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Infinitely many solutions for quasilinear Schrödinger equation with concave-convex nonlinearities

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Abstract

In this work, we study the existence of infinitely many solutions to the following quasilinear Schrödinger equations with a parameter α and a concave-convex nonlinearity:

$$-\Delta_p u + V(x)|u|^{p-2}u - \Delta_p(|u|^{2\alpha})|u|^{2\alpha-2}u = \lambda h_1(x)|u|^{m-2}u + h_2(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N, \tag{0.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, $\lambda \geq 0$, and $1 < m < p < 2\alpha p < q < 2\alpha p^* = \frac{2\alpha pN}{N-p}$. The functions $V(x)$, $h_1(x)$, and $h_2(x)$ satisfy some suitable conditions. Using variational methods and some special techniques, we prove that there exists $\lambda_0 > 0$ such that Eq. (0.1) admits infinitely many high energy solutions in $W^{1,p}(\mathbb{R}^N)$ provided that $\lambda \in [0, \lambda_0]$.

Keywords: Quasilinear Schrödinger equations; Dual approach; High energy solution

1 Introduction and main result

In this paper, we are interested in the existence of infinitely many solutions to a class of quasilinear Schrödinger equations with a parameter α and a concave-convex nonlinearity

$$-\Delta_p u + V(x)|u|^{p-2}u - \Delta_p\big(|u|^{2\alpha}\big)|u|^{2\alpha-2}u = \lambda h_1(x)|u|^{m-2}u + h_2(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)(1 < p < N)$ and $\alpha > \frac{1}{2}$ is a parameter.

For the case $p = 2$, $\alpha = 1$, solutions of (1.1) are standing waves of the following Schrödinger equation:

$$iz_t = -\Delta z + W(x)z - h_1\big(|z|^2\big)z - \Delta g\big(|z|^2\big)g'\big(|z|^2\big)z, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ and $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $h_1, g : \mathbb{R}^+ \to \mathbb{R}$ are real functions.

It is well known that the standing wave solutions of the form $z(t,x) = \exp(-i\omega t)u(x)$ satisfy (1.2) with $g(s) = s$ if and only if the function $u(x)$ solves the equation of elliptic type

$$-\Delta u + V(x)u - \Delta(u^2)u = h(u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $V(x) = W(x) - \omega$, $\omega \in \mathbb{R}$ and $h(u) \equiv h_1(|u|^2)u$.

Quasilinear Schrödinger equations of form (1.2) appear naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of nonlinear term g. The case $g(s) = s$ was used for the superfluid film equation in plasma physics by Kurihura in [11] (see also [12]). In the case $g(s) = (1+s)^{1/2}$, Eq. (1.2) models the self-channeling of a high power ultra short laser in matter, see [7]. Equation (1.2) also appears in plasma physics and fluid mechanics [20], in mechanics [9], and in condensed matter theory [18]. More information on this subject can be found in [15] and the references therein.

For $p = 2$, several methods can be used to solve (1.1), e.g., the existence of positive ground state solution was proved in [17, 19] by using a constrained minimization argument; Eq. (1.1) was transformed to a semilinear one in [4–6, 10, 15] by a change of variables (dual approach); Nehari method was used to get the existence results of ground state solutions in [16, 22]. Especially, in [13, 15–17, 25], the existence of the ground state solutions for the following problem with a parameter $\alpha(>\frac{1}{2})$:

$$-\Delta u + V(x)u - \Delta\left(|u|^{2\alpha}\right)|u|^{2\alpha-2}u = g(x,u), \quad x \in \mathbb{R}^N \tag{1.4}$$

was studied with subcritical nonlinearities $g(x,u)$.

For (1.4), we find in the literature several types of potentials $V(x)$ to obtain a solution. Wu in [25] studied Eq. (1.4) considering the subcritical case and a potential $V(x)$, which is unbounded in $\mathbb{R}^N$ and satisfies the following assumption:

(A_1) The potential $V(x) \in C(\mathbb{R}^N)$ and $0 < V_0 := \inf_{x\in\mathbb{R}^N} V(x)$, and for each $M > 0$, $\operatorname{meas}(\{x \in \mathbb{R}^N : V(x) \le M\}) < \infty$.

In [15–17], Liu et al. proved the existence of a positive solution to problem (1.4) with $V(x) \in C(\mathbb{R}^N)$, $\inf_{x\in\mathbb{R}^N} V(x) > 0$ and the following conditions:

(A_2) $\lim_{|x|\to\infty} V(x) = +\infty$;

(A_3) $0 < V_0 := \inf_{x\in\mathbb{R}^N} V(x) < \lim_{|x|\to\infty} V(x) = V_\infty = \|V\|_{L^\infty(\mathbb{R}^N)} < \infty$;

(A_4) V is radially symmetric, i.e., $V(x) = V(|x|)$;

(A_5) V is periodic in each variable of $x_1, \dots, x_N$.

Similar assumptions also appeared in Severo [24], Ruiz and Siciliano [22], Fang and Szulkin [8]. By the variational principle in a suitable Orlicz space, do Ó and Severo in [3] established the existence of positive standing wave solutions for (1.4) with a concave-convex nonlinearity and the following condition:

(A_6) $0 < V_0 \le V(x)$ in $\mathbb{R}^N$ and $V^{-1}(x) \in L^1(\mathbb{R}^N)$.

Recently, Aires and Souto [1] considered (1.4) with $\alpha = 1$ and the vanishing potential $V(x)$ at infinity.

Clearly, it is well known that assumption (A_1) or (A_2) guarantees that the embedding $W^{1,2}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ is compact for each $2 \le s < \frac{2N}{N-2}$. Similarly, the application of (A_3) in [2, 15, 24] shows that the solution is nontrivial.

It is worth pointing out that the aforementioned authors always assumed that the potential $V(x)$ has some special characteristic. As far as we know, there are few papers that deal with a general bounded potential case for (1.1). Motivated by papers [1, 25], in the present paper we consider problem (1.1) with positive and more general bounded potential $V(x)$ by a dual approach and establish the existence of infinitely many high energy solutions under a concave-convex nonlinearity and different type weight functions $h_1(x)$, $h_2(x)$. It is easy to verify that for a general continuous and bounded function $V(x)$, assumptions (A_1) – (A_6) fail to hold. We shall use mountain pass theorem under the Cerami condition to study Eq. (1.1).

Throughout this paper, we always assume the potential $V(x) \in C(\mathbb{R}^N)$ and the weight function $h_2(x) \geq 0, \not\equiv 0$ in $\mathbb{R}^N$. Furthermore, we let $C, C_1, C_2, \dots$ be positive generic constants that can change from line to line.

The main result in this paper is as follows.

Theorem 1.1 *Assume*:

(H_0) $1 < p < N$, $1 < m < p < 2\alpha p < q < 2\alpha p^* = \frac{2\alpha pN}{N-p}$;

(H_1) *There exist the constants* $V_0, V_1 > 0$ *such that* $V_0 \leq V(x) \leq V_1$ *for all* $x \in \mathbb{R}^N$;

(H_2) $h_1 \in L^\sigma(\mathbb{R}^N)$ *with* $\sigma = \frac{2\alpha p}{2\alpha p - m}$;

In addition, suppose that one of the following two hypotheses holds:

(H_3) $h_2 \in L^\gamma(\mathbb{R}^N) \cap C_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ *with* $\gamma = \frac{2\alpha p^*}{2\alpha p^* - q}$;

(H_4) $h_2(x) \in L^\gamma_{\text{loc}}(\mathbb{R}^N) \cap C_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ *with* $\gamma = \frac{2\alpha p^*}{2\alpha p^* - q}$, *and* $h_2(x) \to 0$ *as* $|x| \to \infty$;

Then there exists a constant $\lambda_0 > 0$ *such that for all* $\lambda \in [0, \lambda_0]$, *Eq.* (1.1) *admits infinitely many high energy solutions in* $u_n \in W^{1,p}(\mathbb{R}^N)$ *such that* $J(v_n) \to \infty$ *as* $n \to \infty$, *where* $v_n = f^{-1}(u_n)$ *and* $f(t)$ *is defined by* (2.5) *later.*

Remark 1.2 Assumptions (H_3) – (H_4) are independent. For example, let $0 < \tau < N/\gamma$ and $k > N$, then the unbounded function

$$h_2(x) = \begin{cases} |x|^{-\tau}, & 0 < |x| < 1, \\ \exp(-|x|^k |\sin|\pi x||^{1/\gamma}), & |x| \geq 1, \end{cases} \tag{1.5}$$

satisfies (H_3), but $h_2(x) \not\to 0$ as $|x| \to \infty$. On the other hand, the unbounded function $h_2(x) = |x|^{-\tau}$, $x \in \mathbb{R}^N \setminus \{0\}$ satisfies (H_4), but fails to verify (H_3).

Remark 1.3 When $p = 2$, $\alpha = 1$, $\lambda = 0$, and $h_2 = \mu > 0$, problem (1.1) becomes

$$-\Delta u + V(x)u - \Delta\left(|u|^2\right)u = \mu|u|^{q-2}u, \quad x \in \mathbb{R}^N, \tag{1.6}$$

with $4 < q < 22^*$. The authors [15] proved that for any $\mu > 0$, Eq. (1.6) has a positive solution under assumptions (A_2) – (A_5). Fang and Szulkin [8] also established the existence of infinitely many solutions to (1.6) provided that $V(x)$ satisfies (A_5). Clearly, if $V(x)$ is continuous in $\mathbb{R}^N$ and verifies (A_5), then $V(x)$ satisfies (H_1). Theorem 1.1 shows that there are infinitely many solutions to (1.6) if (H_1) is true.

This paper is organized as follows. In Sect. 2, with a convenient change of variable, we set up the variational framework for (1.1). In Sect. 3, we verify that the energy functional associated with (1.1) satisfies the Cerami condition. In Sect. 4, the geometric conditions of the mountain pass theorem are verified, and the proof of Theorem 1.1 is given.

2 Variational setting of the equation

Let $E = W^{1,p}(\mathbb{R}^N)$ be the Sobolev spaces with the norm

$$\|u\|_E = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^p + V(x)|u|^p\right) dx\right)^{1/p}. \tag{2.1}$$

By hypothesis (H_1), it is equivalent to the standard norm in E. It is well known that there is a constant $S > 0$ such that

$$S\left(\int_{\mathbb{R}^N} |v|^{p^*} dx\right)^{p/p^*} \le \int_{\mathbb{R}^N} |\nabla v|^p \, dx, \quad \forall v \in C_0^\infty\left(\mathbb{R}^N\right). \tag{2.2}$$

From the approximation argument, we see that (2.2) holds on E.

We observe that the natural energy functional associated with Eq. (1.1) is given by

$$\begin{aligned} I(u) = {}& \frac{1}{p}\int_{\mathbb{R}^N} \left(1 + (2\alpha)^{p-1}|u|^{(2\alpha-1)p}\right)|\nabla u|^p \, dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p \, dx \\ & - \int_{\mathbb{R}^N} G(x,u)\, dx, \end{aligned} \tag{2.3}$$

where

$$G(x,u) = \int_0^u g(x,t)\, dt, \qquad g(x,t) = \lambda h_1(x)|t|^{m-2}t + h_2(x)|t|^{q-2}t, \quad \forall t \in \mathbb{R}. \tag{2.4}$$

It should be pointed out that the functional I is not well defined in general in E. To overcome this difficulty, we employ an argument developed by Colin and Jeanjean [6] for the case $p = 2$ and Severo [24] for $1 < p \le N$. We make the change of variables $u = f(v)$ or $v = f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{h(t)}, \qquad h(t) = \left(1 + (2\alpha)^{p-1}\left|f(t)\right|^{p(2\alpha-1)}\right)^{1/p}, \quad t \ge 0, f(0) = 0 \tag{2.5}$$

and by $f(t) = -f(-t)$ on $(-\infty, 0]$. Then we have the following.

Lemma 2.1 *The function $f(t)$ satisfies the following properties*:

(f_1) *f is uniquely defined, odd, increasing, and invertible in $\mathbb{R}$*;

(f_2) $0 < f'(t) \le 1$, $\forall t \in \mathbb{R}$;

(f_3) $|f(t)| \le |t|$, $\forall t \in \mathbb{R}$;

(f_4) $\frac{f(t)}{t} \to 1$ *as* $t \to 0$;

(f_5) $|f(t)| \le (2\alpha)^{1/2\alpha p}|t|^{1/2\alpha}$, $\forall t \in \mathbb{R}$;

(f_6) $\frac{1}{2}f(t) \le \alpha t f'(t) \le \alpha f(t)$, $\forall t \in \mathbb{R}^+ = [0,\infty)$ *and* $\alpha f(t) \le \alpha t f'(t) \le \frac{1}{2}f(t)$, $\forall t \in \mathbb{R}^- = (-\infty, 0]$;

(f_7) *There exists* $a \in (0, (2\alpha)^{1/2\alpha p}]$ *such that* $\frac{f(t)}{t^{1/2\alpha}} \to a$ *as* $t \to +\infty$;

(f_8) *There exists* $b_0 > 0$ *such that*

$$\left|f(t)\right| \ge \begin{cases} b_0|t| & \text{if } |t| \le 1, \\ b_0|t|^{1/2\alpha} & \text{if } |t| \ge 1; \end{cases}$$

(f_9) *For each* $\tau > 0$, *there exist* $C(\tau) = n$ *if* $\tau = n$ *and* $C(\tau) = n + 1$ *if* $\tau \in (n, n + 1)$, $n \in \mathbb{N}$ *such that*

$$|f(\tau t)| \leq C(\tau)|f(t)|, \quad \forall t \in \mathbb{R}. \tag{2.6}$$

Proof The proof of properties $(f_1) - (f_8)$ can be found in [24](for the case $1 < p \leq N$ and $\alpha = 1$) and in [25] (for the case $p = 2$ and $\frac{1}{2} < \alpha \leq 1$). For the case $1 < p < N$ and $\alpha > \frac{1}{2}$, the proof of $(f_1) - (f_8)$ is similar and omitted. Here we prove (f_9). Note that

$$f(t) = \int_0^t \frac{ds}{h(s)}, \quad t \geq 0. \tag{2.7}$$

Then

$$f(2t) = \int_0^{2t} \frac{ds}{h(s)} = \int_0^t \frac{ds}{h(s)} + \int_t^{2t} \frac{ds}{h(s)}. \tag{2.8}$$

For the second integral in (2.8), we take $s = t + \xi$ and $h(s) \geq (1 + (2\alpha)^{p-1}|f(\xi)|^{p(2\alpha-1)})^{1/p}$. Thus,

$$f(2t) \leq \int_0^t \frac{ds}{h(s)} + \int_0^t \frac{d\xi}{h(\xi)} = 2f(t), \quad t \geq 0. \tag{2.9}$$

Similarly, we have $f(nt) \leq nf(t)$ for $t \geq 0$ and $n \in \mathbb{N}$. Since $f(t)$ is odd and increasing in $\mathbb{R}$, we obtain (2.6). □

So, after the change of variables, we can write $I(u)$ as

$$J(v) \equiv I(f(v)) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p \, dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|f(v)|^p \, dx - \int_{\mathbb{R}^N} G(x, f(v)) \, dx, \tag{2.10}$$

which is well defined on E under assumptions $(H_0) - (H_4)$.

As in [24], we observe that if $v \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ is a critical point of the functional J, that is, $J'(v)\varphi = 0$ for all $\varphi \in W^{1,p}(\mathbb{R}^N)$, where

$$\begin{aligned} J'(v)\varphi = {} & \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla \varphi \, dx + \int_{\mathbb{R}^N} V(x)|f(v)|^{p-2} f(v) f'(v)\varphi \, dx \\ & - \int_{\mathbb{R}^N} g(x, f(v)) f'(v)\varphi \, dx, \end{aligned} \tag{2.11}$$

then v is a weak solution of the equation

$$-\Delta_p v = -V(x)|f(v)|^{p-2} f(v) f'(v) + g(x, f(v)) f'(v), \quad x \in \mathbb{R}^N, \tag{2.12}$$

and $u = f(v)$ is a weak solution of (1.1). By using Theorem 1 in [23], we can conclude that v is locally bounded in $\mathbb{R}^N$. So, we consider the existence of solutions to (2.12) in E.

3 The boundedness of the Cerami sequences

To obtain the existence of solutions to problem (2.12), we need to prove that the functional J defined by (2.10) satisfies the Cerami condition.

We first recall that a sequence $\{v_n\}$ in E is called a Cerami sequence of J if $\{J(v_n)\}$ is bounded and

$$\big(1+\|v_n\|_E\big)\big\|J'(v_n)\big\|_{E^*}\to 0 \quad \text{as } n\to\infty. \tag{3.1}$$

The functional J satisfies the Cerami condition if any Cerami sequence possesses a convergent subsequence in E

Lemma 3.1 *Assume $(H_0)-(H_2)$ and $h_2\geq 0$ in $\mathbb{R}^N$. If $\{v_n\}\subset E$ is a Cerami sequence, then $\{v_n\}$ is bounded in E.*

Proof Without loss of generality, we assume $v_n\not\equiv 0$ for all $n\in\mathbb{N}$. Set $\varphi_n(x)=\frac{f(v_n(x))}{f'(v_n(x))}$. Then, using (f_2) and (f_5) in Lemma 2.1, we have

$$\begin{aligned}&\big|\varphi_n(x)\big|\leq 2\alpha\big|v_n(x)\big|, \qquad \big|\nabla\varphi_n(x)\big|\leq 2\big|\nabla v_n(x)\big| \quad \text{in } \mathbb{R}^N \quad \text{and}\\ &\|\varphi_n\|_E\leq 2\alpha\|v_n\|_E \quad \forall n\in\mathbb{N}.\end{aligned} \tag{3.2}$$

Since $\{v_n\}$ is a Cerami sequence in E, there is a constant $C_1>0$ such that

$$\begin{aligned}C_1&\geq J(v_n)-\frac{1}{q}J'(v_n)\varphi_n\\ &\geq\left(\frac{1}{p}-\frac{2\alpha}{q}\right)\int_{\mathbb{R}^N}|\nabla v_n|^p\,dx+\left(\frac{1}{p}-\frac{1}{q}\right)\int_{\mathbb{R}^N}V\big|f(v_n)\big|^p\,dx\\ &\quad+\lambda\left(\frac{1}{q}-\frac{1}{m}\right)\int_{\mathbb{R}^N}h_1\big|f(v_n)\big|^m\,dx\\ &\geq\left(\frac{1}{p}-\frac{2\alpha}{q}\right)\|\nabla v_n\|_p^p-\lambda\left(\frac{1}{q}-\frac{1}{m}\right)\|h_1\|_\sigma\|\nabla v_n\|_p^m.\end{aligned} \tag{3.3}$$

This estimate and the assumption $m\in(1,p)$ prove that $\{\|\nabla v_n\|_p\}$ is bounded. Moreover,

$$\begin{aligned}C_1&\geq J(v_n)-\frac{1}{2p\alpha}J'(v_n)\varphi_n\geq\frac{2\alpha-1}{2p\alpha}\int_{\mathbb{R}^N}\big(|\nabla v_n|^p\big|f'(v_n)\big|^p+V\big|f(v_n)\big|^p\big)\,dx\\ &\quad+\lambda\left(\frac{1}{2p\alpha}-\frac{1}{m}\right)\int_{\mathbb{R}^N}h_1\big|f(v_n)\big|^m\,dx+\left(\frac{1}{2p\alpha}-\frac{1}{q}\right)\int_{\mathbb{R}^N}h_2\big|f(v_n)\big|^q\,dx\\ &\geq\frac{2\alpha-1}{2p\alpha}\|u_n\|_E^p-\lambda\left(\frac{1}{m}-\frac{1}{2p\alpha}\right)\|h_1\|_\sigma\|\nabla v_n\|_p^m,\end{aligned} \tag{3.4}$$

where $u_n=f(v_n)$. Then $\{\int_{\mathbb{R}^N}V|f(v_n)|^p\,dx\}$ is bounded and so is $\{A_n^p\}$, where

$$A_n^p=\int_{\mathbb{R}^N}\big(|\nabla v_n|^p+V\big|f(v_n)\big|^p\big)\,dx, \quad \forall n\in\mathbb{N}. \tag{3.5}$$

In the following, we show that there exists a constant $C_0>0$ such that

$$\int_{\mathbb{R}^N}\big(|\nabla v_n|^p+V(x)\big|f(v_n)\big|^p\big)\,dx\geq C_0\|v_n\|_E^p, \quad \forall n\in\mathbb{N}. \tag{3.6}$$

We argue by contradiction and assume that, up to a subsequence, $v_n \in E$ such that

$$A_n^p = \int_{\mathbb{R}^N} \left(|\nabla v_n|^p + V(x)\left|f(v_n)\right|^p\right) dx \le \frac{1}{n}\|v_n\|_E^p. \tag{3.7}$$

Hence, $\frac{A_n^p}{\|v_n\|_E^p} \to 0$ as $n \to \infty$. Let $\omega_n(x) = \frac{v_n(x)}{\|v_n\|_E}$, $f_n(x) = \frac{|f(v_n(x))|^p}{\|v_n\|_E^p}$. Then

$$\int_{\mathbb{R}^N} |\nabla \omega_n|^p \, dx + \int_{\mathbb{R}^N} V(x) f_n(x) \, dx \to 0 \quad \text{as } n \to \infty, \tag{3.8}$$

which shows

$$\int_{\mathbb{R}^N} |\nabla \omega_n|^p \, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} V(x) f_n(x) \, dx \to 0 \quad \text{as } n \to \infty. \tag{3.9}$$

Moreover, since

$$\int_{\mathbb{R}^N} \left(|\nabla \omega_n|^p + V(x)\left|\omega_n(x)\right|^p\right) dx = 1, \tag{3.10}$$

we conclude

$$\int_{\mathbb{R}^N} V(x)\left|\omega_n(x)\right|^p dx \to 1. \tag{3.11}$$

Similar to the idea of [25], we assert that for each $\varepsilon > 0$ there exists $\alpha_\varepsilon \ge 1$ independent of n such that $|\Omega_n| < \varepsilon$, where $\Omega_n = \{x \in \mathbb{R}^N : |v_n(x)| \ge \alpha_\varepsilon\}$ and $|\Omega_n| = \operatorname{meas}(\Omega_n)$. Otherwise, there are $\varepsilon_0 > 0$ and subsequence $\{v_{n_k}\} \subset \{v_n\}$ such that $|\Omega_{n_k}| \ge \varepsilon_0$, where

$$\Omega_{n_k} = \left\{x \in \mathbb{R}^N : \left|v_{n_k}(x)\right| \ge k\right\}, \quad \forall k \in \mathbb{N}. \tag{3.12}$$

By (f_8), one sees

$$\begin{aligned} A_{n_k}^p &\ge \int_{\mathbb{R}^N} V(x)\left|f(v_{n_k})\right|^p dx \ge V_0 b_0^{\frac{p}{2\alpha}} \int_{\Omega_{n_k}} |v_{n_k}|^{\frac{p}{2\alpha}} \, dx \\ &\ge C k^{\frac{p}{2\alpha}} |\Omega_{n_k}| \ge C\varepsilon_0 k^{\frac{p}{2\alpha}} \to \infty \end{aligned} \tag{3.13}$$

as $k \to \infty$. This is a contradiction. Hence the assertion is true. Denote $\Omega_n^c = \mathbb{R}^N \setminus \Omega_n$. For $x \in \Omega_n^c$, we have $|v_n(x)| \le \alpha_\varepsilon$. Using (f_8) and (f_9), we get

$$C_2\left|v_n(x)\right|^p \le \left|f\left(\alpha_\varepsilon^{-1} v_n(x)\right)\right|^p \le \left|f\left(v_n(x)\right)\right|^p, \quad x \in \Omega_n^c \tag{3.14}$$

for some $C_2 > 0$. Thus, as $n \to \infty$,

$$\begin{aligned} \int_{\Omega_n^c} V|\omega_n|^p \, dx = \int_{\Omega_n^c} V(x) \frac{|v_n(x)|^p}{\|v_n\|_E^p} \, dx &\le \frac{1}{C_2} \int_{\Omega_n^c} V \frac{|f(v_n)|^p}{\|v_n\|_E^p} \, dx \\ &= \frac{1}{C_2} \int_{\Omega_n^c} V f_n \, dx \to 0 \end{aligned} \tag{3.15}$$

On the other hand, from the integral absolute continuity, it follows that there is $\delta > 0$ such that whenever $\Omega \subset \mathbb{R}^N$ and $|\Omega| < \delta$,

$$\int_\Omega V(x)\big|\omega_n(x)\big|^p\,dx < \frac{1}{2}. \tag{3.16}$$

For this $\delta > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} V(x)\big|\omega_n(x)\big|^p\,dx &= \int_{\Omega_n} V(x)\big|\omega_n(x)\big|^p\,dx + \int_{\Omega_n^c} V(x)\big|\omega_n(x)\big|^p\,dx \\ &< \frac{1}{2} + \int_{\Omega_n^c} V(x)\big|\omega_n(x)\big|^p\,dx \end{aligned}\tag{3.17}$$

Letting $n \to \infty$, one sees from (3.11) and (3.17) that $1 \le \frac{1}{2}$. It is impossible. So, (3.6) is true and $\{v_n\}$ is bounded in E. □

Since the sequence $\{v_n\}$ given by Lemma 3.1 is a bounded sequence in E, there exist a constant $M > 0$ and $v \in E$, and a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, such that $\|v_n\|_E \le M$, $\|v\|_E \le M$ and

$$\begin{aligned}&v_n \rightharpoonup v \quad \text{weakly in } E, \quad v_n \to v \quad \text{in } L^s_{\mathrm{loc}}\big(\mathbb{R}^N\big), \quad \forall s \in [1,p^*), \\ &v_n(x) \to v(x) \quad \text{a.e. in } \mathbb{R}^N.\end{aligned}\tag{3.18}$$

Lemma 3.2 *Assume* (H_0) – (H_2). *If the sequence* $\{v_n\}$ *satisfies* (3.18), *then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} h_1(x)\big|f(v_n)\big|^m\,dx = \int_{\mathbb{R}^N} h_1(x)\big|f(v)\big|^m\,dx \tag{3.19}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} h_1(x)\big|f(v_n)\big|^{m-2}f(v_n)f'(v_n)v_n\,dx = \int_{\mathbb{R}^N} h_1(x)\big|f(v)\big|^{m-2}f(v)f'(v)v\,dx. \tag{3.20}$$

Proof From (3.18), we have $f(v_n(x)) \to f(v(x))$ a.e. in $\mathbb{R}^N$. Then

$$\int_{B_r} h_1\big|f(v_n)\big|^m\,dx \to \int_{B_r} h_1\big|f(v)\big|^m\,dx \tag{3.21}$$

for any $r > 0$, where $B_r = \{x \in \mathbb{R}^N : |x| < r\}$, $B_r^c = \mathbb{R}^N \setminus B_r$. On the other hand, we see from Hölder's inequality and (2.2) that

$$\begin{aligned}\int_{B_r^c} |h_1|\big|f(v_n)\big|^m\,dx &\le \left(\int_{B_r^c} |h_1|^\sigma\,dx\right)^{1/\sigma}\left(\int_{B_r^c} |v_n|^{p^*}\right)^{m/2\alpha p^*} \\ &\le S^{-\frac{m}{2\alpha p}}\|h_1\|_{L^\sigma(B_r^c)}\|\nabla v_n\|_p^{\frac{m}{2\alpha}} \le S^{-\frac{m}{2\alpha p}}M^{\frac{m}{2\alpha}}\|h_1\|_{L^\sigma(B_r^c)} \to 0\end{aligned}\tag{3.22}$$

as $r \to \infty$. By Fatou's lemma, we obtain

$$\begin{aligned}\int_{B_r^c} |h_1|\big|f(v)\big|^m\,dx &\le \liminf_{n\to\infty}\int_{B_r^c} |h_1|\big|f(v_n)\big|^m\,dx \\ &\le S^{-\frac{m}{2\alpha p}}M^{\frac{m}{2\alpha}}\|h_1\|_{L^\sigma(B_r^c)} \to 0 \quad \text{as } r \to \infty.\end{aligned}\tag{3.23}$$

Then, the application of (3.21)–(3.23) gives that (3.19). Similarly, noticing that (f_6) and

$$\begin{aligned}&\big|h_1\big|f(v_n)\big|^{m-2}f(v_n)f'(v_n)v_n\big| \le |h_1|\big|f(v_n)\big|^m,\\&\big|h_1\big|f(v)\big|^{m-2}f(v)f'(v)v\big| \le |h_1|\big|f(v)\big|^m \quad \text{in } \mathbb{R}^N,\end{aligned}$$

we can derive (3.20). □

Lemma 3.3 *Assume* (H_0) – (H_2) *and one of hypotheses* (H_3) *and* (H_4). *If the sequence* $\{v_n\}$ *satisfies* (3.18), *then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} h_2(x)\big|f(v_n)\big|^q\,dx = \int_{\mathbb{R}^N} h_2(x)\big|f(v)\big|^q\,dx \tag{3.24}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} h_2(x)\big|f(v_n)\big|^{q-2}f(v_n)f'(v_n)v_n\,dx = \int_{\mathbb{R}^N} h_2(x)\big|f(v)\big|^{q-2}f(v)f'(v)v\,dx. \tag{3.25}$$

Proof If (H_3) is satisfied, we use a similar argument in the proof of Lemma 3.2 to get limits (3.24) and (3.25). We now assume (H_4). Choose $t \in (0,1)$ such that $q = 2\alpha(pt + (1-t)p^*)$. Then

$$\begin{aligned}\int_{B_r^c} h_2\big|f(v_n)\big|^q\,dx &\le \int_{B_r^c} h_2|v_n|^{\frac{q}{2\alpha}}\,dx\\&\le \left(\int_{B_r^c} V|v_n|^p\,dx\right)^t\left(\int_{B_r^c} |v_n|^{p^*}h_2^{\frac{1}{1-t}}V^{-\frac{t}{1-t}}\,dx\right)^{1-t}\\&\le CV_0^{-t}\sup_{x\in B_r^c}\big|h_2(x)\big|\left(\int_{B_r^c} V|v_n|^p\,dx\right)^t\|\nabla v_n\|_p^{(1-t)p^*}\\&\le C\sup_{x\in B_r^c}\big|h_2(x)\big|\|v_n\|_E^{\frac{q}{2\alpha}}\\&\le CM^{\frac{q}{2\alpha}}\sup_{x\in B_r^c}\big|h_2(x)\big| \to 0 \quad \text{as } r\to\infty\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\int_{B_r^c} h_2\big|f(v)\big|^q\,dx &\le \liminf_{n\to\infty}\int_{B_r^c} h_2\big|f(v_n)\big|^q\,dx\\&\le CM^{\frac{q}{2\alpha}}\sup_{x\in B_r^c}\big|h_2(x)\big| \to 0 \quad \text{as } r\to\infty.\end{aligned} \tag{3.27}$$

Moreover, it follows from (3.18) that for all $r > 0$,

$$\int_{B_r} h_2\big|f(v_n)\big|^q\,dx \to \int_{B_r} h_2\big|f(v)\big|^q\,dx \quad \text{as } n\to\infty. \tag{3.28}$$

Then the application of (3.26)–(3.28) yields (3.24). Similarly, from (f_6), it follows that

$$\begin{aligned}\int_{B_r^c} \left|h_2 \left|f(v_n)\right|^{q-2} f(v_n)f'(v_n)v_n\right| dx &\le \int_{B_r^c} h_2 \left|f(v_n)\right|^q dx \\ &\le CM^{\frac{q}{2\alpha}} \sup_{x\in B_r^c} \left|h_2(x)\right| \to 0 \quad \text{as } r \to \infty \end{aligned} \tag{3.29}$$

and

$$\begin{aligned}\int_{B_r^c} \left|h_2 \left|f(v)\right|^{q-2} f(v)f'(v)v\right| dx &\le \int_{B_r^c} h_2 \left|f(v)\right|^q dx \\ &\le CM^{\frac{q}{2\alpha}} \sup_{x\in B_r^c} \left|h_2(x)\right| \to 0 \quad \text{as } r \to \infty, \end{aligned} \tag{3.30}$$

$$\lim_{n\to\infty} \int_{B_r} h_2 \left|f(v_n)\right|^{q-2} f(v_n)f'(v_n)v_n \, dx = \int_{B_r} h_2 \left|f(v)\right|^{q-2} f(v)f'(v)v \, dx. \tag{3.31}$$

Then we get (3.25) from (3.29)–(3.31). Then the proof of Lemma 3.3 is completed. □

Lemma 3.4 *Assume that all hypotheses in Theorem 1.1 hold. Let $\{v_n\}$ be a Cerami sequence and satisfy (3.18). Then the following statements hold:*

(i). *For each $\varepsilon > 0$, there exists $r_0 \ge 1$ such that $r \ge r_0$,*

$$\limsup_{n\to\infty} \int_{B_{2r}^c} \left(|\nabla v_n|^p + V(x)\left|f(v_n)\right|^p\right) dx < \varepsilon, \tag{3.32}$$

$$\limsup_{n\to\infty} \int_{B_{2r}^c} \left(|\nabla v_n|^p + V(x)\left|f(v_n)\right|^{p-2} f(v_n)f'(v_n)v_n\right) dx < \varepsilon, \tag{3.33}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)\left|f(v_n)\right|^p dx = \int_{\mathbb{R}^N} V(x)\left|f(v)\right|^p dx, \tag{3.34}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)\left|f(v_n)\right|^{p-2} f(v_n)f'(v_n)v_n \, dx = \int_{\mathbb{R}^N} V(x)\left|f(v)\right|^{p-2} f(v)f'(v)v \, dx. \tag{3.35}$$

(ii). *The weak limit $v \in E$ is a critical point for functional J.*

Proof (i). In fact, for $r > 1$, we choose the function $\eta = \eta(|x|) \in C^1(\mathbb{R}^N)$ such that

$$\begin{aligned} &\eta\left(|x|\right) \equiv 1 \quad x \in B_{2r}^c, \quad \eta\left(|x|\right) = 0 \quad x \in B_r \quad \text{and} \quad 0 \le \eta \le 1, \\ &|\nabla \eta| \le \frac{2}{r}, \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{3.36}$$

Since the sequence $\{v_n\}$ is bounded in E, the sequence $\{\eta\varphi_n\}$, where $\varphi_n = \frac{f(v_n)}{f'(v_n)}$, is also bounded in E. Hence, we have $J'(v_n)(\eta\varphi_n) = o_n(1)$, that is,

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla v_n|^p \left(1 + \frac{(2\alpha-1)(2\alpha)^{p-1}|f(v_n)|^{p(2\alpha-1)}}{1 + (2\alpha)^{p-1}|f(v_n)|^{p(2\alpha-1)}}\right)\eta \, dx + \int_{\mathbb{R}^N} V(x)\left|f(v_n)\right|^p \eta \, dx \\ &\quad = -\int_{\mathbb{R}^N} |\nabla v_n|^{p-2} \nabla v_n \nabla \eta \varphi_n \, dx + \int_{\mathbb{R}^N} g\left(x, f(v_n)\right) f(v_n)\eta \, dx + o_n(1). \end{aligned} \tag{3.37}$$

For assumptions (H_2) – (H_4), we have from (3.22) and (3.26) that

$$\int_{\mathbb{R}^N} g\big(x,f(v_n)\big)f(v_n)\eta\,dx = o_n(1) \quad \text{as } n\to\infty. \tag{3.38}$$

Hence, limits (3.37) and (3.38) show that

$$\begin{aligned}
&\int_{B_r^c}\big(|\nabla v_n|^p + V(x)\big|f(v_n)\big|^p\big)\eta\,dx\\
&\quad\le 2\alpha\int_{B_r^c}|\nabla v_n|^{p-1}|v_n||\nabla\eta|\,dx + o_n(1)\\
&\quad\le \frac{4\alpha}{r}\int_{B_{2r}^c\setminus B_r^c}|\nabla v_n|^{p-1}|v_n|\,dx + o_n(1)\\
&\quad\le \frac{4\alpha}{r}\left(\int_{B_{2r}^c\setminus B_r^c}|\nabla v_n|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_{B_{2r}^c\setminus B_r^c}|v_n|^p\,dx\right)^{\frac{1}{p}} + o_n(1)\\
&\quad\le \frac{4\alpha}{r}V_0^{-\frac{1}{p}}\|v_n\|_E^p + o_n(1) \le \frac{4\alpha M}{r}V_0^{-\frac{1}{p}} + o_n(1), \quad \text{as } n\to\infty.
\end{aligned} \tag{3.39}$$

This estimate concludes (3.32). Moreover, limit (3.32) gives

$$\limsup_{n\to\infty}\int_{B_{2r}^c} V(x)\big|f(v_n)\big|^p\big)\,dx < \varepsilon, \tag{3.40}$$

and consequently,

$$\int_{B_{2r}^c} V(x)\big|f(v)\big|^p\big)\,dx \le \varepsilon. \tag{3.41}$$

Since $v_n \to v$ in $L^p(B_{2r})$, we get

$$\lim_{n\to\infty}\int_{B_{2r}} V(x)\big|f(v_n)\big|^p\,dx = \int_{B_{2r}} V(x)\big|f(v)\big|^p\,dx. \tag{3.42}$$

Then, for all $\varepsilon > 0$, limits (3.40)–(3.42) yield

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N} V(x)\big(\big|f(v_n)\big|^p - \big|f(v)\big|^p\big)\,dx\right| \le 3\varepsilon, \tag{3.43}$$

and limit (3.34) holds.

In the following, we prove (3.35). We first note that (f_6) and (3.38) show

$$\int_{\mathbb{R}^N} g\big(x,f(v_n)\big)f(v_n)f'(v_n)v_n\eta\,dx = o_n(1) \quad \text{as } n\to\infty. \tag{3.44}$$

Then the fact $J'(v_n)(\eta v_n) = o_n(1)$ implies that

$$\begin{aligned}\int_{\mathbb{R}^N} \left(|\nabla v_n|^p + V(x)\left|f(v_n)\right|^{p-2} f(v_n) f'(v_n) v_n\right)\eta\, dx &= -\int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \nabla\eta v_n\, dx \\ &\quad + \int_{\mathbb{R}^N} g\big(x, f(v_n)\big) f'(v_n) v_n \eta\, dx + o_n(1) \\ &\le \int_{\mathbb{R}^N} |\nabla v_n|^{p-1}|\nabla\eta||v_n|\, dx + o_n(1) \le \frac{4\alpha M}{r} V_0^{-\frac{1}{p}} + o_n(1). \end{aligned} \tag{3.45}$$

This shows that there exists a constant $r_0 \ge 1$ such that

$$\limsup_{n\to\infty} \int_{B_{2r}^c} \left(|\nabla v_n|^p + V(x)\left|f(v_n)\right|^{p-2} f(v_n) f'(v_n) v_n\right) dx < \varepsilon \tag{3.46}$$

for $r > r_0$. So,

$$\limsup_{n\to\infty} \int_{B_{2r}^c} V(x)\left|f(v_n)\right|^{p-2} f(v_n) f'(v_n) v_n\, dx < \varepsilon, \tag{3.47}$$

and consequently

$$\int_{B_{2r}^c} V(x)\left|f(v)\right|^{p-2} f(v) f'(v) v\, dx \le \varepsilon. \tag{3.48}$$

Since $v_n \to v$ in $L^p(B_{2r})$, we have

$$\limsup_{n\to\infty} \int_{B_{2r}} V(x)\left|f(v_n)\right|^{p-2} f(v_n) f'(v_n) v_n\, dx = \int_{B_{2r}} V(x)\left|f(v)\right|^{p-2} f(v) f'(v) v\, dx, \tag{3.49}$$

and then

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} V(x)\left(\left|f(v_n)\right|^{p-2} f(v_n) f'(v_n) v_n - \left|f(v)\right|^{p-2} f(v) f'(v) v\right) dx \right| \le 3\varepsilon \tag{3.50}$$

for every $\varepsilon > 0$. Therefore, limit (3.35) is true. The proof of part (i) is completed.

(ii). From (3.18), one sees that as $n \to \infty$

$$\int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \nabla\varphi\, dx \to \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla\varphi\, dx \quad \forall\varphi \in C_0^\infty\left(\mathbb{R}^N\right). \tag{3.51}$$

As in the proof of (i), we can derive as $n \to \infty$

$$\int_{\mathbb{R}^N} V(x)\left(\left|f(v_n)\right|^{p-2} f(v_n) f'(v_n) - \left|f(v)\right|^{p-2} f(v) f'(v)\right)\varphi\, dx \to 0 \tag{3.52}$$

and

$$\int_{\mathbb{R}^N} \left(g\big(x, f(v_n)\big) f'(v_n) - g\big(x, f(v)\big) f'(v)\right)\varphi\, dx \to 0. \tag{3.53}$$

Then, from (3.51), (3.52), and (3.53), it follows

$$0 = \lim_{n\to\infty} J'(v_n)\varphi = J'(v)\varphi, \quad \forall\varphi \in C_0^\infty\left(\mathbb{R}^N\right). \tag{3.54}$$

By the dense $C_0^\infty(\mathbb{R}^N)$ in E, we have $J'(v)\varphi = 0$, $\forall \varphi \in E$. In particular, $J'(v)v = 0$. Hence, v is a critical point of J in E. This completes the proof of Lemma 3.4. □

Lemma 3.5 *Assume that all hypotheses in Theorem 1.1 hold. Let $\{v_n\}$ be a Cerami sequence and satisfy (3.18). Then $v_n \to v$ in E, that is, the functional J satisfies the Cerami condition in E.*

Proof From $J'(v_n)v_n = o_n(1)$ as $n \to \infty$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla v_n|^p\,dx = &-\int_{\mathbb{R}^N} V(x)\big|f(v_n)\big|^{p-2} f(v_n)f'(v_n)v_n\,dx \\ &+ \int_{\mathbb{R}^N} g\big(x,f(v_n)\big)f(v_n)f'(v_n)v_n\,dx + o_n(1).\end{aligned} \tag{3.55}$$

Using limits (3.20), (3.25), and (3.35) together with $J'(v)v = 0$, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla v_n|^p\,dx = \int_{\mathbb{R}^N} |\nabla v|^p\,dx. \tag{3.56}$$

The application of Brezis–Lieb lemma in [14] yields

$$\lim_{n\to\infty}\big\|\nabla(v_n - v)\big\|_p^p = \lim_{n\to\infty}\int_{\mathbb{R}^N}\big|\nabla(v_n - v)\big|^p\,dx = 0. \tag{3.57}$$

As in the proof of (3.6), we see that

$$\int_{\mathbb{R}^N}\big(\big|\nabla(v_n - v)\big|^p + V(x)\big|f(v_n - v)\big|^p\big)\,dx \ge C_0\|v_n - v\|_E^p, \quad \forall n \in \mathbb{N}. \tag{3.58}$$

Clearly, it follows from (3.57) and (3.58) that, to conclude $v_n \to v$ in E, it remains to prove

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)\big|f(v_n - v)\big|^p\,dx = 0. \tag{3.59}$$

Indeed, by Fatou's lemma, for any $r > 0$, we have

$$\begin{aligned}&\int_{B_{2r}} V(x)\big|f(v)\big|^p\,dx \le \liminf_{n\to\infty}\int_{B_{2r}} V(x)\big|f(v_n)\big|^p\,dx, \\ &\int_{B_{2r}^c} V(x)\big|f(v)\big|^p\,dx \le \liminf_{n\to\infty}\int_{B_{2r}^c} V(x)\big|f(v_n)\big|^p\,dx.\end{aligned} \tag{3.60}$$

On the other hand, from (3.34), one sees

$$\begin{aligned}&\lim_{n\to\infty}\int_{B_{2r}} V(x)\big|f(v_n)\big|^p\,dx = \int_{B_{2r}} V(x)\big|f(v)\big|^p\,dx, \\ &\lim_{n\to\infty}\int_{B_{2r}^c} V(x)\big|f(v_n)\big|^p\,dx = \int_{B_{2r}^c} V(x)\big|f(v)\big|^p\,dx.\end{aligned} \tag{3.61}$$

Noticing that the function $\phi''(t) > p(p-2\alpha)|f(t)|^{p-2}(f'(t))^2 > 0$ in $\mathbb{R}\setminus\{0\}$, we know that $\phi(t)$ is convex and even in $\mathbb{R}$, where $\phi(t) = |f(t)|^p$. Hence, by (f_9), it follows from (3.40) and

(3.41) that

$$\int_{B_{2r}^c} V(x)\big|f(v_n - v)\big|^p\,dx \le \frac{1}{2}\int_{B_{2r}^c} V(x)\big(\big|f(2v_n)\big|^p + \big|f(2v)\big|^p\big)\,dx$$
$$\le \int_{B_{2r}^c} V(x)\big(\big|f(v_n)\big|^p + \big|f(v)\big|^p\big)\,dx \le 2\varepsilon \tag{3.62}$$

for large n. Since $|f(v_n - v)|^p \le |v_n - v|^p$ and $v_n \to v$ in $L^p(B_{2r})$, we have $\int_{B_{2r}} V(x)|f(v_n - v)|^p\,dx \to 0$ as $n \to \infty$. Altogether, we get (3.59) and $v_n \to v$ in E. This completes the proof of Lemma 3.5. □

4 Proof of Theorem 1.1

We need the following mountain pass theorem to prove our result.

Lemma 4.1 ([21], *Theorem* 9.12). *Let E be an infinite dimensional real Banach space, $J \in C^1(E,\mathbb{R})$ be even and satisfy the Cerami condition, and $J(0) = 0$. If $E = Y \oplus Z$, Y is finite dimensional and J satisfies*

(J_1) There exist constants $\rho, \tau > 0$ such that $J(u) \ge \tau$ on $\partial B_\rho \cap Z$;

(J_2) For each finite dimensional subspace $E_0 \subset E$, there is $R_0 = R_0(E_0)$ such that $J(u) \le 0$ on $E_0 \setminus B_{R_0}$, where $B_r = \{v \in E : \|v\|_E < r\}$.

Then J possesses an unbounded sequence of critical values.

Proof of Theorem 1.1 Clearly, the functional J defined by (2.10) is even in E. By Lemmas 3.1–3.5 in Sect. 3, the functional J satisfies the Cerami condition. Next, we prove that J satisfies (J_1) and (J_2).

From (f_5) and Hölder's inequality, we deduce that

$$\int_{\mathbb{R}^N} |h_1|\big|f(v)\big|^m\,dx \le (2\alpha)^{\frac{m}{2\alpha p}}\int_{\mathbb{R}^N} |h_1||v|^{\frac{m}{2\alpha}}\,dx \le (2\alpha)^{\frac{m}{2\alpha p}}\|h_1\|_\sigma \|v\|_{p^*}^{\frac{m}{2\alpha}}$$
$$\le C_1\|v\|_E^{\frac{m}{2\alpha}}, \quad v \in E \tag{4.1}$$

with some constant $C_1 > 0$. Similarly, if (H_3) is true, then one sees that

$$\int_{\mathbb{R}^N} |h_2|\big|f(v)\big|^q\,dx \le (2\alpha)^{\frac{q}{2\alpha p}}\|h_2\|_\gamma \|v\|_E^{\frac{q}{2\alpha}} \le C_1\|v\|_E^{\frac{q}{2\alpha}}, \quad v \in E. \tag{4.2}$$

If (H_4) holds, one has

$$\int_{B_1} |h_2|\big|f(v)\big|^q\,dx \le (2\alpha)^{\frac{q}{2\alpha p}}\|h_2\|_{L^\gamma(B_1)}\|v\|_E^{\frac{q}{2\alpha}} \le C_1\|v\|_E^{\frac{q}{2\alpha}}, \quad v \in E. \tag{4.3}$$

Moreover, it follows from (f_3), (f_5) and Hölder's inequality that

$$\int_{B_1^c} |h_2|\big|f(v)\big|^q\,dx \le h_0\bigg(\int_{B_1^c}\big|f(v)\big|^p\,dx\bigg)^t\bigg(\int_{B_1^c}\big|f(v)\big|^{2\alpha p^*}\,dx\bigg)^{1-t}$$
$$\le (2\alpha)^{\frac{N}{N-p}}h_0\bigg(\int_{B_1^c}|v|^p\,dx\bigg)^t\bigg(\int_{B_1^c}|v|^{p^*}\bigg)^{1-t}$$
$$\le C_2\|v\|_E^{q_0}, \quad v \in E \tag{4.4}$$

with some $C_2 > 0$ and $h_0 = \|h_2\|_{L^\infty(B_1^c)}$, $q_0 = pt + p^*(1-t)$, $t = \frac{2\alpha p^*-q}{2\alpha p^*-p}$. Clearly, $q_0 > \frac{q}{2\alpha}$. Then (4.3) and (4.4) show that there is a constant $C_3 > 0$ such that

$$\int_{\mathbb{R}^N} |h_2| \left|f(v)\right|^q dx \le C_3 \|v\|_E^{\frac{q}{2\alpha}}, \quad \|v\|_E \le 1. \tag{4.5}$$

As in the proof of (3.6), we can derive

$$\int_{\mathbb{R}^N} \left(|\nabla v|^p + V(x)\left|f(v)\right|^p\right) dx \ge C_0 \|v\|_E^p, \quad \forall \|v\|_E \le 1. \tag{4.6}$$

Then, from (4.1),(4.2), and (4.5), we conclude that

$$J(v) \ge \frac{C_0}{p}\|v\|_E^p - \lambda\beta_1\|v\|_E^{\frac{m}{2\alpha}} - \beta_2\|v\|_E^{\frac{q}{2\alpha}}, \quad \forall \|v\|_E \le 1, \tag{4.7}$$

where $\beta_1 = C_1$, $\beta_2 = \min\{C_1, C_3\}$. Denote

$$h(z) = z^p\Big(\frac{C_0}{p} - \lambda\beta_1 z^{\frac{m}{2\alpha}-p} - \beta_2 z^{\frac{q}{2\alpha}-p}, \quad 0 < z \le 1. \tag{4.8}$$

Choose $z_1 \in (0,1)$ such that

$$\frac{C_0}{p} - \beta_2 z^{\frac{q}{2\alpha}-p} \ge \frac{C_0}{p} - \beta_2 z_1^{\frac{q}{2\alpha}-p} \ge \frac{C_0}{2p}, \quad 0 < z \le z_1. \tag{4.9}$$

This is possible since $\frac{q}{2\alpha} > p$. Moreover, let

$$0 \le \lambda \le \lambda_0 = \frac{C_0}{4p\beta_1} z_1^{p-\frac{m}{2\alpha}}. \tag{4.10}$$

Then

$$\frac{C_0}{2p} - \lambda\beta_1 z_1^{\frac{m}{2\alpha}-p} \ge \frac{C_0}{4p} \quad \text{and} \quad h(z_1) \ge \frac{C_0}{4p} z_1^p \equiv \tau > 0. \tag{4.11}$$

So, it follows from (4.8), (4.10), and (4.11) that there exist $\lambda_0, \tau, \rho > 0$ such that $J(v) \ge \tau$ with $\rho = z_1 = \|v\|_E$ and $\lambda \in [0, \lambda_0]$. Thus condition (J_1) is satisfied.

We now verify (J_2). For any finite dimensional subspace $E_0 \subset E$, we assert that there exists a constant $R_0 > \rho$ such that $J < 0$ on $E_0 \setminus B_{R_0}$. Otherwise, there is a sequence $\{v_n\} \subset E_0$ such that $\|v_n\|_E \to \infty$ and $J(v_n) \ge 0$. Hence,

$$\begin{aligned}
&\frac{1}{p}\int_{\mathbb{R}^N} \left(|\nabla v_n|^p + V(x)\left|f(v_n)\right|^p\right) dx \\
&\quad \ge \int_{\mathbb{R}^N} G\big(x, f(v_n)\big)\, dx \\
&\quad = \frac{\lambda}{m}\int_{\mathbb{R}^N} h_1 \left|f(v_n)\right|^m dx + \frac{1}{q}\int_{\mathbb{R}^N} h_2 \left|f(v_n)\right|^q dx.
\end{aligned} \tag{4.12}$$

Set $\omega_n = \frac{v_n}{\|v_n\|_E}$. Then up to a subsequence, we can assume $\omega_n \rightharpoonup \omega$ in E, $\omega_n(x) \to \omega(x)$ a.e. in $\mathbb{R}^N$. Denote $\Omega = \{x \in \mathbb{R}^N : \omega(x) \ne 0\}$. Assume $|\Omega| > 0$. Clearly, $v_n(x) \to \infty$ in Ω. It follows from (4.1) that

$$\|v_n\|_E^{-p}\int_\Omega |h_1|\big|f(v_n)\big|^m\,dx \le C_1\|v_n\|_E^{\frac{m}{2\alpha}-p} \to 0 \quad \text{as } n\to\infty. \tag{4.13}$$

On the other hand, from (f_7), we derive

$$\|v_n\|_E^{-p}\int_\Omega h_2\big|f(v_n)\big|^q\,dx = \int_\Omega h_2\frac{|f(v_n)|^q}{|v_n|^{\frac{q}{2\alpha}}}|v_n|^{\frac{q}{2\alpha}-p}\omega_n^p\,dx \to \infty \quad \text{as } n\to\infty \tag{4.14}$$

Therefore,

$$\lim_{n\to\infty}\|v_n\|_E^{-p}\int_\Omega G\big(x,f(v_n)\big)\,dx = \infty. \tag{4.15}$$

But it is easy to see that

$$\int_{\mathbb{R}^N}\big(|\nabla v_n|^p + V(x)\big|f(v_n)\big|^p\big)\,dx \le \int_{\mathbb{R}^N}\big(|\nabla v_n|^p + V(x)|v_n|^p\big)\,dx \le \|v_n\|_E^p. \tag{4.16}$$

We have a contradiction from (4.12), (4.15), and (4.16). So, $|\Omega| = 0$ and $\omega(x) = 0$ a.e. on $\mathbb{R}^N$. By the equivalency of all norms in E_0, there exists a constant $\beta > 0$ such that

$$\begin{aligned}&\left(\int_{\mathbb{R}^N}|h_2||v|^q\,dx\right)^{1/q} \ge \beta\|v\|_E, \quad \forall v\in E_0, \quad \text{and}\\ &\int_{\mathbb{R}^N}|h_2||v_n|^q\,dx \ge \beta^q\|v_n\|_E^q, \quad \forall n\in\mathbb{N}.\end{aligned} \tag{4.17}$$

Hence,

$$\beta^q \le \lim_{n\to\infty}\int_{\mathbb{R}^N}|h_2||\omega_n|^q\,dx = 0. \tag{4.18}$$

It is impossible. This shows that there is a constant $R_0 > 0$ such that $J < 0$ on $E_0 \setminus B_{R_0}$. Therefore, the existence of infinitely many solutions $\{v_n\}$ for problem (2.12) follows from Lemma 4.1, and so $u_n = f(v_n)$ is a solution of Problem (1.1) for $n = 1,2,\dots$. We finish the proof of Theorem 1.1. □

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Data availability
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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