Haider et al. *Boundary Value Problems* (2024) 2024:53
https://doi.org/10.1186/s13661-024-01855-1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A comprehensive study on Milne-type inequalities with tempered fractional integrals

Wali Haider[1], Hüseyin Budak[2], Asia Shehzadi[1], Fatih Hezenci[2] and Haibo Chen[1*]

*Correspondence:
math_chb@csu.edu.cn
[1]School of Mathematics and Statistics, Central South University, Changsha 410083, China
Full list of author information is available at the end of the article

Abstract

In the framework of tempered fractional integrals, we obtain a fundamental identity for differentiable convex functions. By employing this identity, we derive several modifications of fractional Milne inequalities, providing novel extensions to the domain of tempered fractional integrals. The research comprehensively examines significant functional classes, including convex functions, bounded functions, Lipschitzian functions, and functions of bounded variation.

Mathematics Subject Classification: 26D15; 26D10; 26D07

Keywords: Inequalities of Milne type; Fractional version; Tempered fractional integrals; Convex functions

1 Introduction and preliminaries

Numerical integration is a crucial computational tool that tackles mathematical complexities without analytical solutions. It is used in various fields like physics, engineering, finance, economics, signal processing, computer graphics, probability, and statistics. Its versatility and efficiency make it essential in addressing various mathematical challenges across diverse fields [13, 37]. Researchers have contributed substantially to numerical integration by developing new formulas and studying error bounds in detail [16, 17]. In mathematical inequality, there is much research on finding new error bounds by using various types of functions, such as convex, bounded, Lipschitzian, and those with bounded variation [2, 3]. Numerous researchers have actively engaged with the realm of fractional calculus, particularly focusing on its implications in the theory of inequalities. In [8, 31], the authors investigated fractional variants of trapezoid-type inequalities. Budak [6] studied midpoint and trapezoid-type inequalities for newly defined quantum integrals. For local fractional integrals, Sarikaya et al. [33] extended Gruss- and Chebysev-type inequalities on the fractal sets. Luo and Du [25] formulated equality and provided various results related to the Simpson-type inequality for Riemann–Liouville fractional integrals. By taking twice-differentiable functions, Hezenci et al. [20] gave a novel version of fractional Simpson-type inequalities. Recently, numerous publications have focused on the formation of significant inequities [1, 4, 21, 38].

The Milne-type inequality, attributed to the British mathematician Edward Arthur Milne in the early twentieth century, is a significant mathematical tool for integral estimation. This inequality, widely acknowledged and bearing Milne's name, holds significance in mathematical inequalities, with applications extending to optimization theory, physics, and engineering [5, 14, 15, 35].

Tempered fractional calculus, essentially evolved from the principles of fractional calculus, can be traced to the innovative study of Buschman [10], which introduced the concept of fractional integration involving weak singular and exponential kernels. Tempered fractional integrals have extensive applications in data processing, image advancement, bioengineering, finance, and other scientific fields [24, 26]. A significant leap in this field is attributed to Mohammed et al. [27], who notably formulated Hermite–Hadamard-type inequalities for convex functions using tempered fractional integrals. Their contributions extend beyond existing results in Riemann–Liouville fractional integrals, leveraging the methodology proposed by Sarikaya et al. [32] and Sarikaya and Yildirim [34]. This technique provides a wide range of inequalities, including trapezoidal and midpoint inequalities, in the setting of tempered fractional integrals. Cao et al. [11] have established the equivalence of tempered fractional and substantial derivatives under certain conditions. Additionally, they obtained definitions and analyzed their properties. By employing tempered fractional integrals in the perspective of twice-differentiable functions, Hezenci and Budak [19] provided a novel identity. By involving this identity, they investigated various results related to left Hermite–Hadamard-type inequalities for tempered fractional integrals.

Numerous noteworthy fractional integrals, such as the Riemann–Liouville, Caputo, Grünwald–Letnikov, and Weyl fractional integrals, have been found in the literature. Numerous scholarly articles have been devoted to expanding and broadening the scope of these integrals. In particular, the relationships between Riemann–Liouville and tempered fractional integrals in [28], Ψ-Hilfer derivative [23], and $(\Psi - k)$-fractional operators [29] were reported in the setting of tempered fractional integral.

The following section outlines the fundamental preliminaries required to prove our key results.

Definition 1 (See [27]) For the real numbers $\alpha > 0$ and $x, \lambda \geq 0$, the λ-incomplete gamma function is described as:

$$\Upsilon_{\lambda}(\alpha, x) := \int_0^x \xi^{\alpha-1} e^{-\lambda\xi}\, d\xi.$$

If $\lambda = 1$, it is equivalent to the incomplete gamma function [12]:

$$\Upsilon(\alpha, x) := \int_0^x \xi^{\alpha-1} e^{-\xi}\, d\xi.$$

Here, $0 < \alpha < \infty$ and $\lambda \geq 0$.

Remark 1 (See [27]) For the real numbers $\alpha > 0$ and $x, \lambda \geq 0$, we attain

I. $\Upsilon_{\lambda(\varsigma-\omega)}(\alpha, 1) = \int_0^1 \xi^{\alpha-1} e^{-\lambda(\varsigma-\omega)\xi}\, d\xi = \frac{1}{(\varsigma-\omega)^{\alpha}} \Upsilon_{\lambda}(\alpha, \varsigma-\omega)$.

II. $\int_0^1 \Upsilon_{\lambda(\varsigma-\omega)}(\alpha, x)\, dx = \frac{\Upsilon_{\lambda}(\alpha,\varsigma-\omega)}{(\varsigma-\omega)^{\alpha}} - \frac{\Upsilon_{\lambda}(\alpha+1,\varsigma-\omega)}{(\varsigma-\omega)^{\alpha+1}}$.

Definition 2 (See [18, 22]) A function $\mathcal{F}$ is defined on the interval $[\omega,\varsigma]$, considering $\mathcal{F}\in L_1[\omega,\varsigma]$, the Riemann–Liouville fractional integral of order $\alpha>0$, is defined as follows:

$$J_{\omega+}^{\alpha}\mathcal{F}(x)=\frac{1}{\Gamma(\alpha)}\int_{\omega}^{x}(x-\xi)^{\alpha-1}\mathcal{F}(\xi)\,d\xi,\quad x>\omega$$

and

$$J_{\varsigma-}^{\alpha}\mathcal{F}(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{\varsigma}(\xi-x)^{\alpha-1}\mathcal{F}(\xi)\,d\xi,\quad x<\varsigma.$$

Here, $\Gamma(\alpha)$ is the Gamma function and $J_{\omega+}^{0}\mathcal{F}(x)=J_{\varsigma-}^{0}\mathcal{F}(x)=\mathcal{F}(x)$.

Definition 3 (See [24, 26]) Tempered fractional integral operators are defined as:

$$\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}(x)=\frac{1}{\Gamma(\alpha)}\int_{\omega}^{x}(x-\xi)^{\alpha-1}e^{-\lambda(x-\xi)}\mathcal{F}(\xi)\,d\xi,\quad x\in[\omega,\varsigma]$$

and

$$\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{\varsigma}(\xi-x)^{\alpha-1}e^{-\lambda(\xi-x)}\mathcal{F}(\xi)\,d\xi,\quad x\in[\omega,\varsigma].$$

Here, $\mathcal{F}\in L_1[\omega,\varsigma]$, $\alpha>0$ and $\lambda\geq 0$.

In the case of $\lambda=0$, in Definition 3, immediately we acquire Definition 2. For an extensive review of tempered fractional integrals and their diverse cases, the following books offer extensive insights [26, 30, 36].

Budak et al. [9] investigated fractional forms of Milne-type inequalities for functions with bounded variation, Lipschitz, and differentiable convex functions. They were the first to investigate these inequalities, focusing on fractional integrals.

Leveraging previous investigations, we derive tempered fractional variations of Milne-type inequalities using differentiable convex mappings. We examine new bounds by involving differentiable convex mappings within the context of the tempered fractional integral. These resulting inequalities are versatile and can be transformed into Riemann–Liouville fractional Milne-type inequalities when $\lambda=0$. Under these specified conditions, if we assume $\alpha=1$, the inequalities are reduced to basic Milne-type inequalities.

2 Main results

Initially, we establish an identity employing tempered fractional integrals. Subsequently, utilizing this particular identity, we derive new Milne-type inequalities with tempered fractional integrals.

Lemma 1 *Consider $\mathcal{F}:[\omega,\varsigma]\to\mathbb{R}$ is an absolutely continuous function on the interval (ω,ς) with $\mathcal{F}'\in L_1[\omega,\varsigma]$. In this case, the following equality is valid*:

$$\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right] \\ -\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}(\alpha,\frac{\varsigma-\omega}{2})}\left[\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right] \quad (1)$$

$$= \frac{(\varsigma-\omega)^{\alpha+1}}{2^{\alpha+2}\,\Upsilon_{\lambda}\,(\alpha,\frac{\varsigma-\omega}{2})}\int_0^1\left(\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\,\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}\,(\alpha,1)\right)$$
$$\times\left[\mathcal{F}'\left(\left(\frac{1-\xi}{2}\right)\omega+\left(\frac{1+\xi}{2}\right)\varsigma\right)-\mathcal{F}'\left(\left(\frac{1+\xi}{2}\right)\omega+\left(\frac{1-\xi}{2}\right)\varsigma\right)\right]d\xi.$$

Proof Through the utilization of the integration by parts technique, we acquire

$$\begin{aligned}
I_1 &= \int_0^1\left(\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\,\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}\,(\alpha,1)\right)\left[\mathcal{F}'\left(\left(\frac{1+\xi}{2}\right)\omega+\left(\frac{1-\xi}{2}\right)\varsigma\right)\right]d\xi \\
&= -\frac{2}{\varsigma-\omega}\left[\left(\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\,\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}\,(\alpha,1)\right)\mathcal{F}\left(\left(\frac{1+\xi}{2}\right)\omega+\left(\frac{1-\xi}{2}\right)\varsigma\right)\right|_0^1 \\
&\quad+\frac{2}{\varsigma-\omega}\int_0^1\xi^{\alpha-1}e^{-\lambda(\frac{\varsigma-\omega}{2})\xi}\mathcal{F}'\left(\left(\frac{1+\xi}{2}\right)\omega+\left(\frac{1-\xi}{2}\right)\varsigma\right)d\xi \\
&= -\frac{2}{\varsigma-\omega}\left[\left(\frac{4}{3}\,\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}\,(\alpha,1)\right)\mathcal{F}(\omega)+\left(\frac{1}{3}\,\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}\,(\alpha,1)\right)\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right] \\
&\quad+\frac{2^{\alpha+1}\Gamma(\alpha)}{(\varsigma-\omega)^{\alpha+1}}\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right).
\end{aligned} \tag{2}$$

By taking the same steps, we derive

$$\begin{aligned}
I_2 &= \frac{2}{\varsigma-\omega}\left[\left(\frac{4}{3}\,\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}\,(\alpha,1)\right)\mathcal{F}(\varsigma)-\left(\frac{1}{3}\,\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}\,(\alpha,1)\right)\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right] \\
&\quad-\frac{2^{\alpha+1}\Gamma(\alpha)}{(\varsigma-\omega)^{\alpha+1}}\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right).
\end{aligned} \tag{3}$$

By (2) and (3), this yields

$$\begin{aligned}
\frac{(\varsigma-\omega)^{\alpha+1}}{2^{\alpha+2}\,\Upsilon_{\lambda}\,(\alpha,\frac{\varsigma-\omega}{2})}[I_2-I_1] &= \frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right] \\
&\quad-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}\,(\alpha,\frac{\varsigma-\omega}{2})}\left[\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right].
\end{aligned} \tag{4}$$

The proof of Lemma 1 is concluded. □

Theorem 1 *Consider the conditions outlined in Lemma 1 and the convexity of the function $|\mathcal{F}'|$ on the interval $[\omega,\varsigma]$, then, we attain the following Milne-type inequalities for tempered fractional integrals*:

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]\right. \\
&\quad\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}\,(\alpha,\frac{\varsigma-\omega}{2})}\left[\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\right| \\
&\leq\frac{(\varsigma-\omega)^{\alpha+1}}{2^{\alpha+2}\,\Upsilon_{\lambda}\,(\alpha,\frac{\varsigma-\omega}{2})}\Omega_1(\alpha,\lambda)\left[\left|\mathcal{F}'(\omega)\right|+\left|\mathcal{F}'(\varsigma)\right|\right],
\end{aligned} \tag{5}$$

where

$$\Omega_1(\alpha,\lambda)=\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\,\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}\,(\alpha,1)\right|d\xi.$$

Proof By applying the absolute value in Lemma 1 and considering the convexity of $|\mathcal{F}'|$, we acquire

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]\right. \\
&\qquad\left.-\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,\frac{\varsigma-\omega}{2})}\left[\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\right| \qquad (6)\\
&\quad\leq\frac{(\varsigma-\omega)^{\alpha+1}}{2^{\alpha+2}\Upsilon_{\lambda}(\alpha,\frac{\varsigma-\omega}{2})}\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,1)\right| \\
&\quad\quad\times\left[\left|\mathcal{F}'\left(\left(\frac{1-\xi}{2}\right)\omega+\left(\frac{1+\xi}{2}\right)\varsigma\right)\right|+\left|\mathcal{F}'\left(\left(\frac{1+\xi}{2}\right)\omega+\left(\frac{1-\xi}{2}\right)\varsigma\right)\right|\right]d\xi \\
&\quad\leq\frac{(\varsigma-\omega)^{\alpha+1}}{2^{\alpha+2}\Upsilon_{\lambda}(\alpha,\frac{\varsigma-\omega}{2})}\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,1)\right| \\
&\quad\quad\times\left[\frac{1+\xi}{2}\left|\mathcal{F}'(\omega)\right|+\frac{1-\xi}{2}\left|\mathcal{F}'(\varsigma)\right|+\frac{1-\xi}{2}\left|\mathcal{F}'(\omega)\right|+\frac{1+\xi}{2}\left|\mathcal{F}'(\varsigma)\right|\right]d\xi \\
&\quad=\frac{(\varsigma-\omega)^{\alpha+1}}{2^{\alpha+2}\Upsilon_{\lambda}(\alpha,\frac{\varsigma-\omega}{2})}\Omega_1(\alpha,\lambda)\left[\left|\mathcal{F}'(\omega)\right|+\left|\mathcal{F}'(\varsigma)\right|\right].
\end{aligned}$$

Consequently, the proof is concluded. □

Remark 2 Assume $\lambda = 0$ in Theorem 1, we derive the subsequent Milne's rule-type inequality:

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varsigma-\omega)^{\alpha}}\left[J_{\varsigma-}^{\alpha}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+J_{\omega+}^{\alpha}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\right| \\
&\quad\leq\frac{\alpha(\varsigma-\omega)}{4}\Omega_1(\alpha,0)\left[\left|\mathcal{F}'(\omega)\right|+\left|\mathcal{F}'(\varsigma)\right|\right],
\end{aligned}$$

which is defined in [9, Theorem 1].

Remark 3 When setting $\lambda = 0$ and $\alpha = 1$ in Theorem 1, we attain the subsequent Milne's rule-type inequality:

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]-\frac{1}{\varsigma-\omega}\int_{\omega}^{\varsigma}\mathcal{F}(\xi)\,d\xi\right| \\
&\quad\leq\frac{5(\varsigma-\omega)}{24}\left(\left|\mathcal{F}'(\omega)\right|+\left|\mathcal{F}'(\varsigma)\right|\right),
\end{aligned}$$

which is obtained in [7].

Theorem 2 *Consider the conditions outlined in Lemma 1 and the convexity of the function $|\mathcal{F}'|^q, q > 1$ on the interval $[\omega, \varsigma]$. Then, we have the following Milne-type inequalities for tempered fractional integrals*:

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]\right. \\
&\qquad\left.-\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,\frac{\varsigma-\omega}{2})}\left[\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\right| \qquad (7)
\end{aligned}$$

$$\leq \left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1} \frac{\Phi_1^{\mathrm{p}}(\alpha,\lambda)}{2\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}\left[\left(\frac{3|\mathcal{F}'(\omega)|^q+|\mathcal{F}'(\varsigma)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|\mathcal{F}'(\omega)|^q+3|\mathcal{F}'(\varsigma)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1} \frac{4\Phi_1^{\mathrm{p}}(\alpha,\lambda)}{2\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}\left[\left|\mathcal{F}'(\omega)\right|^q+\left|\mathcal{F}'(\varsigma)\right|^q\right].$$

Here, $q^{-1}+p^{-1}=1$ *and*

$$\Phi_1^{\mathrm{p}}(\alpha,\lambda)=\left(\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\,\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}\,(\alpha,1)\right|^p\right)^{\frac{1}{p}}d\xi.$$

Proof By utilizing Hölder's inequality in (6), this yields

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right] - \frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}\left[\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\right|\\
&\leq \frac{(\varsigma-\omega)^{\alpha+1}}{2^{\alpha+2}\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}\left\{\left(\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\,\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}\,(\alpha,1)\right|^p d\xi\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\int_0^1\left|\mathcal{F}'\left(\left(\frac{1+\xi}{2}\right)\omega+\left(\frac{1-\xi}{2}\right)\varsigma\right)\right|^q d\xi\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\,\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}\,(\alpha,1)\right|^p d\xi\right)^{\frac{1}{p}}\\
&\quad\left.\times\left(\int_0^1\left|\mathcal{F}'\left(\left(\frac{1-\xi}{2}\right)\omega+\left(\frac{1+\xi}{2}\right)\varsigma\right)\right|^q d\xi\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By utilizing the convexity of $|\mathcal{F}'|^q$, we attain

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right] - \frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}\left[\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\right|\\
&\leq \frac{(\varsigma-\omega)^{\alpha+1}}{2^{\alpha+2}\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}\left\{\left(\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\,\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}\,(\alpha,1)\right|^p d\xi\right)^{\frac{1}{p}}\right.\\
&\quad\times\left[\left(\int_0^1\left[\frac{1+\xi}{2}\left|\mathcal{F}'(\omega)\right|^q+\frac{1-\xi}{2}\left|\mathcal{F}'(\varsigma)\right|^q\right]d\xi\right)^{\frac{1}{q}}\right.\\
&\quad\left.\left.+\left(\int_0^1\left[\frac{1-\xi}{2}\left|\mathcal{F}'(\omega)\right|^q+\frac{1+\xi}{2}\left|\mathcal{F}'(\varsigma)\right|^q\right]d\xi\right)^{\frac{1}{q}}\right]\right\}\\
&=\left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1} \frac{\Phi_1^{\mathrm{p}}(\alpha,\lambda)}{2\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}\left[\left(\frac{3|\mathcal{F}'(\omega)|^q+|\mathcal{F}'(\varsigma)|^q}{4}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{|\mathcal{F}'(\omega)|^q+3|\mathcal{F}'(\varsigma)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

The first inequality of (7) is proved. For the proof of the second inequality, let $\omega_1 = 3|\mathcal{F}'(\omega)|^q, \varsigma_1 = |\mathcal{F}'(\varsigma)|^q$, $\omega_2 = |\mathcal{F}'(\omega)|^q$, and $\varsigma_2 = 3|\mathcal{F}'(\varsigma_2)|^q$. Leveraging the provided information that

$$\sum_{\kappa=1}^{n}(\omega_\kappa + \varsigma_\kappa)^s \leq \sum_{\kappa=1}^{n}\omega_\kappa^s + \sum_{\kappa=1}^{n}\varsigma_\kappa^s, \quad 0 \leq s < 1$$

and $1 + 3^{\frac{1}{q}} \leq 4$, the required result can be derived directly. With this, the proof of Theorem 2 is accomplished. □

Remark 4 Setting $\lambda = 0$ in Theorem 2, we obtain the subsequent Milne's rule-type inequality:

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega) - \mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + 2\mathcal{F}(\varsigma)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varsigma-\omega)^\alpha}\left[J_{\varsigma-}^{\alpha}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + J_{\omega+}^{\alpha}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\right| \\
&\quad \leq \frac{\alpha(\varsigma-\omega)}{4}\left(\Phi_1^{\mathrm{p}}(\alpha,0)\right)\left[\left(\frac{3|\mathcal{F}'(\omega)|^q + |\mathcal{F}'(\varsigma)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|\mathcal{F}'(\omega)|^q + 3|\mathcal{F}'(\varsigma)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\quad \leq \frac{\alpha(\varsigma-\omega)}{4}\left(4\Phi_1^{\mathrm{p}}(\alpha,0)\right)\left[\left|\mathcal{F}'(\omega)\right| + \left|\mathcal{F}'(\varsigma)\right|\right],
\end{aligned}$$

which is presented by Budak et al. in [9, Theorem 2].

Remark 5 Assume $\lambda = 0$ and $\alpha = 1$ in Theorem 2, we deduce the subsequent Milne's rule-type inequality:

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega) - \mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + 2\mathcal{F}(\varsigma)\right] - \frac{1}{\varsigma-\omega}\int_a^b \mathcal{F}(\xi)\,d\xi\right| \\
&\quad \leq \frac{\varsigma-\omega}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathcal{F}'(\omega)|^q + |\mathcal{F}'(\varsigma)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|\mathcal{F}'(\omega)|^q + 3|\mathcal{F}'(\varsigma)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\quad \leq \frac{\varsigma-\omega}{12}\left(\frac{4^{p+2}-4}{3(p+1)}\right)^{\frac{1}{p}}\left[\left|\mathcal{F}'(\omega)\right| + \left|\mathcal{F}'(\varsigma)\right|\right],
\end{aligned}$$

which is obtained in [9, Corollary 1].

Theorem 3 *Consider the conditions outlined in Lemma 1 and the convexity of the function $|\mathcal{F}'|^q, q \geq 1$ on the interval $[\omega, \varsigma]$. Then, we have the following Milne-type inequalities for tempered fractional integrals*:

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega) - \mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + 2\mathcal{F}(\varsigma)\right]\right. \\
&\qquad \left. - \frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\,(\alpha, \frac{\varsigma-\omega}{2})}\left[\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + \mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\right| \qquad (8) \\
&\quad \leq \left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1}\frac{(\Omega_1(\alpha,\lambda))^{1-\frac{1}{q}}}{2\,\Upsilon_\lambda\,(\alpha, \frac{\varsigma-\omega}{2})}\left\{\left(\frac{(\Omega_1(\alpha,\lambda) + \Omega_2(\alpha,\lambda))}{2}\left|\mathcal{F}'(\omega)\right|^q\right.\right. \\
&\qquad \left. + \frac{(\Omega_1(\alpha,\lambda) - \Omega_2(\alpha,\lambda))}{2}\left|\mathcal{F}'(\varsigma)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\frac{(\Omega_1(\alpha,\lambda)-\Omega_2(\alpha,\lambda))}{2}\left|\mathcal{F}'(\omega)\right|^q+\frac{(\Omega_1(\alpha,\lambda)+\Omega_2(\alpha,\lambda))}{2}\left|\mathcal{F}'(\varsigma)\right|^q\right)^{\frac{1}{q}}\Bigg\},$$

where $\Omega_1(\alpha,\lambda)$ *is defined in Theorem* 1, *and*

$$\Omega_2(\alpha,\lambda)=\int_0^1\xi\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,1)\right|d\xi.$$

Proof By employing the Power mean inequality in (6), we acquire

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]\\
&\quad-\frac{\Gamma(\alpha)}{2\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}\left[\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\Bigg|\\
&\quad\le\left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1}\frac{1}{2\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}\Bigg\{\left(\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,1)\right|d\xi\right)^{1-\frac{1}{q}}\\
&\quad\times\Bigg[\left(\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,1)\right|\left|\mathcal{F}'\left(\left(\frac{1+\xi}{2}\right)\omega+\left(\frac{1-\xi}{2}\right)\varsigma\right)\right|^q d\xi\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,1)\right|\right.\\
&\quad\times\left.\left|\mathcal{F}'\left(\left(\frac{1-\xi}{2}\right)\omega+\left(\frac{1+\xi}{2}\right)\varsigma\right)\right|^q d\xi\right)^{\frac{1}{q}}\Bigg]\Bigg\}.
\end{aligned}$$

Considering the convexity of $|\mathcal{F}'|^q$ on the interval $[\omega,\varsigma]$, we derive

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]\\
&\quad-\frac{\Gamma(\alpha)}{2\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}\left[\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\Bigg|\\
&\quad\le\left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1}\frac{1}{2\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}\Bigg\{\left(\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,1)\right|d\xi\right)^{1-\frac{1}{q}}\\
&\quad\times\Bigg[\left(\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,1)\right|\right.\\
&\quad\times\left.\left(\frac{1+\xi}{2}\left|\mathcal{F}'(\omega)\right|^q+\frac{1-\xi}{2}\left|\mathcal{F}'(\varsigma)\right|^q\right)d\xi\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,1)\right|\right.\\
&\quad\times\left.\left(\frac{1-\xi}{2}\left|\mathcal{F}'(\omega)\right|^q+\frac{1+\xi}{2}\left|\mathcal{F}'(\varsigma)\right|^q\right)d\xi\right)^{\frac{1}{q}}\Bigg]\Bigg\}\\
&\quad\le\left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1}\frac{(\Omega_1(\alpha,\lambda))^{1-\frac{1}{q}}}{2\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}\Bigg\{\left(\frac{(\Omega_1(\alpha,\lambda)+\Omega_2(\alpha,\lambda))}{2}\left|\mathcal{F}'(\omega)\right|^q\right.\\
&\quad+\left.\frac{(\Omega_1(\alpha,\lambda)-\Omega_2(\alpha,\lambda))}{2}\left|\mathcal{F}'(\varsigma)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\frac{(\Omega_1(\alpha,\lambda)-\Omega_2(\alpha,\lambda))}{2}\left|\mathcal{F}'(\omega)\right|^q+\frac{(\Omega_1(\alpha,\lambda)+\Omega_2(\alpha,\lambda))}{2}\left|\mathcal{F}'(\varsigma)\right|^q\right)^{\frac{1}{q}}\Bigg\}.$$

This concludes the proof. □

Remark 6 Taking $\lambda = 0$ in Theorem 3, we acquire the subsequent Milne's rule-type inequality:

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varsigma-\omega)^{\alpha}}\left[J^{\alpha}_{\varsigma-}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+J^{\alpha}_{\omega+}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\right|\\
&\quad\le\frac{\alpha(\varsigma-\omega)}{4}\left(\Omega_1(\alpha,0)\right)^{1-\frac{1}{q}}\Bigg\{\Bigg(\frac{(\Omega_1(\alpha,0)+\Omega_2(\alpha,0))}{2}\left|\mathcal{F}'(\omega)\right|^q\\
&\qquad+\frac{(\Omega_1(\alpha,0)-\Omega_2(\alpha,0))}{2}\left|\mathcal{F}'(\varsigma)\right|^q\Bigg)^{\frac{1}{q}}\\
&\qquad+\left(\frac{(\Omega_1(\alpha,0)-\Omega_2(\alpha,0))}{2}\left|\mathcal{F}'(\omega)\right|^q+\frac{(\Omega_1(\alpha,0)+\Omega_2(\alpha,0))}{2}\left|\mathcal{F}'(\varsigma)\right|^q\right)^{\frac{1}{q}}\Bigg\},
\end{aligned}$$

which is found in [9, Theorem 3].

Remark 7 If we take $\lambda = 0$ and $\alpha = 1$ in Theorem 3, we deduce the subsequent Milne's rule-type inequality:

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]-\frac{1}{\varsigma-\omega}\int_{\omega}^{\varsigma}\mathcal{F}(\xi)\,d\xi\right|\\
&\quad\le\frac{5(\varsigma-\omega)}{24}\left[\left(\frac{4|\mathcal{F}'(\omega)|^q+|\mathcal{F}'(\varsigma)|^q}{5}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}'(\omega)|^q+4|\mathcal{F}'(\varsigma)|^q}{5}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

which is obtained in [7, Remark 3.2].

Theorem 4 *Consider the conditions outlined in Lemma 1 to hold. If there exist $m, M \in \mathbb{R}$ such that $m \le \mathcal{F}'(\xi) \le M$ for $\xi \in [\omega,\varsigma]$, then, we obtain the following Milne-type inequalities for tempered fractional integrals*:

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]\\
&\qquad-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}(\alpha,\frac{\varsigma-\omega}{2})}\left[\mathcal{J}^{(\alpha,\lambda)}_{\omega+}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\mathcal{J}^{(\alpha,\lambda)}_{\varsigma-}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\Bigg|\\
&\quad\le\left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1}\frac{\Omega_1(\alpha,\lambda)}{2\,\Upsilon_{\lambda}(\alpha,\frac{\varsigma-\omega}{2})}(M-m),
\end{aligned}\tag{9}$$

where $\Omega_1(\alpha,\lambda)$ is defined as in Theorem 1.

Proof From Lemma 1, it is easy to write

$$\begin{aligned}
&\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]\\
&\qquad-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}(\alpha,\frac{\varsigma-\omega}{2})}\left[\mathcal{J}^{(\alpha,\lambda)}_{\omega+}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\mathcal{J}^{(\alpha,\lambda)}_{\varsigma-}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]
\end{aligned}\tag{10}$$

$$
\begin{aligned}
&\leq \left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1} \frac{1}{2\Upsilon_{\lambda}(\alpha, \frac{\varsigma-\omega}{2})} \left\{ \int_0^1 \left(\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi) + \frac{1}{3}\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,1) \right) \right. \\
&\quad \times \left[\mathcal{F}'\left(\left(\frac{1-\xi}{2}\right)\omega + \left(\frac{1+\xi}{2}\right)\varsigma \right) - \frac{m+M}{2} \right] d\xi \\
&\quad + \int_0^1 \left(\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi) + \frac{1}{3}\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,1) \right) \\
&\quad \left. \times \left[\frac{m+m}{2} - \mathcal{F}'\left(\left(\frac{1+\xi}{2}\right)\omega + \left(\frac{1-\xi}{2}\right)\varsigma \right) \right] d\xi \right\}.
\end{aligned}
$$

By employing the properties of modulus in equation (10), we can derive

$$
\begin{aligned}
&\left| \frac{1}{3}\left[2\mathcal{F}(\omega) - \mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + 2\mathcal{F}(\varsigma) \right] \right. \\
&\quad \left. - \frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha, \frac{\varsigma-\omega}{2})} \left[\mathcal{J}^{(\alpha,\lambda)}_{\omega+}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + \mathcal{J}^{(\alpha,\lambda)}_{\varsigma-}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) \right] \right| \\
&\leq \left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1} \frac{1}{2\Upsilon_{\lambda}(\alpha, \frac{\varsigma-\omega}{2})} \left\{ \int_0^1 \left| \Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi) + \frac{1}{3}\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,1) \right| \right. \\
&\quad \times \left| \mathcal{F}'\left(\left(\frac{1-\xi}{2}\right)\omega + \left(\frac{1+\xi}{2}\right)\varsigma \right) - \frac{m+M}{2} \right| d\xi \\
&\quad + \int_0^1 \left| \Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi) + \frac{1}{3}\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,1) \right| \\
&\quad \left. \times \left| \frac{m+m}{2} - \mathcal{F}'\left(\left(\frac{1+\xi}{2}\right)\omega + \left(\frac{1-\xi}{2}\right)\varsigma \right) \right| d\xi \right\}.
\end{aligned}
$$

Based on the given assumption $m \leq \mathcal{F}'(\xi) \leq M$ for $\xi \in [\omega, \varsigma]$, this yields

$$
\left| \mathcal{F}'\left(\left(\frac{1-\xi}{2}\right)\omega + \left(\frac{1+\xi}{2}\right)\varsigma \right) - \frac{m+M}{2} \right| \leq \frac{M-m}{2} \tag{11}
$$

and

$$
\left| \frac{m+m}{2} - \mathcal{F}'\left(\left(\frac{1+\xi}{2}\right)\omega + \left(\frac{1-\xi}{2}\right)\varsigma \right) \right| \leq \frac{M-m}{2}. \tag{12}
$$

With the utilization of inequalities (11) and (12), we achieve

$$
\begin{aligned}
&\left| \frac{1}{3}\left[2\mathcal{F}(\omega) - \mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + 2\mathcal{F}(\varsigma) \right] \right. \\
&\quad \left. - \frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha, \frac{\varsigma-\omega}{2})} \left[\mathcal{J}^{(\alpha,\lambda)}_{\omega+}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + \mathcal{J}^{(\alpha,\lambda)}_{\varsigma-}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) \right] \right| \\
&\leq \left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1} \frac{1}{2\Upsilon_{\lambda}(\alpha, \frac{\varsigma-\omega}{2})} \left[\int_0^1 \left| \Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi) + \frac{1}{3}\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,1) \right| d\xi \right] (M-m) \\
&= \left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1} \frac{\Omega_1(\alpha,\lambda)}{2\Upsilon_{\lambda}(\alpha, \frac{\varsigma-\omega}{2})} (M-m).
\end{aligned}
$$

Hence, the proof is effectively concluded. □

Corollary 1 *Under the conditions of Theorem 4, if there exist $M \in \mathbb{R}^+$ such that $|\mathcal{F}'(\xi)| \leq M$, for all $\xi \in [\omega, \varsigma]$, then we attain*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{F}(\omega) - \mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + 2\mathcal{F}(\varsigma)\right]\right.\\&\left.\quad - \frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda(\alpha, \frac{\varsigma-\omega}{2})}\left[\mathcal{J}^{(\alpha,\lambda)}_{\omega+}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + \mathcal{J}^{(\alpha,\lambda)}_{\varsigma-}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\right|\\&\leq \left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1}\frac{\Omega_1(\alpha,\lambda)}{\Upsilon_\lambda(\alpha, \frac{\varsigma-\omega}{2})}M.\end{aligned}$$

Remark 8 If we set $\lambda = 0$, in Corollary 1, we have

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{F}(\omega) - \mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + 2\mathcal{F}(\varsigma)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varsigma-\omega)^\alpha}\left[J^\alpha_{\omega+}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + J^\alpha_{\varsigma-}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\right|\\&\leq \frac{\alpha(\varsigma-\omega)}{2}\Omega_1(\alpha,0)M,\end{aligned}$$

which is obtained in [9, Corollary 3].

Remark 9 Putting $\lambda = 0$ and $\alpha = 1$ in Corollary 1, this yields

$$\left|\frac{1}{3}\left[2\mathcal{F}(\omega) - \mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + 2\mathcal{F}(\varsigma)\right] - \frac{1}{\varsigma-\omega}\int_\omega^\varsigma \mathcal{F}(\xi)\,d\xi\right| \leq \frac{5(\varsigma-\omega)}{12}M,$$

which was obtained by Alomari and Liu [3].

Remark 10 Taking $\lambda = 0$ in Theorem 4, we attain

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{F}(\omega) - \mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + 2\mathcal{F}(\varsigma)\right] - \frac{\Gamma(\alpha+1)}{2(\varsigma-\omega)^\alpha}\left[J^\alpha_{\varsigma-}\mathcal{F}(\omega) + J^\alpha_{\omega+}\mathcal{F}(\varsigma)\right]\right|\\&\leq \frac{\alpha(\varsigma-\omega)}{4}\Omega_1(\alpha,0)(M-m),\end{aligned}$$

which is presented in [9, Theorem 4].

Remark 11 Assume $\lambda = 0$ and $\alpha = 1$ in Theorem 4, we deduce the subsequent inequality:

$$\left|\frac{1}{3}\left[2\mathcal{F}(\omega) - \mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + 2\mathcal{F}(\varsigma)\right] - \frac{1}{\varsigma-\omega}\int_\omega^\varsigma \mathcal{F}(\xi)\,d\xi\right| \leq \frac{5(\varsigma-\omega)}{24}(M-m),$$

which is defined in [9, Corollary 2].

Theorem 5 *Consider the conditions outlined in Lemma 1 to hold. If $\mathcal{F}'$ is an* L*-Lipschitzian function on the interval $[\omega, \varsigma]$, then we obtain the following Milne-type inequalities for tempered fractional integrals*:

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathcal{F}(\omega) - \mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + 2\mathcal{F}(\varsigma)\right]\right.\\&\left.\quad - \frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda(\alpha, \frac{\varsigma-\omega}{2})}\left[\mathcal{J}^{(\alpha,\lambda)}_{\omega+}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right) + \mathcal{J}^{(\alpha,\lambda)}_{\varsigma-}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\right| \qquad (13)\end{aligned}$$

$$\leq \left(\frac{\varsigma-\omega}{2}\right)^{\alpha+2} \frac{\Omega_2(\alpha,\lambda)}{\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}\mathsf{L}.$$

Here, $\Omega_2(\alpha,\lambda)$ *is defined as in Theorem 3.*

Proof Utilizing the modulus in Lemma 1, we attain

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]-\frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\,(\alpha,\varsigma-\omega)}\left[\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}(\omega)+\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}(\varsigma)\right]\right| \\
&\quad\leq \left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1}\frac{1}{2\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}\left[\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\,\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}\,(\alpha,1)\right|\right. \\
&\qquad\left.\times\left|\mathcal{F}'\left(\left(\frac{1-\xi}{2}\right)\omega+\left(\frac{1+\xi}{2}\right)\varsigma\right)-\mathcal{F}'\left(\left(\frac{1+\xi}{2}\right)\omega+\left(\frac{1-\xi}{2}\right)\varsigma\right)\right|d\xi\right].
\end{aligned}$$

Since $|\mathcal{F}'|$ is an L-Lipschitzian function, we can conclude

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]-\frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\,(\alpha,\varsigma-\omega)}\left[\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}(\omega)+\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}(\varsigma)\right]\right| \\
&\quad\leq \left(\frac{\varsigma-\omega}{2}\right)^{\alpha+1}\frac{1}{2\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}\int_0^1\left|\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}(\alpha,\xi)+\frac{1}{3}\,\Upsilon_{\lambda(\frac{\varsigma-\omega}{2})}\,(\alpha,1)\right|\mathsf{L}(\varsigma-\omega)\xi\,d\xi \\
&\quad\leq \left(\frac{\varsigma-\omega}{2}\right)^{\alpha+2}\frac{\Omega_2(\alpha,\lambda)}{\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}\mathsf{L}.
\end{aligned}$$

Consequently, the demonstration is concluded. □

Remark 12 By setting $\lambda = 0$, in Theorem 5, we attain

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]-\frac{\Gamma(\alpha+1)}{2(\varsigma-\omega)^\alpha}\left[J_{\varsigma-}^{\alpha}\mathcal{F}(\omega)+J_{\omega+}^{\alpha}\mathcal{F}(\varsigma)\right]\right| \\
&\quad\leq \frac{\alpha(\varsigma-\omega)^2}{4}\Omega_2(\alpha,0)\mathsf{L},
\end{aligned}$$

which is defined in [9, Theorem 5].

Remark 13 By setting $\lambda = 0$ and $\alpha = 1$ in Theorem 5, this yields

$$\left|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]-\frac{1}{\varsigma-\omega}\int_\omega^\varsigma \mathcal{F}(\xi)\,d\xi\right| \leq \frac{(\varsigma-\omega)^2}{8}\mathsf{L},$$

which is presented in [9, Corollary 4].

Theorem 6 *Assume* $\mathcal{F}:[\omega,\varsigma]\to\mathbb{R}$ *exhibits a bounded variation over the interval* $[\omega,\varsigma]$. *Then, we have the following Milne-type inequality that is specifically tailored for tempered fractional integrals*:

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]\right. \\
&\qquad\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}\left[\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\right| \qquad (14)
\end{aligned}$$

$$\le \frac{1}{2\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}\left[\max\left\{\frac{\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}{3},\left|\Upsilon_\lambda\left(\alpha,\frac{\varsigma-\omega}{2}\right)+\frac{\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}{3}\right|\right\}\right]\bigvee_{\omega}^{\varsigma}(\mathcal{F}),$$

where $\bigvee_{\omega}^{\varsigma}(\mathcal{F})$ *indicate the total variation of* $\mathcal{F}$ *on* $[\omega,\varsigma]$.

Proof Define the mappings

$$\mathsf{T}_\alpha(\varkappa)=\begin{cases}-\Upsilon_\lambda\,(\alpha,\frac{\omega+\varsigma}{2}-\varkappa)-\frac{\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}{3}, & \omega\le\varkappa\le\frac{\omega+\varsigma}{2}\\ \Upsilon_\lambda(\alpha,\varkappa-\frac{\omega+\varsigma}{2})+\frac{\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}{3} & \frac{\omega+\varsigma}{2}<\varkappa\le\varsigma.\end{cases}$$

By employing the integration by parts technique, we acquire

$$\begin{aligned}
&\int_\omega^\varsigma \mathsf{T}_\alpha(\varkappa)\,d\mathcal{F}(\varkappa)\\
&=\int_\omega^{\frac{\omega+\varsigma}{2}}\left(-\Upsilon_\lambda\left(\alpha,\frac{\omega+\varsigma}{2}-\varkappa\right)-\frac{\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}{3}\right)d\mathcal{F}(\varkappa)\\
&\quad+\int_{\frac{\omega+\varsigma}{2}}^{\varsigma}\left(\Upsilon_\lambda\left(\alpha,\varkappa-\frac{\omega+\varsigma}{2}\right)+\frac{\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}{3}\right)d\mathcal{F}(\varkappa)\\
&=\left(-\Upsilon_\lambda\left(\alpha,\frac{\omega+\varsigma}{2}-\varkappa\right)-\frac{\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}{3}\right)\mathcal{F}(\varkappa)\Big|_\omega^{\frac{\omega+\varsigma}{2}}\\
&\quad-\int_\omega^{\frac{\omega+\varsigma}{2}}\left(\frac{\omega+\varsigma}{2}-\varkappa\right)^{\alpha-1}e^{-\lambda(\frac{\omega+\varsigma}{2}-\varkappa)}\mathcal{F}(\varkappa)\,d\mathcal{F}(\varkappa)\\
&\quad+\left(\Upsilon_\lambda\left(\alpha,\varkappa-\frac{\omega+\varsigma}{2}\right)+\frac{\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}{3}\right)\mathcal{F}(\varkappa)\Big|_{\frac{\omega+\varsigma}{2}}^{\varsigma}\\
&\quad-\int_{\frac{\omega+\varsigma}{2}}^{\varsigma}\left(\varkappa-\frac{\omega+\varsigma}{2}\right)^{\alpha-1}e^{-\lambda(\varkappa-\frac{\omega+\varsigma}{2})}\mathcal{F}(\varkappa)\,d\mathcal{F}(\varkappa)\\
&=-\frac{\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}{3}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\frac{4\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}{3}\mathcal{F}(\omega)-\Gamma(\alpha)\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\\
&\quad+\frac{4\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}{3}\mathcal{F}(\varsigma)-\frac{\Upsilon_\lambda(\alpha,\frac{\varsigma-\omega}{2})}{3}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)-\Gamma(\alpha)\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\\
&=\frac{2\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]\\
&\quad-\Gamma(\alpha)\left[\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right].
\end{aligned}$$

That is,

$$\begin{aligned}
&\frac{1}{3}\left[2\mathcal{F}(\omega)-\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+2\mathcal{F}(\varsigma)\right]\\
&\quad-\frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}\left[\mathcal{J}_{\omega+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)+\mathcal{J}_{\varsigma-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\omega+\varsigma}{2}\right)\right]\\
&=\frac{1}{2\,\Upsilon_\lambda\,(\alpha,\frac{\varsigma-\omega}{2})}\int_\omega^\varsigma \mathsf{T}_\alpha(\varkappa)\,d\mathcal{F}(\varkappa).
\end{aligned}$$

It is a well-established fact that if $\mathsf{g}, \mathcal{F} : [\omega, \varsigma] \to \mathbb{R}$ satisfy the conditions where g is continuous on $[\omega, \varsigma]$ and $\mathcal{F}$ is of bounded variation on $[\omega, \varsigma]$, then $\int_{\omega}^{\varsigma} \mathsf{g}(\xi)\, d\mathcal{F}(\xi)$ exists and

$$\left| \int_{\omega}^{\varsigma} \mathsf{g}(\xi)\, d\mathcal{F}(\xi) \right| \le \sup_{\xi \in [\omega, \varsigma]} \left| \mathsf{g}(\xi) \right| \bigvee_{\omega}^{\varsigma} (\mathcal{F}). \tag{15}$$

However, employing (15), we have

$$\begin{aligned}
&\left| \frac{1}{3} \left[2\mathcal{F}(\omega) - \mathcal{F}\left(\frac{\omega + \varsigma}{2} \right) + 2\mathcal{F}(\varsigma) \right] \right. \\
&\qquad \left. - \frac{\Gamma(\alpha)}{2\, \Upsilon_{\lambda}\, (\alpha, \frac{\varsigma - \omega}{2})} \left[\mathcal{J}_{\omega+}^{(\alpha,\lambda)} \mathcal{F}\left(\frac{\omega + \varsigma}{2} \right) + \mathcal{J}_{\varsigma-}^{(\alpha,\lambda)} \mathcal{F}\left(\frac{\omega + \varsigma}{2} \right) \right] \right| \\
&\quad \le \frac{1}{2\, \Upsilon_{\lambda}\, (\alpha, \frac{\varsigma - \omega}{2})} \left| \int_{\omega}^{\varsigma} \mathsf{T}_{\alpha}(\varkappa)\, d\mathcal{F}(\varkappa) \right| \\
&\quad \le \frac{1}{2\, \Upsilon_{\lambda}\, (\alpha, \frac{\varsigma - \omega}{2})} \left[\left| \int_{\omega}^{\frac{\omega+\varsigma}{2}} \left(\Upsilon_{\lambda}\left(\alpha, \frac{\omega + \varsigma}{2} - \varkappa \right) + \frac{\Upsilon_{\lambda}(\alpha, \frac{\varsigma - \omega}{2})}{3} \right) d\mathcal{F}(\varkappa) \right| \right. \\
&\qquad \left. + \left| \int_{\frac{\omega+\varsigma}{2}}^{\varsigma} \left(\Upsilon_{\lambda}\left(\alpha, \varkappa - \frac{\omega + \varsigma}{2} \right) + \frac{\Upsilon_{\lambda}(\alpha, \frac{\varsigma - \omega}{2})}{3} \right) d\mathcal{F}(\varkappa) \right| \right] \\
&\quad \le \frac{1}{2\, \Upsilon_{\lambda}\, (\alpha, \frac{\varsigma - \omega}{2})} \left[\sup_{\varkappa \in [\omega, \frac{\omega+\varsigma}{2}]} \left| \Upsilon_{\lambda}\left(\alpha, \frac{\omega + \varsigma}{2} - \varkappa \right) + \frac{\Upsilon_{\lambda}(\alpha, \frac{\varsigma - \omega}{2})}{3} \right| \bigvee_{\omega}^{\frac{\omega+\varsigma}{2}} (\mathcal{F}) \right. \\
&\qquad \left. + \sup_{\varkappa \in [\frac{\omega+\varsigma}{2}, \varsigma]} \left| \Upsilon_{\lambda}\left(\alpha, \varkappa - \frac{\omega + \varsigma}{2} \right) + \frac{\Upsilon_{\lambda}(\alpha, \frac{\varsigma - \omega}{2})}{3} \right| \bigvee_{\frac{\omega+\varsigma}{2}}^{\varsigma} (\mathcal{F}) \right] \\
&\quad \le \frac{1}{2\, \Upsilon_{\lambda}\, (\alpha, \frac{\varsigma - \omega}{2})} \left[\max \left\{ \frac{\Upsilon_{\lambda}(\alpha, \frac{\varsigma - \omega}{2})}{3}, \left| \Upsilon_{\lambda}\left(\alpha, \frac{\varsigma - \omega}{2} \right) + \frac{\Upsilon_{\lambda}(\alpha, \frac{\varsigma - \omega}{2})}{3} \right| \bigvee_{\omega}^{\frac{\omega+\varsigma}{2}} (\mathcal{F}) \right\} \right. \\
&\qquad \left. + \max \left\{ \frac{\Upsilon_{\lambda}(\alpha, \frac{\varsigma - \omega}{2})}{3}, \left| \Upsilon_{\lambda}\left(\alpha, \frac{\varsigma - \omega}{2} \right) + \frac{\Upsilon_{\lambda}(\alpha, \frac{\varsigma - \omega}{2})}{3} \right| \bigvee_{\frac{\omega+\varsigma}{2}}^{\varsigma} (\mathcal{F}) \right\} \right] \\
&\quad \le \frac{1}{2\, \Upsilon_{\lambda}\, (\alpha, \frac{\varsigma - \omega}{2})} \left[\max \left\{ \frac{\Upsilon_{\lambda}(\alpha, \frac{\varsigma - \omega}{2})}{3}, \left| \Upsilon_{\lambda}\left(\alpha, \frac{\varsigma - \omega}{2} \right) + \frac{\Upsilon_{\lambda}(\alpha, \frac{\varsigma - \omega}{2})}{3} \right| \right\} \right] \bigvee_{\omega}^{\varsigma} (\mathcal{F}).
\end{aligned}$$

Hence, the proof is finalized. □

Remark 14 Setting $\lambda = 0$, in Theorem 6, we find

$$\left| \frac{1}{3} \left[2\mathcal{F}(\omega) - \mathcal{F}\left(\frac{\omega + \varsigma}{2} \right) + 2\mathcal{F}(\varsigma) \right] - \frac{\Gamma(\alpha + 1)}{2(\varsigma - \omega)^{\alpha}} \left[J_{\varsigma-}^{\alpha} \mathcal{F}(\omega) + J_{\omega+}^{\alpha} \mathcal{F}(\varsigma) \right] \right| \le \frac{2}{3} \bigvee_{\omega}^{\varsigma} (\mathcal{F}),$$

which is obtained in [9, Theorem 6].

Remark 15 Assume $\lambda = 0$ and $\alpha = 1$ in Theorem 6, this yields

$$\left| \frac{1}{3} \left[2\mathcal{F}(\omega) - \mathcal{F}\left(\frac{\omega + \varsigma}{2} \right) + 2\mathcal{F}(\varsigma) \right] - \frac{1}{\varsigma - \omega} \int_{\omega}^{\varsigma} \mathcal{F}(\xi)\, d\xi \right| \le \frac{2}{3} \bigvee_{\omega}^{\varsigma} (\mathcal{F}),$$

which is proposed in [3].

3 Conclusions

In summary, our investigation yields new Milne-type inequalities using tempered fractional integrals. We employ essential function classes, to establish a fundamental identity, and derived novel fractional Milne inequalities, extending formulations for tempered fractional integrals. Our findings contribute insights for future studies, generalize prior work, and suggest potential applications in diverse mathematical contexts. The insights and methodologies revealed by our findings concerning Milne-type inequalities through tempered fractional integrals have the potential to serve as a foundation for future explorations in this domain. Readers can further expand on our findings by applying a different kind of fractional integral.

Author contributions
Manuscript preparation: W.H.,; Literature consultation: A.S.,; supervision: H.C. and H.B.; computed and scrutinized the results F.H. All authors read and approved the final manuscript.

Funding
This research received no external funding.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author details
[1]School of Mathematics and Statistics, Central South University, Changsha 410083, China. [2]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey.

Received: 17 January 2024 Accepted: 22 March 2024 Published online: 10 April 2024

References

1. Ali, M.A., Budak, H., Michal, F., Sundas, K.: A new version of q-Hermite-Hadamard's midpoint and trapezoid type inequalities for convex functions. Math. Slovaca **73**(2), 369–386 (2023)
2. Alomari, M.W.: A companion of the generalized trapezoid inequality and applications. J. Math. Appl. **36**, 5–15 (2013)
3. Alomari, M.W., Liu, Z.: New error estimations for the Milne's quadrature formula in terms of at most first derivatives. Konuralp J. Math. **1**(1), 17–23 (2013)
4. Bohner, M., Kashuri, A., Mohammed, P., Valdes, J.E.N.: Hermite-Hadamard-type inequalities for conformable integrals. Hacet. J. Math. Stat. **51**(3), 775–786 (2022)
5. Bosch, P., Rodríguez, J.M., Sigarreta, J.M.: On new Milne-type inequalities and applications. J. Inequal. Appl. **2023**(1), 3 (2023)
6. Budak, H.: Some trapezoid and midpoint type inequalities for newly defined quantum integrals. Proyecciones **40**(1), 199–215 (2021)
7. Budak, H., Hyder, A.A.: Enhanced bounds for Riemann-Liouville fractional integrals: novel variations of Milne inequalities. AIMS Math. **8**(12), 30760–30776 (2023)
8. Budak, H., Kara, H., Sarikaya, M.Z., Kiriş, M.E.: New extensions of the Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals. Miskolc Math. Notes **21**(2), 665–678 (2020)
9. Budak, H., Kösem, P., Kara, H.: On new Milne-type inequalities for fractional integrals. J. Inequal. Appl. **2023**(1), 10 (2023)
10. Buschman, R.G.: Decomposition of an integral operator by use of Mikusiński calculus. SIAM J. Math. Anal. **3**(1), 83–85 (1972)
11. Cao, J., Li, C., Chen, Y.: On tempered and substantial fractional calculus. In: pp. 1–6 (2014)
12. Chaudhry, M.A., Zubair, S.M.: Generalized incomplete gamma functions with applications. J. Comput. Appl. Math. **55**(1), 99–124 (1994)
13. Davis, P.J., Rabinowitz, P.: Methods of Numerical Integration. Courier Corporation (2007)
14. Demir, İ.: A new approach of Milne-type inequalities based on proportional Caputo-hybrid operator: a new approach for Milne-type inequalities. J. Adv. Appl. Comput. Math. **10**, 102–119 (2023)
15. Desta, H.D., Budak, H., Kara, H.: New perspectives on fractional Milne-type inequalities: insights from twice-differentiable functions. Univers. J. Math. Appl. **7**(1), 30–37 (2024)
16. Dragomir, S.: On the trapezoid quadrature formula and applications. Kragujev. J. Math. **23**(23), 25–36 (2001)

17. Dragomir, S.S., Agarwal, R.: Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl. Math. Lett. **11**(5), 91–95 (1998)
18. Gorenflo, R., Mainardi, F.: Fractional Calculus: Integral and Differential Equations of Fractional Order. Springer, Vienna (1997)
19. Hezenci, F., Budak, H.: Midpoint-type inequalities via twice-differentiable functions on tempered fractional integrals. J. Inequal. Appl. **2023**, 150 (2023)
20. Hezenci, F., Budak, H., Kara, H.: New version of fractional Simpson type inequalities for twice differentiable functions. Adv. Differ. Equ. **2021**, 460 (2021)
21. Hezenci, F., Budak, H., Kösem, P.: A new version of newtons inequalities for Riemann-Liouville fractional integrals. Rocky Mt. J. Math. **53**(1), 49–64 (2023)
22. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations, vol. 204. Elsevier (2006)
23. Kucche, K.D., Mali, A.D., Fernandez, A., Fahad, H.M.: On tempered Hilfer fractional derivatives with respect to functions and the associated fractional differential equations. Chaos Solitons Fractals **163**, 112547 (2022)
24. Li, C., Deng, W., Zhao, L.: Well-posedness and numerical algorithm for the tempered fractional ordinary differential equations (2015). arXiv preprint arXiv:1501.00376
25. Luo, C., Du, T.: Generalized Simpson type inequalities involving Riemann-Liouville fractional integrals and their applications. Filomat **34**(3), 751–760 (2020)
26. Meerschaert, M.M., Sabzikar, F., Chen, J.: Tempered fractional calculus. J. Comput. Phys. **293**, 14 (2015)
27. Mohammed, P.O., Sarikaya, M.Z., Baleanu, D.: On the generalized Hermite–Hadamard inequalities via the tempered fractional integrals. Symmetry **12**(4), 595 (2020)
28. Rahman, G., Nisar, K.S., Abdeljawad, T.: Tempered fractional integral inequalities for convex functions. Mathematics **8**(4), 500 (2020)
29. Salim, A., Lazreg, J.E., Benchohra, M.: A novel study on tempered (Φ, ψ)-Hilfer fractional operators (2023)
30. Samko, S.G.: Fractional Integrals and Derivatives. Theory and Applications (1993)
31. Sarikaya, M.Z., Budak, H.: Some Hermite-Hadamard type integral inequalities for twice differentiable mappings via fractional integrals. Facta Univ., Ser. Math. Inform. **29**(4), 371–384 (2015)
32. Sarikaya, M.Z., Set, E., Yaldiz, H., Başak, N.: Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**(9–10), 2403–2407 (2013)
33. Sarikaya, M.Z., Tunc, T., Budak, H.: On generalized some integral inequalities for local fractional integrals. Appl. Math. Comput. **276**, 316–323 (2016)
34. Sarikaya, M.Z., Yildirim, H.: On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. Miskolc Math. Notes **17**(2), 1049–1059 (2016)
35. Siala, I.B., Budakb, H., Alic, M.A.: Some Milne's rule type inequalities in quantum calculus. Filomat **37**(27), 9119–9134 (2023)
36. Srivastava, H.M., Buschman, R.G.: Convolution Integral Equations, with Special Function Kernels. Wiley, New York (1977)
37. Stoer, J., Bulirsch, R., Bartels, R., Gautschi, W., Witzgall, C.: Introduction to Numerical Analysis (Vol. 1993). Springer, New York (1980)
38. You, X., Hezenci, F., Budak, H., Kara, H.: New Simpson type inequalities for twice differentiable functions via generalized fractional integrals. AIMS Math. **7**(3), 3959–3971 (2022)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.