \tag{3.3}$$

So, we have

$$\max_{t\in[0,T]} |u'(t)| \le M, \quad \text{uniformly for } u \in D. \tag{3.4}$$

Let $m_1 = \min\{\gamma_0, D_1\}$ and $m_2 = \{\gamma_1, D_2\}$ be two constants, where D_1 and D_2 are the constants determined in Remark 2.3. Then we get that every possible positive T-periodic solution $x(t)$ to equation (1.5) satisfies

$$m_1 < x(t) < m_2, \quad |x'(t)| < M, \quad \text{for all } t \in [0, T]. \tag{3.5}$$

Furthermore, we have

$$\left(\overline{\varphi} m_1^{\mu} - \frac{1}{m_1^{\gamma}}\right)\left(\overline{\varphi} m_2^{\mu} - \frac{1}{m_2^{\gamma}}\right) < 0, \tag{3.6}$$

by using Lemma 2.1, thus equation (1.5) has at least one positive T-periodic solution.

On the other hand, by Lemmas 2.6 and 2.7, we get the same conclusion as in Theorem 3.1, which can be proved similarly. Thus, the proofs are omitted. □

Theorem 3.2 *Assume $\overline{\varphi} > 0$ and $\overline{e} = 0$ for a.e. $t \in [0, T]$ and suppose that the assumptions of* (2.30) *and* (2.31) *in Lemma* 2.6, *as well as the assumption* (2.37) *in Lemma* 2.7, *hold. Then for each $\mu \in [0, +\infty)$, equation* (1.5) *has at least one positive T-periodic solution.*

4 Example

In this section, we present two examples to demonstrate the main results.

Example 4.1 Considering the following equation:

$$x''(t) + \left[\frac{3}{x^4} + \left(\frac{25\pi}{6} + 5\right)x^{\frac{3}{2}}\right]x'(t) + (1 + 2\cos t)x^{\frac{3}{2}}(t) - \frac{1}{x^2(t)} = \sin t. \tag{4.1}$$

Corresponding to equation (1.5), in (4.1), $e(t) = \sin(t)$, $\varphi(t) = 1 + 2\cos t$, $T = 2\pi$. Obviously, $\overline{\varphi} = 1 > 0$, and $\overline{e} = 0$ for all $t \in [0, T]$ with $\overline{\varphi_+} = \frac{5}{6} + \frac{1}{\pi}$ and $\overline{\varphi_-} = \frac{1}{\pi} - \frac{1}{6}$. Since $F(x) = -\frac{1}{x^3} + (\frac{5\pi}{3} + 2)x^{\frac{5}{2}}$, we can easily verify that equation (4.1) satisfies

$$B_0 = \inf_{[A_1, +\infty)} \left(F(x) - T\overline{\varphi_-} x^{\frac{3}{2}}\right) > -\infty, \tag{4.2}$$

$$\lim_{x \to 0^+} (F(x) + \frac{2\pi}{x^2}) = -\infty, \tag{4.3}$$

and

$$\lim_{x \to +\infty} \left(F(x) - T\overline{\varphi_+} x^{\frac{3}{2}}\right) = +\infty. \tag{4.4}$$

Obviously, (4.2), (4.3), and (4.4) imply that assumptions (2.15), (2.16), and (2.24) hold. Thus, by using Theorem 3.1, equation (4.1) has at least one positive 2π-periodic solution.

Example 4.2 Now consider

$$x''(t) - \left[\frac{3}{x^4} + \left(5 - \frac{5\pi}{6}\right)x^{\frac{3}{2}}\right]x'(t) + (1 + 2\cos t)x^{\frac{3}{2}}(t) - \frac{1}{x^2(t)} = \sin t. \tag{4.5}$$

Corresponding to equation (1.5), here, $e(t) = \sin t$, $\varphi(t) = 1 + 2\cos t$, $T = 2\pi$. Clearly, $\overline{\varphi} = 1 > 0$, and $\overline{e} = 0$ for all $t \in [0, T]$ with $\overline{\varphi_+} = \frac{5}{6} + \frac{1}{\pi}$ and $\overline{\varphi_-} = \frac{1}{\pi} - \frac{1}{6}$. Since $F(x) = \frac{1}{x^3} - (2 - \frac{\pi}{3})x^{\frac{5}{2}}$, we can easily verify that (4.1) satisfies

$$C_0 = \sup_{[A_1, +\infty)} \left(F(x) + T\overline{\varphi_-} x^{\frac{3}{2}}\right) < +\infty, \tag{4.6}$$

$$\lim_{x \to 0^+} \left(F(x) - \frac{2\pi}{x^2}\right) = +\infty, \tag{4.7}$$

and

$$\lim_{x\to+\infty}\left(F(x)+2\pi\overline{\varphi_-}x^{\frac{3}{2}}+\frac{2\pi}{x^2}\right)=-\infty. \tag{4.8}$$

Obviously, (4.6), (4.7), (4.8) imply that assumptions (2.30), (2.31), and (2.37) hold. Thus, by using Theorem 3.2, equation (4.5) has at least one positive 2π-periodic solution.

Remark 4.3 In (4.5), since $\mu=\frac{3}{2}>1$ and $\varphi(t)=1+2\cos t$ is a sign-changing function, the result of Example 4.2 can be obtained neither by using the main results of [23], nor by using the theorems of [23]. In this sense, the theorems of the present paper are new results on the existence of positive periodic solutions for singular Liénard equations.

Acknowledgements
Acknowledge my teacher.

Author contributions
Yu Zhu have equally contributed to obtaining new results in this article and also read and approved the final manuscript.

Funding
This work is supported by Anhui Province higher discipline top talent academic funding project (No.gxbjZD2022097) and KJ2021A1231

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Received: 5 April 2024 Accepted: 19 June 2024 Published online: 08 July 2024

References

1. Torres, P.J.: Mathematical Models with Singularities-a Zoo of Singular Creatures. Atlantis Press, Amsterdam (2015). ISBN:978-94-6239-105-5
2. Plesset, M.S.: The dynamics of cavitation bubbles. J. Appl. Mech. **16**, 228–231 (1949)
3. Jebelean, P., Mawhin, J.: Periodic solutions of singular nonlinear differential perturbations of the ordinary *p*-Laplacian. Adv. Nonlinear Stud. **2**(3), 299–312 (2002)
4. Lei, J., Zhang, M.: Twist property of periodic motion of an atom near a charged wire. Lett. Math. Phys. **60**(1), 9–17 (2002)
5. Huang, J., Ruan, S., Song, J.: Bifurcations in a predator–pry system of Leslite type with deneralized Holling type functional response. J. Differ. Equ. **257**(6), 1721–1751 (2014)
6. Yu, X., Lu, S.: A multiplicity result for periodic solutions of Liénard equations with an attractive singularity. Appl. Math. Comput. **346**, 183–192 (2019)
7. Nagumo, M.: On the periodic solution of an ordinary differential equation of second order. In: Zenkokou Shijou Suudaku Danwakai, pp. 54–61 (1994). (in Japanese). English translation in Mitio Nagumo Collected papers, Springer, Berlin (1993)
8. Torres, P.J.: Weak singularities may help periodic solutions to exist. J. Differ. Equ. **232**, 277–284 (2007)
9. Li, X., Zhang, Z.: Periodic solutions for second order differential equations with a singular nonlinearity. Nonlinear Anal. **69**, 3866–3876 (2008)
10. Jiang, D., Chu, J., Zhang, M.: Multiplicity of positive periodic solutions to superlinear repulsive singular equations. J. Differ. Equ. **211**, 282–302 (2005)
11. Chu, J., Torres, P.J., Zhang, M.: Periodic solutions of second order non-autonomous singular dynamical systems. J. Differ. Equ. **239**, 196–212 (2007)
12. Cheng, Z., Ren, J.: Multiplicity results of positive solutions foe fourth-order nonlinear differential equation with singularity. Math. Methods Appl. Sci. **38**, 5284–5304 (2016)
13. Cheng, Z., Ren, J.: Positive solutions for fourth-order singular nonlinear differential equation with variable-coefficient. Math. Methods Appl. Sci. **39**, 2251–2274 (2016)
14. Hakl, R., Torres, P.J.: On periodic solutions of second-order differential equations with attractive–repulsive singularities. J. Differ. Equ. **248**, 111–126 (2010)
15. Habets, P., Sanchez, L.: Periodic solutions of some Liénard equations with singularities. Proc. Am. Math. Soc. **109**, 1135–1144 (1990)
16. Zhang, M.: Periodic solutions of Liénard equations with singular forces of repulsive type. J. Math. Anal. Appl. **203**(1), 254–269 (1996)

17. Kong, F., Lu, S., Luo, Z.: Positive periodic solutions of singular higher order delay differential equations. Results Math. **72**(1), 1–16 (2017)
18. Lu, S., Zhong, T., Gao, Y.: Periodic solutions of *p*-Laplacian equations with singularities. Adv. Differ. Equ. **2016**, 146 (2016). https://doi.org/10.1186/s13662-016-0875-6
19. Martins, R.: Existence of periodic solutions for second-order differential equations with singularities and the strong force condition. J. Math. Anal. Appl. **317**, 1–13 (2006)
20. Yu, X., Lu, S.: A multiplicity result for periodic solutions of Liénard equations with an attractive singularity. Appl. Math. Comput. **346**, 183–192 (2019)
21. Wang, Z.: Periodic solutions of Lińard equation with a singularity and a deviating argument. Nonlinear Anal., Real World Appl. **16**(1), 227–234 (2014)
22. Hakl, R., Torres, P.J., Zamora, M.: Periodic solutions of singular second order differential equations: the repulsive case. Topol. Methods Nonlinear Anal. **39**, 199–220 (2012)
23. Lu, S., Guo, Y., Chen, L.: Periodic solutions for Liénard equation with an indefinite singularity. Nonlinear Anal., Real World Appl. **45**, 542–556 (2019)
24. Manásevich, R., Mawhin, J.: Periodic solutions for nonlinear systems with *p*-Laplacian-like operator. J. Differ. Equ. **145**, 367–393 (1998)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.