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The well posedness of solutions for the 2D magnetomicropolar boundary layer equations in an analytic framework

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Abstract
In this paper, we prove the existence and uniqueness of solutions to the 2D magnetomicropolar boundary layer equations on the half-plane by using the classical bootstrap argument in an analytic framework.

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1 Introduction

The magnetomicropolar boundary layer equations were derived from the 2D incompressible magnetomicropolar equations on the half-space when we consider the asymptotic behavior of solutions as the parameters tend to zero, see [13]. In this paper, we shall consider the existence and uniqueness of solutions for the following 2D magnetomicropolar boundary layer equations on the upper half-plane $\mathbb{R}^2_+ = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}_+\}$, which read as

$$\begin{cases} \partial_t u_1 + u_1\partial_x u_1 + u_2\partial_y u_1 - 2\kappa\partial_y w_1 - \mu\partial_y^2 u_1 - b_1\partial_x b_1 - b_2\partial_y b_1 = 0, \\ \partial_t b_1 + u_1\partial_x b_1 + u_2\partial_y b_1 - \nu\partial_y^2 b_1 - b_2\partial_y u_1 = 0, \\ \partial_t w_1 + u_1\partial_x w_1 + u_2\partial_y w_1 + 2\kappa\partial_y u_1 - \gamma\partial_y^2 w_1 = 0, \\ \partial_x u_1 + \partial_y u_2 = 0, \qquad \partial_x b_1 + \partial_y b_2 = 0, \end{cases} \tag{1.1}$$

where the unknown functions (u_1, u_2), (b_1, b_2), and w_1 stand for the velocity field, the magneto field and the microrotational velocity, respectively. The positive constants μ, κ, ν, γ are associated with the properties of the materials. μ is the Newtonian kinematic viscosity coefficient, κ is the microrotation viscosity coefficient, ν is the resistivity coefficient, and γ is the spin viscosity coefficient.

The problem (1.1) is subject to the initial data and the boundary conditions

$$\begin{cases} u_1(t,x,y)|_{t=0} = u_0(x,y), \qquad b_1(t,x,y)|_{t=0} = b_0(x,y), \\ w_1(t,x,y)|_{t=0} = w_0(x,y), \\ u_1(t,x,y)|_{y=0} = u_2(t,x,y)|_{y=0} = b_1(t,x,y)|_{y=0} \\ \qquad\qquad = b_2(t,x,y)|_{y=0} = w_1(t,x,y)|_{y=0} = 0. \end{cases} \tag{1.2}$$

The far field is represented by $(\bar{u}, \bar{b}, \bar{w})$

$$\lim_{y\to\infty}(u_1, b_1, w_1) = (\bar{u}, \bar{b}, \bar{w}). \tag{1.3}$$

To start with, let us briefly review some known results of the problem (1.1). In particular, when the magneto field (b_1, b_2) and the microrotational velocity w_1 do not exist, the problem (1.1) reduces to the classical Prandtl equations that were first introduced formally by Prandtl [24] in 1904. This system is the foundation of the boundary layer theory. It says that the flow about a solid body can be divided into two regions: a very thin layer in the neighborhood of the body (the boundary layer) where viscous friction plays an essential part, and the remaining region outside this layer where friction may be neglected (the outer flow). Formally, the asymptotic limit of the Navier–Stokes equations can be denoted by the Prandtl equations within the boundary layer and by the Euler equations away from boundary layer. About sixty years later, Oleinik [22] established the first systematic work in strictly mathematics, in which she pointed out that the local-in-time well posedness of solutions to the 2D Prandtl system can be proved by using the Crocco transformation under the monotonicity condition on the tangential velocity field in the normal variable to the boundary. This result together with a detailed description of the boundary layer theory was showed in the classical book by Oleinik–Samokhin [23]. Recently, the well-posedness result was reproved by using an energy method in the framework of Sobolev spaces in [1] and [21] independently by observing the cancelation mechanism in the convection terms. By imposing an additional favorable condition on the pressure, a global-in-time weak solution was obtained in [31].

There have been some results in the analytic framework with analytic radius $\tau(t)$ for the 2D and 3D boundary layer equations. First, Kukavica et al. [11] investigated the local well posedness of solutions to the 2D Prandtl and hydrostatic Euler equations by the energy method. Kukavica and Vicol [12] considered the local well posedness of solutions to the 2D Prandtl boundary layer equations with general initial data by using analytic energy estimates. Later, Ignatova and Vicol [10] studied the almost global existence and uniqueness of the solution for the 2D Prandtl equations in an exponential weighted space by applying analytic energy estimates. Xie and Yang [29] studied the global existence of solutions to the 2D MHD boundary layer equations in the mixed Prandtl and Hartmann regime when the initial datum is a small perturbation of the Hartmann profile, and the solutions in the analytic norm exponentially decay in time. Recently, Liu and Zhang [19] established the global existence and the asymptotic decay estimate of solutions to the 2D MHD boundary layer equations with small initial data. Inspired by [10], Xie and Yang [30] investigated the lifespan of the solution to the 2D MHD boundary layer system by using the cancelation mechanism and obtained that the lifespan of the solution has a lower bound. Dong and Qin [5] obtained the global well posedness of solutions to the 2D Prandtl–Hartmann equations in the analytic framework by the standard energy method. Motivated by [12], Lin and Zhang [13] studied the local existence of solutions for the 2D incompressible magnetomicropolar boundary layer equations when the initial data are analytic in the x variable by using the energy method. Lin and Zhang [14] proved the lifespan of solutions to the 3D Prandtl system with the initial data that lie within ε of a stable shear flow in Chemin–Lerner-type spaces by using anisotropic Littlewood–Paley energy estimates. There are also some results for the boundary layer equations by using the methods that are different from the energy method. For interested readers, we refer to [2–4, 7, 16, 20, 32] and the references therein for the recent progress.

For the MHD boundary layer equations, there have been some results in the Sobolev framework for the 2D MHD boundary layer equations. Liu, Xie, and Yang [17] investigated the local existence and uniqueness of solutions in a weighted Sobolev space for the 2D nonlinear MHD boundary layer equations by using the energy method. As a continuation of [17], the same authors [18] proved the validity of the Prandtl boundary layer expansion and gave a L^∞ estimate on the error by multiscale analysis. Liu et al. [15] proved the local well posedness of solutions to the 2D MHD boundary layer equations in Sobolev spaces. Finally, they obtained the linear instability of the 2D MHD boundary layer when the tangential magneto field is degenerate at one point. So far, besides the well posedness of solutions in Sobolev framework and analytic framework, there also have some results on the vanishing limits [26–28] for the incompressible MHD systems. Huang, Liu, and Yang [9] attained the local well posedness of solutions to the 2D compressible MHD boundary layer in weighted Sobolev spaces by applying the classical iteration scheme. Gao, Huang, and Yao [6] investigated the local well posedness of solutions for 2D incompressible MHD boundary layer equations in weighted conormal Sobolev spaces.

Motivated by [10, 30], we investigate the well posedness of solutions to the 2D magnetomicropolar boundary layer equations (1.1)–(1.3) in an analytic framework by the standard energy method. Similar to the classical Prandtl equations, the difficulty of solving the problem (1.1)–(1.3) in the Sobolev framework is the loss of the x-derivative in the term $v\partial_y u$. To overcome this, we show the Gaussian weighted Poincaré inequality (see Lemma 2.3). In other words, the Poincaré inequality does not hold in unbounded domains. In addition, we also construct a set of functions (see (2.16)) to eliminate a technical difficulty. Compared to the existence and uniqueness of solutions to the classical Prandtl equations in weighted Sobolev spaces where the monotonicity condition of the tangential velocity plays a key role or the 2D MHD boundary layer equations where the initial tangential magnetic field has a lower bound plays an important role, these conditions are not needed for the well posedness of solutions to the 2D magnetomicropolar boundary layer equations (1.1)–(1.3) in the analytic framework. Compared to the result in [30], some new estimates of the microrotational velocity w are needed for the well-posedness of solutions for the 2D magnetomicropolar boundary layer equations in this paper.

Finally, the rest of the paper is organized as follows. In Sect. 2 we show the main result in this paper and give some lemmas that will be used frequently. In Sect. 3 we prove the existence and uniqueness of solutions for problem (1.1)–(1.3) by using the classical bootstrap argument.

Hereafter, let letter C be a general positive constant, which may vary from line to line at each step.

2 Main result and preliminary estimates

Without loss of generality, taking $\bar{b} = \bar{w} = 1$ and $\kappa = \mu = \nu = 1$. First, a shear flow $(u^s(t,y), 0, 1, 0, 1)$ is a trivial solution to problem (1.1)–(1.3) with $u^s(t,y)$ being the solution of the heat equations

$$\begin{cases} \partial_t u^s - \partial_y^2 u^s = 0, & (t,y) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ u^s|_{y=0} = 0, \quad \text{and} \quad \lim_{y\to+\infty} u^s = 1, & t \in \mathbb{R}_+, \\ u^s|_{t=0} = u_0^s(y), \quad y \in \mathbb{R}_+. \end{cases} \tag{2.1}$$

As in [30], we assume the shear flow $u^s(t,y)$ has the following properties:

$$\left\| \partial_y^i u^s \right\|_{L_y^\infty} \le \frac{C}{\langle t\rangle^{i/2}}, \quad i = 1,2, \qquad \int_0^\infty \left|\partial_y u^s\right| dy < C, \qquad \left\| \theta_\alpha \partial_y^2 u^s \right\|_{L_y^2} \le \frac{C}{\langle t\rangle^{3/4}}, \tag{2.2}$$

where C is a positive constant, $\theta_\alpha = \exp(\frac{\alpha z^2}{4})$ and notation $\langle t\rangle = (1+t)$.

Motivated by [10, 30], we use the following Gaussian weighted function θ_α to introduce the function space of the solutions,

$$\theta_\alpha = \exp\left(\frac{\alpha z^2}{4}\right), \quad \text{with } z = \frac{y}{\sqrt{\langle t\rangle}}, \alpha \in [1/4, 1/2],$$

which combined with

$$M_m = \frac{\sqrt{(m+1)}}{m!},$$

define the Sobolev weighted seminorms by

$$\begin{cases} X_m = X_m(f,\tau) = \|\theta_\alpha \partial_x^m f\|_{L^2(\mathbb{R}_+^2)} \tau^m M_m, \\ D_m = D_m(f,\tau) = \|\theta_\alpha \partial_y \partial_x^m f\|_{L^2(\mathbb{R}_+^2)} \tau^m M_m, \\ Y_m = Y_m(f,\tau) = \|\theta_\alpha \partial_x^m f\|_{L^2(\mathbb{R}_+^2)} \tau^{m-1} m M_m. \end{cases} \tag{2.3}$$

Then, the following space of analytic functions in the tangential variable x and weighted Sobolev in the normal variable y is defined by

$$X_{\tau,\alpha} = \left\{ f(t,x,y) \in L^2\left(\mathbb{R}_+^2; \theta_\alpha \, dx\, dy\right) : \|f\|_{X_{\tau,\alpha}} < \infty \right\},$$

with $\tau > 0$ and the norms

$$\begin{cases} \|f\|_{X_{\tau,\alpha}} = \sum_{m\ge 0} X_m(f,\tau), \\ \|f\|_{D_{\tau,\alpha}} = \sum_{m\ge 0} D_m(f,\tau), \\ \|f\|_{Y_{\tau,\alpha}} = \sum_{m\ge 1} Y_m(f,\tau). \end{cases} \tag{2.4}$$

We can now state our main result as follows.

Theorem 2.1 *Let the initial data* $(u_1(0), b_1(0), w_1(0))$ *satisfy*

$$\left(u_1(0) - u^s(0), b_1(0) - 1, w_1(0) - 1\right) \in X_{\tau_0,\alpha}, \tag{2.5}$$

with $\frac{1}{4} \leq \alpha \leq \frac{1}{2}$ and assume that there exists an analyticity radius τ_0 such that

$$\frac{8C_2}{7} \leq \tau_0^{\frac{3}{2}}. \tag{2.6}$$

Then, there exists a unique solution (u, v, b, g, w) to the magnetomicropolar boundary layer problem (1.1)–(1.3) such that

$$\left(u_1 - u^s, b_1 - 1, w_1 - 1\right) \in X_{\tau,\alpha}, \tag{2.7}$$

with analyticity radius τ larger than $\frac{\tau_0}{4}$ in the time interval $[0, T]$, where $T = \min\{(\frac{7}{8C_2}\tau_0^{\frac{3}{2}})^{\frac{3}{4}} - 1, T_1\}$ and C_2, T_1 be given by (3.59) and (3.60), respectively.

The following two main estimates on the functions in the norms defined in the previous section will be used frequently, whose proofs are given in [8, 10, 30].

Lemma 2.2 (*Agmon inequality* [10]) *Let $f \in H^1(\mathbb{R}^2_+)$, then*

$$\|f\|_{L^\infty_x L^2_y} \leq C\|f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2_+)}\|f\|^{\frac{1}{2}}_{H^1(\mathbb{R}^2_+)}.$$

Lemma 2.3 ([8, 10]) *Let f be a function such that $f|_{y=0} = 0$ (or $\partial_y f|_{y=0} = 0$) and $f|_{y=\infty} = 0$. Then, for $m \geq 0$ and $t \geq 0$, it holds that*

$$\begin{cases} \frac{\alpha}{\langle t\rangle}\|\theta_\alpha \partial_x^m f\|^2_{L^2_y} \leq \|\theta_\alpha \partial_y \partial_x^m f\|^2_{L^2_y}, \\ \sum_{m\geq 0} \frac{\|\theta_\alpha \partial_y \partial_x^m f\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha \partial_x^m f\|_{L^2(\mathbb{R}^2_+)}}\tau^m M_m \geq \frac{\alpha^{\frac{1}{2}}\beta}{2\langle t\rangle^{\frac{1}{2}}}\|f\|_{D_{\tau,\alpha}} + \frac{\alpha(1-\beta)}{\langle t\rangle}\|f\|_{X_{\tau,\alpha}}, \end{cases} \tag{2.8}$$

for $\alpha \in [1/4, 1/2]$ and $\beta \in (0, 1/2)$.

Lemma 2.4 ([30]) *Let f be a function such that $f|_{y=0} = 0$. Then, it holds that*

$$\begin{cases} \|f\|_{L^2_x L^\infty_y} \leq C\|\theta_\alpha f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2_+)}\|\theta_\alpha \partial_y f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2_+)} \leq C\langle t\rangle^{\frac{1}{4}}\|\theta_\alpha \partial_y f\|_{L^2(\mathbb{R}^2_+)}, \\ \|f\|_{L^\infty_{xy}} \leq C\|\theta_\alpha f\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2_+)}\|\theta_\alpha \partial_x f\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2_+)}\|\theta_\alpha \partial_y f\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2_+)}\|\theta_\alpha \partial_x \partial_y f\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2_+)}. \end{cases} \tag{2.9}$$

Next, we are going to establish the uniform estimates of solutions for problem (1.1)–(1.3). For our purpose, we first rewrite the solutions to problem (1.1)–(1.3) as a perturbation (u, v, b, g, w) of the $(u^s, 0, 1, 0, 1)$ by standing for

$$\begin{cases} u_1 = \tilde{u} + u^s, \qquad b_1 = \tilde{b} + 1, \qquad w_1 = \tilde{w} + 1, \\ u_2 = \tilde{v}, \qquad b_2 = \tilde{g}. \end{cases} \tag{2.10}$$

Then, equations (1.1)–(1.3) can be converted to

$$\begin{cases} \partial_t \tilde{u} + (u^s + \tilde{u})\partial_x \tilde{u} + \tilde{v}\partial_y(\tilde{u} + u^s) - 2\partial_y \tilde{w} - \partial_y^2 \tilde{u} - (\tilde{b} + 1)\partial_x \tilde{b} - \tilde{g}\partial_y \tilde{b} = 0, \\ \partial_t \tilde{b} + (u^s + \tilde{u})\partial_x \tilde{b} + \tilde{v}\partial_y \tilde{b} - \partial_y^2 \tilde{b} - (\tilde{b} + 1)\partial_x \tilde{u} - g\partial_y(\tilde{u} + u^s) = 0, \\ \partial_t \tilde{w} + (u^s + \tilde{u})\partial_x \tilde{w} + \tilde{v}\partial_y \tilde{w} + 2\partial_y(\tilde{u} + u^s) - \partial_y^2 \tilde{w} = 0, \\ \partial_x \tilde{u} + \partial_y \tilde{v} = 0, \qquad \partial_x \tilde{b} + \partial_y \tilde{g} = 0. \end{cases} \tag{2.11}$$

The initial and boundary data of $(\tilde{u},\tilde{v},\tilde{b},\tilde{g},\tilde{w})$ are given by

$$\begin{cases} \tilde{u}(t,x,y)|_{t=0} = \tilde{u}_0(x,y) - u^s(0,y), \\ \tilde{b}(t,x,y)|_{t=0} = \tilde{b}_0(x,y) - 1, \qquad \tilde{w}(t,x,y)|_{t=0} = \tilde{w}_0(x,y) - 1, \\ \tilde{u}|_{y=0} = \tilde{v}|_{y=0} = 0, \qquad \tilde{b}|_{y=0} = \tilde{g}|_{y=0} = 0, \qquad \tilde{w}|_{y=0} = 0, \end{cases} \tag{2.12}$$

with the corresponding far-field condition

$$\lim_{y\to\infty} (\tilde{u},\tilde{b},\tilde{w}) = \mathbf{0}. \tag{2.13}$$

As in [30], integrating equation $(2.11)_2$ over $[0,y]$ yields that

$$\partial_t \int_0^y \tilde{b}\, d\tilde{y} + \tilde{v}(1+\tilde{b}) - \big(u^s + \tilde{u}\big)\tilde{g} = \partial_y^2 \int_0^y \tilde{b}\, d\tilde{y}, \tag{2.14}$$

where we have used the boundary conditions $b|_{y=0} = v|_{y=0} = g|_{y=0} = 0$.
Define

$$\psi(t,y) = \int_0^y \tilde{b}\, d\tilde{y},$$

which yields

$$\partial_t \psi + \tilde{v}(1+\tilde{b}) - \big(u^s + \tilde{u}\big)\tilde{g} = \partial_y^2 \psi. \tag{2.15}$$

We now introduce the new unknown functions to eliminate the difficult term $\tilde{v}\partial_y u^s$ in the system as follows:

$$u = \tilde{u} - \partial_y u^s \psi, \qquad b = \tilde{b}, \qquad w = \tilde{w}, \tag{2.16}$$

where u^s is a solution to the heat equation, that is,

$$\partial_t u^s - \partial_y^2 u^s = 0, \qquad \partial_t\partial_y u^s - \partial_y^3 u^s = 0.$$

Then, (u,b,w) satisfies the following equations

$$\begin{cases} \partial_t u + (u^s + \tilde{u})\partial_x u + \tilde{v}\partial_y u - 2\partial_y w - \partial_y^2 u - (b+1)\partial_x b - g\partial_y b \\ \quad - 2\partial_y^2 u^s b + \tilde{v}\partial_y^2 u^s \psi = 0, \\ \partial_t b + (u^s + \tilde{u})\partial_x b + \tilde{v}\partial_y b - \partial_y^2 b - (b+1)\partial_x u - g\partial_y u - g\partial_y^2 u^s \psi = 0, \\ \partial_t w + (u^s + \tilde{u})\partial_x w + \tilde{v}\partial_y w + 2\partial_y(u + u^s) + 2\partial_y u^s b + 2\partial_y^2 u^s \psi - \partial_y^2 w = 0. \end{cases} \tag{2.17}$$

The boundary conditions of (u,b,w) are given by

$$\begin{cases} (u,b,w)|_{y=0} = 0, \\ (u,b,w)|_{y=\infty} = 0. \end{cases} \tag{2.18}$$

Next, we will prove the existence of solution (u,b,w) to equations (2.17) and (2.18) with the corresponding initial data

$$\begin{cases} u(0,x,y) = \tilde{u}(0,x,y) - \partial_y u^s(0,y)\int_0^y \tilde{b}(0,x,\tilde{y})\,d\tilde{y}, \\ \tilde{b}(0,x,\tilde{y}) = b(0,x,\tilde{y}), \qquad \tilde{w}(0,x,\tilde{y}) = w(0,x,\tilde{y}). \end{cases} \tag{2.19}$$

We note that

$$\big\|u(0,x,y)\big\|_{X_{2\tau_0,\alpha}} \le \big\|\tilde{u}(0,x,y)\big\|_{X_{2\tau_0,\alpha}} + \big\|\tilde{b}(0,x,y)\big\|_{X_{2\tau_0,\alpha}}, \tag{2.20}$$

for $\alpha \in [1/4, 1/2]$.

Recalling the definition of $(\tilde{u},\tilde{b},\tilde{w})$ by

$$\begin{cases} u(t,x,y) = \tilde{u}(t,x,y) - \partial_y u^s(t,y)\int_0^y \tilde{b}(t,x,\tilde{y})\,d\tilde{y}, \\ \tilde{b}(t,x,\tilde{y}) = b(t,x,\tilde{y}), \qquad \tilde{w}(t,x,\tilde{y}) = w(t,x,\tilde{y}), \end{cases} \tag{2.21}$$

we find that the existence of solution $(\tilde{u},\tilde{b},\tilde{w})$ to problem (2.11)–(2.13) followed by the solution (u,b,w) to problem (2.17) and (2.18) is obtained, and satisfies the following estimates

$$\begin{cases} \|\tilde{u}(t,x,y)\|_{X_{2\tau,\alpha}} \le \|u(t,x,y)\|_{X_{2\tau,\alpha}} + \|b(t,x,y)\|_{X_{2\tau,\alpha}}, \\ \|\tilde{b}(t,x,y)\|_{X_{2\tau,\alpha}} = \|b(t,x,y)\|_{X_{2\tau,\alpha}}, \qquad \|\tilde{w}(t,x,y)\|_{X_{2\tau,\alpha}} = \|w(t,x,y)\|_{X_{2\tau,\alpha}}. \end{cases} \tag{2.22}$$

Hence, we only need to prove the existence of solution (u,b,w) to problem (2.17)–(2.19) in the analytic space as shown in the following three subsections.

3 Uniform estimate

3.1 A priori estimate on the velocity field

In this subsection, we will prove the estimate of solution to problem (2.17)–(2.19) on the velocity field u.

Lemma 3.1 *It holds that for any $t \in [0,T]$*

$$\begin{aligned} &\frac{d}{dt}\|u\|_{X_{\tau,\alpha}} + \sum_{m\ge 0}\tau^m M_m \frac{\|\theta_\alpha \partial_x^m \partial_y u\|_{L^2(\mathbb{R}_+^2)}^2}{\|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}} - \frac{\alpha}{2\langle t\rangle}\|u\|_{X_{\tau,\alpha}} - \frac{C}{\langle t\rangle}\|b\|_{X_{\tau,\alpha}} \\ &\quad \le \dot{\tau}(t)\|u\|_{Y_{\tau,\alpha}} + \frac{C}{\tau(t)^{\frac{1}{2}}}\big(\langle t\rangle^{-1/4}\big(\|u\|_{X_{\tau,\alpha}} + \|b\|_{X_{\tau,\alpha}}\big) + \langle t\rangle^{1/4}\big(\|u\|_{D_{\tau,\alpha}} + \|b\|_{D_{\tau,\alpha}}\big)\big) \\ &\qquad \times \big(\|u\|_{Y_{\tau,\alpha}} + \|b\|_{Y_{\tau,\alpha}}\big) + \|w\|_{X_{\tau,\alpha}} - \frac{1}{4}\sum_{m\ge 0}\frac{\|\theta_\alpha \partial_x^m \partial_y w\|_{L^2(\mathbb{R}_+^2)}^2 \tau^m M_m}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}_+^2)}}. \end{aligned}$$

Proof For $m \ge 0$, applying the operator ∂_x^m on $(2.17)_1$ and multiplying the resulting equation by $\theta_\alpha^2 \partial_x^m u$, we derive that

$$\begin{aligned} &\int_{\mathbb{R}_+^2} \partial_x^m\big(\partial_t u + \big(u^s + \tilde{u}\big)\partial_x u + \tilde{v}\partial_y u - 2\partial_y w - \partial_y^2 u - (b+1)\partial_x b \\ &\quad - g\partial_y b - 2\partial_y^2 u^s b + \tilde{v}\partial_y^2 u^s \psi\big)\theta_\alpha^2 \partial_x^m u\,dx\,dy = 0. \end{aligned} \tag{3.1}$$

We now deal with each term in (3.1) as follows. For the first term, integrating it by parts with respect to time t, we have

$$\begin{aligned}\int_{\mathbb{R}_+^2} \partial_x^m \partial_t u \theta_\alpha^2 \partial_x^m u \, dx\, dy &= \frac{1}{2}\int_{\mathbb{R}_+^2} \partial_t \big(\partial_x^m u\big)^2 \theta_\alpha^2 \, dx\, dy \\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}_+^2} \big(\partial_x^m u\big)^2 \theta_\alpha^2 \, dx\, dy - \int_{\mathbb{R}_+^2} \big(\partial_x^m u\big)^2 \theta_\alpha \frac{d}{dt}\theta_\alpha \, dx\, dy \\ &= \frac{1}{2}\frac{d}{dt}\big\|\theta_\alpha \partial_x^m u\big\|_{L^2(\mathbb{R}_+^2)}^2 + \frac{\alpha}{4\langle t\rangle}\big\|\theta_\alpha z \partial_x^m u\big\|_{L^2(\mathbb{R}_+^2)}^2.\end{aligned}$$

Therefore, we obtain

$$\frac{\int_{\mathbb{R}_+^2} \partial_x^m \partial_t u \theta_\alpha^2 \partial_x^m u \, dx\, dy}{\|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}} = \frac{1}{2}\frac{d}{dt}\big\|\theta_\alpha \partial_x^m u\big\|_{L^2(\mathbb{R}_+^2)} + \frac{\alpha}{4\langle t\rangle}\frac{\|\theta_\alpha z \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}^2}{\|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}}. \tag{3.2}$$

For the fifth term, integrating it by parts over $\mathbb{R}_+^2$, we obtain

$$\begin{aligned}-\int_{\mathbb{R}_+^2} \partial_y^2 \partial_x^m u \theta_\alpha^2 \partial_x^m u \, dx\, dy &= \big\|\theta_\alpha \partial_x^m \partial_y u\big\|_{L^2(\mathbb{R}_+^2)}^2 + \int_{\mathbb{R}_+^2} \partial_y \partial_x^m u \partial_y\big(\theta_\alpha^2\big)\partial_x^m u \, dx\, dy \\ &= \big\|\theta_\alpha \partial_x^m \partial_y u\big\|_{L^2(\mathbb{R}_+^2)}^2 - \frac{1}{2}\int_{\mathbb{R}_+^2} \big(\partial_x^m u\big)^2 \partial_y^2\big(\theta_\alpha^2\big) \, dx\, dy \\ &= \big\|\theta_\alpha \partial_x^m \partial_y u\big\|_{L^2(\mathbb{R}_+^2)}^2 - \frac{\alpha}{2\langle t\rangle}\big\|\theta_\alpha \partial_x^m u\big\|_{L^2(\mathbb{R}_+^2)}^2 \\ &\quad - \frac{\alpha^2}{2\langle t\rangle}\big\|\theta_\alpha z \partial_x^m u\big\|_{L^2(\mathbb{R}_+^2)}^2,\end{aligned}$$

where we have used the fact that $\partial_y^2(\theta_\alpha^2) = \frac{\alpha}{\langle t\rangle}\theta_\alpha^2 + \frac{\alpha^2}{\langle t\rangle}z^2\theta_\alpha^2$ and the boundary condition $\partial_x^m u|_{y=0} = 0$.

Thus,

$$\begin{aligned}&\frac{-\int_{\mathbb{R}_+^2} \partial_y^2 \partial_x^m u \theta_\alpha^2 \partial_x^m u \, dx\, dy}{\|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}} \\ &\quad = \frac{\|\theta_\alpha \partial_x^m \partial_y u\|_{L^2(\mathbb{R}_+^2)}^2}{\|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}} - \frac{\alpha}{2\langle t\rangle}\big\|\theta_\alpha \partial_x^m u\big\|_{L^2(\mathbb{R}_+^2)} - \frac{\alpha^2}{2\langle t\rangle}\frac{\|\theta_\alpha z \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}^2}{\|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}}.\end{aligned} \tag{3.3}$$

Next, we establish the estimates of the remainder terms in (3.1),

$$\begin{aligned}R_1 &\triangleq \int_{\mathbb{R}_+^2} \partial_x^m\big(\big(\tilde{u} + u^s\big)\partial_x u\big)\theta_\alpha^2 \partial_x^m u \, dx\, dy = \sum_{i=0}^{m}\binom{m}{i}\int_{\mathbb{R}_+^2} \partial_x^{m-i}\tilde{u}\partial_x^{i+1}u\theta_\alpha^2\partial_x^m u \, dx\, dy \\ &\le \sum_{i=0}^{[m/2]}\binom{m}{i}\big\|\partial_x^{m-i}\tilde{u}\big\|_{L_x^2 L_y^\infty}\big\|\theta_\alpha \partial_x^{i+1}u\big\|_{L_x^\infty L_y^2}\big\|\theta_\alpha \partial_x^m u\big\|_{L^2(\mathbb{R}_+^2)} \\ &\quad + \sum_{i=[m/2]+1}^{m}\binom{m}{i}\big\|\partial_x^{m-i}u\big\|_{L_{xy}^\infty}\big\|\theta_\alpha \partial_x^{i+1}u\big\|_{L^2(\mathbb{R}_+^2)}\big\|\theta_\alpha \partial_x^m u\big\|_{L^2(\mathbb{R}_+^2)}.\end{aligned} \tag{3.4}$$

For $0 \le i \le [m/2]$, by using $(2.9)_1$ and Lemma 2.2 (Agmon inequality in x), we deduce

$$\begin{aligned}\|\partial_x^{m-i}\tilde{u}\|_{L_x^2L_y^\infty} &= \|\partial_x^{m-i}(u+\partial_y u^s\psi)\|_{L_x^2L_y^\infty} \le \|\partial_x^{m-i}u\|_{L_x^2L_y^\infty} + \|\partial_y u^s\partial_x^{m-i}\psi\|_{L_x^2L_y^\infty} \\ &\le C\|\theta_\alpha\partial_x^{m-i}u\|_{L^2(\mathbb{R}_+^2)}^{\frac12}\|\theta_\alpha\partial_y\partial_x^{m-i}u\|_{L^2(\mathbb{R}_+^2)}^{\frac12} + C\langle t\rangle^{-\frac14}\|\theta_\alpha\partial_x^{m-i}b\|_{L^2(\mathbb{R}_+^2)} \end{aligned}\tag{3.5}$$

and

$$\|\theta_\alpha\partial_x^{i+1}u\|_{L_x^\infty L_y^2} \le C\|\theta_\alpha\partial_x^{i+1}u\|_{L^2(\mathbb{R}_+^2)}^{\frac12}\|\theta_\alpha\partial_x^{i+2}u\|_{L^2(\mathbb{R}_+^2)}^{\frac12}. \tag{3.6}$$

For $[m/2] \le i \le m$, by using $(2.9)_2$, we arrive at

$$\begin{aligned}&\|\partial_x^{m-i}\tilde{u}\|_{L_{xy}^\infty} \\ &\le C\|\theta_\alpha\partial_x^{m-i}u\|_{L^2(\mathbb{R}_+^2)}^{\frac14}\|\theta_\alpha\partial_x^{m-i+1}u\|_{L^2(\mathbb{R}_+^2)}^{\frac14}\|\theta_\alpha\partial_y\partial_x^{m-i}u\|_{L^2(\mathbb{R}_+^2)}^{\frac14}\|\theta_\alpha\partial_x^{m-i+1}\partial_y u\|_{L^2(\mathbb{R}_+^2)}^{\frac14} \\ &\quad + C\langle t\rangle^{-\frac14}\|\theta_\alpha\partial_x^{m-i}b\|_{L^2(\mathbb{R}_+^2)}^{\frac12}\|\theta_\alpha\partial_x^{m-i+1}b\|_{L^2(\mathbb{R}_+^2)}^{\frac12}. \end{aligned}\tag{3.7}$$

From now on, we apply X_i, Y_i, D_i to stand for the seminorms of function u, $\bar{X}_i$, $\bar{Y}_i$, $\bar{D}_i$ to stand for the seminorms of function b, $\tilde{X}_i$, $\tilde{Y}_i$, and $\tilde{D}_i$ denote the seminorms of function w. Therefore,

$$\begin{aligned}&\sum_{m\ge0}\frac{|R_1|\tau^m M_m}{\|\theta_\alpha\partial_x^m u\|_{L^2(\mathbb{R}_+^2)}} \\ &\le \frac{C}{\tau\langle t\rangle^{\frac12}}\Bigg(\sum_{m\ge0}\sum_{i=0}^{[m/2]}\binom{m}{i}\big(X_{m-i}^{\frac12}D_{m-i}^{\frac12} + \langle t\rangle^{-\frac14}\bar{X}_{m-i}^{\frac12}\big)Y_{i+1}^{\frac12}Y_{i+2}^{\frac12} \\ &\quad + \sum_{m\ge0}\sum_{i=[m/2]+1}^{m}\binom{m}{i}\big(X_{m-i}^{\frac14}X_{m-i+1}^{\frac14}D_{m-i}^{\frac14}D_{m-i+1}^{\frac14} + +\langle t\rangle^{-\frac14}\bar{X}_{m-i}^{\frac12}\bar{X}_{m-i+1}^{\frac12}\big)Y_{i+1}\Bigg). \end{aligned}\tag{3.8}$$

Similarly, we also have

$$\begin{aligned}R_2 &\triangleq \int_{\mathbb{R}_+^2}\partial_x^m(\tilde{v}\partial_y u)\theta_\alpha^2\partial_x^m u\,dx\,dy = \sum_{i=0}^m\binom{m}{i}\int_{\mathbb{R}_+^2}\partial_x^{m-i}\tilde{v}\partial_x^i\partial_y u\theta_\alpha^2\partial_x^m u\,dx\,dy \\ &\le \sum_{i=0}^{[m/2]}\binom{m}{i}\|\partial_x^{m-i}\tilde{v}\|_{L_x^2L_y^\infty}\|\theta_\alpha\partial_x^i\partial_y u\|_{L_x^\infty L_y^2}\|\theta_\alpha\partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \\ &\quad + \sum_{i=[m/2]+1}^{m}\binom{m}{i}\|\partial_x^{m-i}\tilde{v}\|_{L_{xy}^\infty}\|\theta_\alpha\partial_x^i\partial_y u\|_{L^2(\mathbb{R}_+^2)}\|\theta_\alpha\partial_x^m u\|_{L^2(\mathbb{R}_+^2)}. \end{aligned}\tag{3.9}$$

For $0 \le i \le [m/2]$, by using $(2.9)_1$ and Lemma 2.2 (Agmon inequality in x), we derive

$$\begin{aligned}\|\partial_x^{m-i}\tilde{v}\|_{L_x^2L_y^\infty} &\le \langle t\rangle^{\frac14}\|\theta_\alpha\partial_y\partial_x^{m-i}\tilde{v}\|_{L^2(\mathbb{R}_+^2)} \\ &\le \langle t\rangle^{\frac14}\|\theta_\alpha\partial_x^{m-i+1}\tilde{u}\|_{L^2(\mathbb{R}_+^2)} \\ &\le \langle t\rangle^{\frac14}\big(\|\theta_\alpha\partial_x^{m-i+1}u\|_{L^2(\mathbb{R}_+^2)} + \|\partial_y u^s\partial_x^{m-i+1}\psi\|_{L^2(\mathbb{R}_+^2)}\big)\end{aligned}$$

$$
\begin{aligned}
&\leq \langle t\rangle^{\frac{1}{4}}\left(\left\|\theta_\alpha \partial_x^{m-i+1} u\right\|_{L^2(\mathbb{R}_+^2)}+\langle t\rangle^{\frac{1}{2}}\left\|\partial_x^{m-i+1} \psi\right\|_{L^2(\mathbb{R}_+^2)}\right) \\
&\leq \langle t\rangle^{\frac{1}{4}}\left(\left\|\theta_\alpha \partial_x^{m-i+1} u\right\|_{L^2(\mathbb{R}_+^2)}+\left\|\partial_x^{m-i+1} b\right\|_{L^2(\mathbb{R}_+^2)}\right)
\end{aligned} \tag{3.10}
$$

and

$$
\left\|\theta_\alpha \partial_x^i \partial_y u\right\|_{L_x^\infty L_y^2} \leq \left\|\theta_\alpha \partial_x^i \partial_y u\right\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{2}}\left\|\theta_\alpha \partial_x^{i+1} \partial_y u\right\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{2}}. \tag{3.11}
$$

For $[m/2] \leq i \leq m$, by using $(2.9)_2$ and Lemma 2.3, we discover

$$
\begin{aligned}
\left\|\partial_x^{m-i} \tilde{v}\right\|_{L_x^\infty L_y^\infty} &\leq \langle t\rangle^{\frac{1}{4}}\left(\left\|\theta_\alpha \partial_x^{m-i+1} u\right\|_{L_y^2 L_x^\infty}+\left\|\partial_x^{m-i+1} b\right\|_{L_y^2 L_x^\infty}\right) \\
&\leq \langle t\rangle^{\frac{1}{4}}\left(\left\|\theta_\alpha \partial_x^{m-i+1} u\right\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{2}}\left\|\theta_\alpha \partial_x^{m-i+2} u\right\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{2}}\right. \\
&\quad \left.+\left\|\partial_x^{m-i+1} b\right\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{2}}\left\|\partial_x^{m-i+2} b\right\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{2}}\right).
\end{aligned} \tag{3.12}
$$

Hence,

$$
\begin{aligned}
&\sum_{m \geq 0} \frac{|R_2| \tau^m M_m}{\|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}} \\
&\quad \leq \frac{C}{\tau(t)^{\frac{1}{2}}}\left(\sum_{m \geq 0} \sum_{i=0}^{[m/2]}\binom{m}{i}\left(\langle t\rangle^{\frac{1}{4}} Y_{m-i+1}+\langle t\rangle^{\frac{1}{4}} \bar{Y}_{m-i+1}\right) D_i^{\frac{1}{2}} D_{i+1}^{\frac{1}{2}}\right. \\
&\quad \left.+\sum_{m \geq 0} \sum_{i=[m/2]+1}^m\binom{m}{i}\left(\langle t\rangle^{\frac{1}{4}} Y_{m-i+1}^{\frac{1}{2}} Y_{m-i+2}^{\frac{1}{2}}+\langle t\rangle^{\frac{1}{4}} \bar{Y}_{m-i+1}^{\frac{1}{2}} \bar{Y}_{m-i+2}^{\frac{1}{2}}\right) D_i\right).
\end{aligned} \tag{3.13}
$$

For the following term, we apply the Hölder inequality, leading to

$$
R_3 \overset{\triangle}{=} \int_{\mathbb{R}_+^2} \partial_x^m \partial_y w \theta_\alpha^2 \partial_x^m u \, dx \, dy \leq \left\|\theta_\alpha \partial_x^m \partial_y w\right\|_{L^2(\mathbb{R}_+^2)}\left\|\theta_\alpha \partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)}.
$$

Therefore,

$$
\begin{aligned}
&\sum_{m \geq 0} \frac{|R_3| \tau^m M_m}{\|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}} \\
&\quad \leq \frac{1}{4} \sum_{m \geq 0} \frac{\|\theta_\alpha \partial_x^m \partial_y w\|_{L^2(\mathbb{R}_+^2)}^2 \tau^m M_m}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}_+^2)}}+\sum_{m \geq 0}\left\|\theta_\alpha \partial_x^m w\right\|_{L^2(\mathbb{R}_+^2)} \tau^m M_m .
\end{aligned} \tag{3.14}
$$

Also, $b = \tilde{b}$ such that

$$
\begin{aligned}
R_4 &\overset{\triangle}{=} \int_{\mathbb{R}_+^2} \partial_x^m\left((b+1) \partial_x b\right) \theta_\alpha^2 \partial_x^m u \, dx \, dy=\sum_{i=0}^m\binom{m}{i} \int_{\mathbb{R}_+^2} \partial_x^{m-i} b \partial_x^{i+1} b \theta_\alpha^2 \partial_x^m u \, dx \, dy \\
&\leq \sum_{i=0}^{[m/2]}\binom{m}{i}\left\|\partial_x^{m-i} b\right\|_{L_x^2 L_y^\infty}\left\|\theta_\alpha \partial_x^{i+1} b\right\|_{L_x^\infty L_y^2}\left\|\theta_\alpha \partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)} \\
&\quad +\sum_{i=[m/2]+1}^m\binom{m}{i}\left\|\partial_x^{m-i} b\right\|_{L_{xy}^\infty}\left\|\theta_\alpha \partial_x^{i+1} b\right\|_{L^2(\mathbb{R}_+^2)}\left\|\theta_\alpha \partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)}.
\end{aligned} \tag{3.15}
$$

For $0 \le i \le [m/2]$, by using $(2.9)_1$ and Lemma 2.2 (Agmon inequality in x), we conclude

$$\|\partial_x^{m-i} b\|_{L_x^2 L_y^\infty} \le \|\theta_\alpha \partial_x^{m-i} b\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{2}} \|\theta_\alpha \partial_y \partial_x^{m-i} b\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{2}} \tag{3.16}$$

and

$$\|\theta_\alpha \partial_x^{i+1} b\|_{L_x^\infty L_y^2} \le \|\theta_\alpha \partial_x^{i+1} b\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{2}} \|\theta_\alpha \partial_x^{i+2} b\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{2}}. \tag{3.17}$$

For $[m/2] \le i \le m$, by using $(2.9)_2$, we arrive at

$$\begin{aligned}\|\partial_x^{m-i} b\|_{L_{xy}^\infty} \le{}& \|\theta_\alpha \partial_x^{m-i} b\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{4}} \|\theta_\alpha \partial_x^{m-i+1} b\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{4}} \\ &\times \|\theta_\alpha \partial_y \partial_x^{m-i} b\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{4}} \|\theta_\alpha \partial_x^{m-i+1} \partial_y b\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{4}}.\end{aligned} \tag{3.18}$$

Therefore,

$$\begin{aligned}\sum_{m\ge 0} \frac{|R_4|\tau^m M_m}{\|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}} \le{}& \frac{C}{\tau(t)^{\frac{1}{2}}} \Bigg(\sum_{m\ge 0} \sum_{i=0}^{[m/2]} \binom{m}{i} \bar{X}_{m-i}^{\frac{1}{2}} \bar{D}_{m-i}^{\frac{1}{2}} \bar{Y}_{i+1}^{\frac{1}{2}} \bar{Y}_{i+2}^{\frac{1}{2}} \\ &+ \sum_{m\ge 0} \sum_{i=[m/2]+1}^{m} \binom{m}{i} \bar{X}_{m-i}^{\frac{1}{4}} \bar{X}_{m-i+1}^{\frac{1}{4}} \bar{D}_{m-i}^{\frac{1}{4}} \bar{D}_{m-i+1}^{\frac{1}{4}} \bar{Y}_{i+1} \Bigg).\end{aligned} \tag{3.19}$$

As a consequence, we obtain

$$\begin{aligned}\sum_{m\ge 0} \frac{|R_5|\tau^m M_m}{\|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}} \le{}& \frac{C}{\tau(t)^{\frac{1}{2}}} \Bigg(\sum_{m\ge 0} \sum_{i=0}^{[m/2]} \binom{m}{i} \langle t\rangle^{\frac{1}{4}} \bar{Y}_{m-i+1} \bar{D}_i^{\frac{1}{2}} \bar{D}_{i+1}^{\frac{1}{2}} \\ &+ \sum_{m\ge 0} \sum_{i=[m/2]+1}^{m} \binom{m}{i} \langle t\rangle^{\frac{1}{4}} \bar{Y}_{m-i+1}^{\frac{1}{2}} \bar{Y}_{m-i+2}^{\frac{1}{2}} \bar{D}_i \Bigg).\end{aligned} \tag{3.20}$$

Similar to [30], we have

$$\left| \int_{\mathbb{R}_+^2} \partial_y^2 u^s \partial_x^m b \theta_\alpha^2 \partial_x^m u \, dx\, dy \right| \le C\langle t\rangle^{-1} \|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}_+^2)} \|\theta_\alpha \partial_x^m b\|_{L^2(\mathbb{R}_+^2)}, \tag{3.21}$$

which implies

$$\sum_{m\ge 0} \frac{|R_6|\tau^m M_m}{\|\theta_\alpha \partial_x^m b\|_{L^2(\mathbb{R}_+^2)}} \le C\langle t\rangle^{-1} \sum_{m\ge 0} \|\theta_\alpha \partial_x^m b\|_{L^2(\mathbb{R}_+^2)} \tau^m M_m, \tag{3.22}$$

where we used the fact that $\|\partial_y^2 u^s\|_{L_y^\infty} \le C\langle t\rangle^{-1}$.

Analogously, we have

$$\begin{aligned} R_7 \triangleq{}& \left| \int_{\mathbb{R}_+^2} \partial_y^2 u^s \partial_x^m (\tilde{v}\psi) \theta_\alpha^2 \partial_x^m u \, dx\, dy \right| \\ \le{}& \sum_{i=0}^{[m/2]} \binom{m}{i} \|\partial_x^{m-i} \tilde{v}\|_{L_x^2 L_y^\infty} \|\theta_\alpha \partial_y^2 u^s\|_{L_y^2} \|\partial_x^i \psi\|_{L_{xy}^\infty} \|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}\end{aligned}$$

$$+\sum_{i=[m/2]+1}^{m}\binom{m}{i}\left\|\partial_x^{m-i}\tilde{v}\right\|_{L^\infty_{xy}}\left\|\theta_\alpha\partial_y^2u^s\right\|_{L^2_y}\left\|\partial_x^i\psi\right\|_{L^2_xL^\infty_y}\left\|\theta_\alpha\partial_x^mu\right\|_{L^2(\mathbb{R}^2_+)}. \tag{3.23}$$

For $0\le i\le[m/2]$, by using $(2.9)_1$ and Lemma 2.2 (Agmon inequality in x), leads to

$$\left\|\partial_x^{m-i}\tilde{v}\right\|_{L^2_xL^\infty_y}\le\langle t\rangle^{\frac14}\left(\left\|\theta_\alpha\partial_x^{m-i+1}u\right\|_{L^2(\mathbb{R}^2_+)}+\left\|\partial_x^{m-i+1}b\right\|_{L^2(\mathbb{R}^2_+)}\right) \tag{3.24}$$

and

$$\left\|\theta_\alpha\partial_y^2u^s\right\|_{L^2_y}\le C\langle t\rangle^{-\frac34}.$$

By using $(2.9)_2$ and Lemma 2.3, we obtain

$$\left\|\partial_x^i\psi\right\|_{L^\infty_{xy}}\le C\langle t\rangle^{\frac14}\left\|\theta_\alpha\partial_x^ib\right\|^{\frac12}_{L^2(\mathbb{R}^2_+)}\left\|\theta_\alpha\partial_x^{i+1}b\right\|^{\frac12}_{L^2(\mathbb{R}^2_+)}. \tag{3.25}$$

For $[m/2]\le i\le m$, by using $(2.9)_2$ and Lemma 2.3, we find

$$\begin{aligned}\left\|\partial_x^{m-i}\tilde{v}\right\|_{L^\infty_xL^\infty_y}\le{}& C\langle t\rangle^{\frac14}\Big(\left\|\theta_\alpha\partial_x^{m-i+1}u\right\|^{\frac12}_{L^2(\mathbb{R}^2_+)}\left\|\theta_\alpha\partial_x^{m-i+2}u\right\|^{\frac12}_{L^2(\mathbb{R}^2_+)}\\&+\left\|\partial_x^{m-i+1}b\right\|^{\frac12}_{L^2(\mathbb{R}^2_+)}\left\|\partial_x^{m-i+2}b\right\|^{\frac12}_{L^2(\mathbb{R}^2_+)}\Big)\end{aligned} \tag{3.26}$$

and

$$\left\|\partial_x^i\psi\right\|_{L^2L^\infty_y}\le C\langle t\rangle^{\frac14}\left\|\theta_\alpha\partial_x^ib\right\|_{L^2(\mathbb{R}^2_+)}. \tag{3.27}$$

Accordingly,

$$\begin{aligned}&\sum_{m\ge0}\frac{|R_7|\tau^mM_m}{\|\theta_\alpha\partial_x^mu\|_{L^2(\mathbb{R}^2_+)}}\\&\quad\le\frac{C}{\tau(t)^{\frac12}}\Bigg(\sum_{m\ge0}\sum_{i=0}^{[m/2]}\binom{m}{i}\langle t\rangle^{-\frac14}(Y_{m-i+1}+\bar{Y}_{m-i+1})\bar{X}_i^{\frac12}\bar{X}_{i+1}^{\frac12}\\&\qquad+\sum_{m\ge0}\sum_{i=[m/2]+1}^{m}\binom{m}{i}\langle t\rangle^{-\frac14}\left(Y^{\frac12}_{m-i+1}Y^{\frac12}_{m-i+2}+\bar{Y}^{\frac12}_{m-i+1}\bar{Y}^{\frac12}_{m-i+2}\right)\bar{X}_i\Bigg).\end{aligned} \tag{3.28}$$

Collecting the previous estimates (3.8), (3.13), (3.14), (3.19), (3.22), and (3.28) and summing up $m\ge0$ gives

$$\begin{aligned}&\frac{d}{dt}\|u\|_{X_{\tau,\alpha}}+\sum_{m\ge0}\frac{\alpha(1-2\alpha)}{4\langle t\rangle}\tau^mM_m\frac{\|\theta_\alpha z\partial_x^mu\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha\partial_x^mu\|_{L^2(\mathbb{R}^2_+)}}+\sum_{m\ge0}\tau^mM_m\frac{\|\theta_\alpha\partial_x^m\partial_yu\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha\partial_x^mu\|_{L^2(\mathbb{R}^2_+)}}\\&\quad-\frac{\alpha}{2\langle t\rangle}\|u\|_{X_{\tau,\alpha}}-\frac{C}{\langle t\rangle}\|b\|_{X_{\tau,\alpha}}\\&\quad\le\dot{\tau}(t)\|u\|_{Y_{\tau,\alpha}}+\frac{C}{\tau(t)^{\frac12}}\left(\langle t\rangle^{-1/4}\left(\|u\|_{X_{\tau,\alpha}}+\|b\|_{X_{\tau,\alpha}}\right)+\langle t\rangle^{1/4}\left(\|u\|_{D_{\tau,\alpha}}+\|b\|_{D_{\tau,\alpha}}\right)\right)\\&\qquad\times\left(\|u\|_{Y_{\tau,\alpha}}+\|b\|_{Y_{\tau,\alpha}}\right)+\|w\|_{X_{\tau,\alpha}}-\frac14\sum_{m\ge0}\frac{\|\theta_\alpha\partial_x^m\partial_yw\|^2_{L^2(\mathbb{R}^2_+)}\tau^mM_m}{\|\theta_\alpha\partial_x^mw\|_{L^2(\mathbb{R}^2_+)}},\end{aligned} \tag{3.29}$$

where we have used the fact that for any positive sequences $\{a_j\}_{j\geq 0}$ and $\{b_j\}_{j\geq 0}$,

$$\sum_{m\geq 0}\sum_{j\geq 0}^{m} a_j b_{m-j} \leq \sum_{j\geq 0} a_j \sum_{j\geq 0} b_j.$$

Choosing suitable $\alpha \leq \frac{1}{2}$ in (3.29) gives

$$\begin{aligned}
&\frac{d}{dt}\|u\|_{X_{\tau,\alpha}} + \sum_{m\geq 0}\tau^m M_m \frac{\|\theta_\alpha \partial_x^m \partial_y u\|_{L^2(\mathbb{R}_+^2)}^2}{\|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}} - \frac{\alpha}{2\langle t\rangle}\|u\|_{X_{\tau,\alpha}} - \frac{C}{\langle t\rangle}\|b\|_{X_{\tau,\alpha}} \\
&\quad \leq \dot{\tau}(t)\|u\|_{Y_{\tau,\alpha}} + \frac{C}{\tau(t)^{\frac{1}{2}}}\Big(\langle t\rangle^{-1/4}\big(\|u\|_{X_{\tau,\alpha}} + \|b\|_{X_{\tau,\alpha}}\big) + \langle t\rangle^{1/4}\big(\|u\|_{D_{\tau,\alpha}} + \|b\|_{D_{\tau,\alpha}}\big)\Big) \\
&\qquad \times \big(\|u\|_{Y_{\tau,\alpha}} + \|b\|_{Y_{\tau,\alpha}}\big) + \|w\|_{X_{\tau,\alpha}} - \frac{1}{4}\sum_{m\geq 0}\frac{\|\theta_\alpha \partial_x^m \partial_y w\|_{L^2(\mathbb{R}_+^2)}^2 \tau^m M_m}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}_+^2)}}. \qquad (3.30)
\end{aligned}$$

The proof is then complete. □

3.2 A priori estimate on the magnetic field

In this subsection, we will derive the estimate of solution to problem (2.17)–(2.19) on the magneto field b.

Lemma 3.2 *It holds that for any $t \in [0, T]$*

$$\begin{aligned}
&\frac{d}{dt}\|b\|_{X_{\tau,\alpha}} + \sum_{m\geq 0}\tau^m M_m \frac{\|\theta_\alpha \partial_x^m \partial_y b\|_{L^2(\mathbb{R}_+^2)}^2}{\|\theta_\alpha \partial_x^m b\|_{L^2(\mathbb{R}_+^2)}} - \frac{\alpha}{2\langle t\rangle}\|b\|_{X_{\tau,\alpha}} \\
&\quad \leq \dot{\tau}(t)\|b\|_{Y_{\tau,\alpha}} + \frac{C}{\tau(t)^{\frac{1}{2}}}\Big(\langle t\rangle^{-1/4}\big(\|u\|_{X_{\tau,\alpha}} + \|b\|_{X_{\tau,\alpha}}\big) + \langle t\rangle^{1/4}\big(\|u\|_{D_{\tau,\alpha}} + \|b\|_{D_{\tau,\alpha}}\big)\Big) \\
&\qquad \times \big(\|u\|_{Y_{\tau,\alpha}} + \|b\|_{Y_{\tau,\alpha}}\big) + \|w\|_{X_{\tau,\alpha}}.
\end{aligned}$$

Proof For $m \geq 0$, applying the operator ∂_x^m on $(2.17)_2$ and multiplying the resulting equation by $\theta_\alpha^2 \partial_x^m b$, we derive that

$$\begin{aligned}
&\int_{\mathbb{R}_+^2} \partial_x^m\big(\partial_t b + \big(u^s + \tilde{u}\big)\partial_x b + \tilde{v}\partial_y b - \partial_y^2 b - (b+1)\partial_x u \\
&\quad - g\partial_y u - g\partial_y^2 u^s \psi\big)\theta_\alpha^2 \partial_x^m b \, dx\, dy = 0. \qquad (3.31)
\end{aligned}$$

Similar to the estimates of (3.1), we now deal with each term in (3.31) as follows. For the first term, we have

$$\frac{\int_{\mathbb{R}_+^2} \partial_x^m \partial_t b \theta_\alpha^2 \partial_x^m b \, dx\, dy}{\|\theta_\alpha \partial_x^m b\|_{L^2(\mathbb{R}_+^2)}} = \frac{1}{2}\frac{d}{dt}\big\|\theta_\alpha \partial_x^m b\big\|_{L^2(\mathbb{R}_+^2)} + \frac{\alpha}{4\langle t\rangle}\frac{\|\theta_\alpha z \partial_x^m b\|_{L^2(\mathbb{R}_+^2)}^2}{\|\theta_\alpha \partial_x^m b\|_{L^2(\mathbb{R}_+^2)}}. \qquad (3.32)$$

For the third term,

$$
\begin{aligned}
&\frac{-\int_{\mathbb{R}_+^2}\partial_y^2\partial_x^m b\theta_\alpha^2\partial_x^m b\,dx\,dy}{\|\theta_\alpha\partial_x^m b\|_{L^2(\mathbb{R}_+^2)}}\\
&\quad=\frac{\|\theta_\alpha\partial_x^m\partial_y b\|_{L^2(\mathbb{R}_+^2)}^2}{\|\theta_\alpha\partial_x^m b\|_{L^2(\mathbb{R}_+^2)}}-\frac{\alpha}{2\langle t\rangle}\big\|\theta_\alpha\partial_x^m b\big\|_{L^2(\mathbb{R}_+^2)}-\frac{\alpha^2}{2\langle t\rangle}\frac{\|\theta_\alpha z\partial_x^m b\|_{L^2(\mathbb{R}_+^2)}^2}{\|\theta_\alpha\partial_x^m b\|_{L^2(\mathbb{R}_+^2)}},
\end{aligned}
\tag{3.33}
$$

where we have used the fact that $\partial_y^2(\theta_\alpha^2)=\frac{\alpha}{\langle t\rangle}\theta_\alpha^2+\frac{\alpha^2}{\langle t\rangle}z^2\theta_\alpha^2$ and boundary condition $\partial_x^m b|_{y=0}=0$.

Indeed, similar to Sect. 3.1, the nonlinear terms of (3.31) can be estimated as below

$$
\begin{aligned}
&\sum_{m\ge0}\frac{|R_8|\tau^m M_m}{\|\theta_\alpha\partial_x^m b\|_{L^2(\mathbb{R}_+^2)}}\\
&\quad\le\frac{C}{\tau(t)^{\frac12}}\Bigg(\sum_{m\ge0}\sum_{i=0}^{[m/2]}\binom{m}{i}\big(X_{m-i}^{\frac12}D_{m-i}^{\frac12}+\langle t\rangle^{-1/4}\bar X_{m-i}\big)\bar Y_{i+1}^{\frac12}\bar Y_{i+2}^{\frac12}\\
&\qquad+\sum_{m\ge0}\sum_{i=[m/2]+1}^{m}\binom{m}{i}\big(X_{m-i}^{\frac14}X_{m-i+1}^{\frac14}D_{m-i}^{\frac14}D_{m-i+1}^{\frac14}+\langle t\rangle^{-1/4}\bar X_{m-i}^{\frac12}\bar X_{m-i+1}^{\frac12}\big)\bar Y_{i+1}\Bigg),
\end{aligned}
\tag{3.34}
$$

$$
\begin{aligned}
&\sum_{m\ge0}\frac{|R_9|\tau^m M_m}{\|\theta_\alpha\partial_x^m b\|_{L^2(\mathbb{R}_+^2)}}\\
&\quad\le\frac{C}{\tau(t)^{\frac12}}\Bigg(\sum_{m\ge0}\sum_{i=0}^{[m/2]}\binom{m}{i}\big(\langle t\rangle^{\frac14}Y_{m-i+1}+\langle t\rangle^{\frac14}\bar Y_{m-i+1}\big)\bar D_i^{\frac12}\bar D_{i+1}^{\frac12}\\
&\qquad+\sum_{m\ge0}\sum_{i=[m/2]+1}^{m}\binom{m}{i}\big(\langle t\rangle^{\frac14}Y_{m-i+1}^{\frac12}Y_{m-i+2}^{\frac12}+\langle t\rangle^{\frac14}\bar Y_{m-i+1}^{\frac12}\bar Y_{m-i+2}^{\frac12}\big)\bar D_i\Bigg),
\end{aligned}
\tag{3.35}
$$

$$
\begin{aligned}
&\sum_{m\ge0}\frac{|R_{10}|\tau^m M_m}{\|\theta_\alpha\partial_x^m b\|_{L^2(\mathbb{R}_+^2)}}\\
&\quad\le\frac{C}{\tau(t)^{\frac12}}\Bigg(\sum_{m\ge0}\sum_{i=0}^{[m/2]}\binom{m}{i}\bar X_{m-i}^{\frac12}\bar D_{m-i}^{\frac12}Y_{i+1}^{\frac12}Y_{i+2}^{\frac12}\\
&\qquad+\sum_{m\ge0}\sum_{i=[m/2]+1}^{m}\binom{m}{i}\bar X_{m-i}^{\frac14}\bar X_{m-i+1}^{\frac14}\bar D_{m-i}^{\frac14}\bar D_{m-i+1}^{\frac14}Y_{i+1}\Bigg),
\end{aligned}
\tag{3.36}
$$

$$
\begin{aligned}
&\sum_{m\ge0}\frac{|R_{11}|\tau^m M_m}{\|\theta_\alpha\partial_x^m b\|_{L^2(\mathbb{R}_+^2)}}\\
&\quad\le\frac{C}{\tau(t)^{\frac12}}\Bigg(\sum_{m\ge0}\sum_{i=0}^{[m/2]}\binom{m}{i}\langle t\rangle^{\frac14}\bar Y_{m-i+1}D_i^{\frac12}D_{i+1}^{\frac12}\\
&\qquad+\sum_{m\ge0}\sum_{i=[m/2]+1}^{m}\binom{m}{i}\langle t\rangle^{\frac14}\bar Y_{m-i+1}^{\frac12}\bar Y_{m-i+2}^{\frac12}D_i\Bigg)
\end{aligned}
\tag{3.37}
$$

and

$$\sum_{m\geq 0}\frac{|R_{12}|\tau^m M_m}{\|\theta_\alpha\partial_x^m b\|_{L^2(\mathbb{R}^2_+)}}$$
$$\leq \frac{C}{\tau(t)^{\frac{1}{2}}}\left(\sum_{m\geq 0}\sum_{i=0}^{[m/2]}\binom{m}{i}\langle t\rangle^{-\frac{1}{4}}\bar{Y}_{m-i+1}\bar{X}_i^{\frac{1}{2}}\bar{X}_{i+1}^{\frac{1}{2}}\right.$$
$$\left.+\sum_{m\geq 0}\sum_{i=[m/2]+1}^{m}\binom{m}{i}\langle t\rangle^{-\frac{1}{4}}\bar{Y}_{m-i+1}^{\frac{1}{2}}\bar{Y}_{m-i+2}^{\frac{1}{2}}\bar{X}_i\right). \tag{3.38}$$

Combining the above estimates (3.31)–(3.38) and summing over $m\geq 0$ yields

$$\frac{d}{dt}\|b\|_{X_{\tau,\alpha}}+\sum_{m\geq 0}\frac{\alpha(1-2\alpha)}{4\langle t\rangle}\tau^m M_m\frac{\|\theta_\alpha z\partial_x^m b\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha\partial_x^m b\|_{L^2(\mathbb{R}^2_+)}}$$
$$+\sum_{m\geq 0}\tau^m M_m\frac{\|\theta_\alpha\partial_x^m\partial_y b\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha\partial_x^m b\|_{L^2(\mathbb{R}^2_+)}}-\frac{\alpha}{2\langle t\rangle}\|b\|_{X_{\tau,\alpha}}$$
$$\leq \dot{\tau}(t)\|b\|_{Y_{\tau,\alpha}}+\frac{C}{\tau(t)^{\frac{1}{2}}}\left(\langle t\rangle^{-1/4}\left(\|u\|_{X_{\tau,\alpha}}+\|b\|_{X_{\tau,\alpha}}\right)+\langle t\rangle^{1/4}\left(\|u\|_{D_{\tau,\alpha}}+\|b\|_{D_{\tau,\alpha}}\right)\right)$$
$$\times\left(\|u\|_{Y_{\tau,\alpha}}+\|b\|_{Y_{\tau,\alpha}}\right)+\|w\|_{X_{\tau,\alpha}}. \tag{3.39}$$

Analogously, by choosing $\alpha\leq\frac{1}{2}$, we conclude

$$\frac{d}{dt}\|b\|_{X_{\tau,\alpha}}+\sum_{m\geq 0}\tau^m M_m\frac{\|\theta_\alpha\partial_x^m\partial_y b\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha\partial_x^m b\|_{L^2(\mathbb{R}^2_+)}}-\frac{\alpha}{2\langle t\rangle}\|b\|_{X_{\tau,\alpha}}$$
$$\leq \dot{\tau}(t)\|b\|_{Y_{\tau,\alpha}}+\frac{C}{\tau(t)^{\frac{1}{2}}}\left(\langle t\rangle^{-1/4}\left(\|u\|_{X_{\tau,\alpha}}+\|b\|_{X_{\tau,\alpha}}\right)+\langle t\rangle^{1/4}\left(\|u\|_{D_{\tau,\alpha}}+\|b\|_{D_{\tau,\alpha}}\right)\right)$$
$$\times\left(\|u\|_{Y_{\tau,\alpha}}+\|b\|_{Y_{\tau,\alpha}}\right)+\|w\|_{X_{\tau,\alpha}}. \tag{3.40}$$

The proof is hence complete. □

3.3 A priori estimate on the microrotational velocity

In this subsection, we will establish the estimate of solution to problem (2.17)–(2.19) on the microrotational velocity w.

Lemma 3.3 *It holds that for any $t\in[0,T]$*

$$\frac{d}{dt}\|w\|_{X_{\tau,\alpha}}+\sum_{m\geq 0}\tau^m M_m\frac{\|\theta_\alpha\partial_x^m\partial_y w\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha\partial_x^m w\|_{L^2(\mathbb{R}^2_+)}}$$
$$\leq \dot{\tau}(t)\|w\|_{Y_{\tau,\alpha}}+\frac{C}{\tau(t)^{\frac{1}{2}}}\left(\langle t\rangle^{-1/4}\left(\|u\|_{X_{\tau,\alpha}}+\|b\|_{X_{\tau,\alpha}}\right)+\langle t\rangle^{1/4}\left(\|u\|_{D_{\tau,\alpha}}+\|w\|_{D_{\tau,\alpha}}\right)\right)$$
$$\times\left(\|u\|_{Y_{\tau,\alpha}}+\|b\|_{Y_{\tau,\alpha}}+\|w\|_{Y_{\tau,\alpha}}\right)+\|u\|_{X_{\tau,\alpha}}+\langle t\rangle^{-1/2}\left(\|u\|_{X_{\tau,\alpha}}+\|w\|_{X_{\tau,\alpha}}\right)$$
$$-\frac{1}{4}\sum_{m\geq 0}\frac{\|\theta_\alpha\partial_x^m\partial_y u\|^2_{L^2(\mathbb{R}^2_+)}\tau^m M_m}{\|\theta_\alpha\partial_x^m u\|_{L^2(\mathbb{R}^2_+)}}.$$

Proof For $m \geq 0$, applying the operator ∂_x^m on $(2.17)_3$ and multiplying the resulting equation by $\theta_\alpha^2 \partial_x^m w$, we derive that

$$\begin{aligned}&\int_{\mathbb{R}_+^2} \partial_x^m \big(\partial_t w + \big(u^s + \tilde{u}\big)\partial_x w + \tilde{v}\partial_y w + 2\partial_y \big(u + u^s\big)\\&\quad + 2\partial_y u^s b + 2\partial_y^2 u^s \psi - \partial_y^2 w\big)\theta_\alpha^2 \partial_x^m w \, dx\, dy = 0.\end{aligned} \tag{3.41}$$

Similar to the estimates of (3.1), we now deal with each term in (3.41) as follows. For the first term, we have

$$\frac{\int_{\mathbb{R}_+^2} \partial_x^m \partial_t w \theta_\alpha^2 \partial_x^m w \, dx\, dy}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}^+)}} = \frac{1}{2}\frac{d}{dt}\big\|\theta_\alpha \partial_x^m w\big\|_{L^2(\mathbb{R}^+)} + \frac{\alpha}{4\langle t\rangle}\frac{\|\theta_\alpha z \partial_x^m w\|_{L^2(\mathbb{R}^+)}^2}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}^+)}}. \tag{3.42}$$

For the last term,

$$\begin{aligned}&\frac{-\int_{\mathbb{R}_+^2} \partial_y^2 \partial_x^m w \theta_\alpha^2 \partial_x^m w \, dx\, dy}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}^+)}}\\&\quad = \frac{\|\theta_\alpha \partial_x^m \partial_y w\|_{L^2(\mathbb{R}^+)}^2}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}^+)}} - \frac{\alpha}{2\langle t\rangle}\big\|\theta_\alpha \partial_x^m w\big\|_{L^2(\mathbb{R}^+)} - \frac{\alpha^2}{2\langle t\rangle}\frac{\|\theta_\alpha z \partial_x^m w\|_{L^2(\mathbb{R}^+)}^2}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}^+)}},\end{aligned} \tag{3.43}$$

where we have used the fact that $\partial_y^2(\theta_\alpha^2) = \frac{\alpha}{\langle t\rangle}\theta_\alpha^2 + \frac{\alpha^2}{\langle t\rangle}z^2\theta_\alpha^2$ and boundary condition $\partial_x^m w|_{y=0} = 0$.

Similar to Sects. 3.1 and 3.2 we have

$$\begin{aligned}&\sum_{m\geq 0}\frac{|R_{12}|\tau^m M_m}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}_+^2)}}\\&\quad \leq \frac{C}{\tau(t)^{\frac{1}{2}}}\Bigg(\sum_{m\geq 0}\sum_{i=0}^{[m/2]}\binom{m}{i}\big(X_{m-i}^{\frac{1}{2}}D_{m-i}^{\frac{1}{2}} + \langle t\rangle^{-1/4}\bar{X}_{m-i}\big)\tilde{Y}_{i+1}^{\frac{1}{2}}\tilde{Y}_{i+2}^{\frac{1}{2}}\\&\qquad + \sum_{m\geq 0}\sum_{i=[m/2]+1}^{m}\binom{m}{i}\big(X_{m-i}^{\frac{1}{4}}X_{m-i+1}^{\frac{1}{4}}D_{m-i}^{\frac{1}{4}}D_{m-i+1}^{\frac{1}{4}} + \langle t\rangle^{-1/4}\bar{X}_{m-i}^{\frac{1}{2}}\bar{X}_{m-i+1}^{\frac{1}{2}}\big)\tilde{Y}_{i+1}\Bigg),\end{aligned} \tag{3.44}$$

$$\begin{aligned}&\sum_{m\geq 0}\frac{|R_{13}|\tau^m M_m}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}_+^2)}}\\&\quad \leq \frac{C}{\tau(t)^{\frac{1}{2}}}\Bigg(\sum_{m\geq 0}\sum_{i=0}^{[m/2]}\binom{m}{i}\big(\langle t\rangle^{\frac{1}{4}}Y_{m-i+1} + \langle t\rangle^{\frac{1}{4}}\bar{Y}_{m-i+1}\big)\tilde{D}_i^{\frac{1}{2}}\tilde{D}_{i+1}^{\frac{1}{2}}\\&\qquad + \sum_{m\geq 0}\sum_{i=[m/2]+1}^{m}\binom{m}{i}\big(\langle t\rangle^{\frac{1}{4}}Y_{m-i+1}^{\frac{1}{2}}Y_{m-i+2}^{\frac{1}{2}} + \langle t\rangle^{\frac{1}{4}}\bar{Y}_{m-i+1}^{\frac{1}{2}}\bar{Y}_{m-i+2}^{\frac{1}{2}}\big)\tilde{D}_i\Bigg),\end{aligned} \tag{3.45}$$

also,

$$\begin{aligned}&\sum_{m\geq 0}\frac{|R_{14}|\tau^m M_m}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}_+^2)}}\\&\quad \leq \frac{1}{4}\sum_{m\geq 0}\frac{\|\theta_\alpha \partial_x^m \partial_y u\|_{L^2(\mathbb{R}_+^2)}^2 \tau^m M_m}{\|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}} + \sum_{m\geq 0}\big\|\theta_\alpha \partial_x^m u\big\|_{L^2(\mathbb{R}_+^2)}\tau^m M_m\end{aligned} \tag{3.46}$$

and

$$\left|\int_{\mathbb{R}^2_+} \partial_y u^s \partial_x^m w \theta_\alpha^2 \partial_x^m w \,dx\,dy\right| \le C\langle t\rangle^{-1/2} \left\|\theta_\alpha \partial_x^m w\right\|^2_{L^2(\mathbb{R}^2_+)}, \tag{3.47}$$

imply

$$\sum_{m\ge 0} \frac{|R_{15}|\tau^m M_m}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}^2_+)}} \le C\langle t\rangle^{-1/2} \sum_{m\ge 0} \tilde{X}_m, \tag{3.48}$$

where we have used the fact that $\|\partial_y u^s\|_{L^\infty_y} \le C\langle t\rangle^{-1/2}$.

Finally, we deal with the remainder term in (3.41) as follows

$$\begin{aligned} |R_{16}| &\le 2\left\|\partial_y^2 u^s\right\|_{L^\infty_y} \left\|\theta_\alpha \partial_x^m \psi\right\|_{L^2(\mathbb{R}^2_+)} \left\|\theta_\alpha \partial_x^m w\right\|_{L^2(\mathbb{R}^2_+)} \\ &\le 2\langle t\rangle^{-1} \left\|\theta_\alpha \partial_x^m \psi\right\|_{L^2(\mathbb{R}^2_+)} \left\|\theta_\alpha \partial_x^m w\right\|_{L^2(\mathbb{R}^2_+)} \\ &\le 2\langle t\rangle^{-1/2} \left\|\theta_\alpha \partial_x^m b\right\|_{L^2(\mathbb{R}^2_+)} \left\|\theta_\alpha \partial_x^m w\right\|_{L^2(\mathbb{R}^2_+)}, \end{aligned}$$

implies

$$\sum_{m\ge 0} \frac{|R_{16}|\tau^m M_m}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}^2_+)}} \le C\langle t\rangle^{-1/2} \sum_{m\ge 0} \bar{X}_m, \tag{3.49}$$

where we have used the fact that $\|\partial_y^2 u^s\|_{L^\infty_y} \le C\langle t\rangle^{-1}$.

Combining the estimates (3.41)–(3.49) and summing over $m \ge 0$ yields

$$\begin{aligned} &\frac{d}{dt}\|w\|_{X_{\tau,\alpha}} + \sum_{m\ge 0} \frac{\alpha(1-2\alpha)}{4\langle t\rangle} \tau^m M_m \frac{\|\theta_\alpha z \partial_x^m w\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}^2_+)}} + \sum_{m\ge 0} \tau^m M_m \frac{\|\theta_\alpha \partial_x^m \partial_y w\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}^2_+)}} \\ &\le \dot{\tau}(t)\|w\|_{Y_{\tau,\alpha}} + \frac{C}{\tau(t)^{\frac{1}{2}}}\Big(\langle t\rangle^{-1/4}\big(\|u\|_{X_{\tau,\alpha}} + \|b\|_{X_{\tau,\alpha}}\big) + \langle t\rangle^{1/4}\big(\|u\|_{D_{\tau,\alpha}} + \|w\|_{D_{\tau,\alpha}}\big)\Big) \\ &\quad \times \big(\|u\|_{Y_{\tau,\alpha}} + \|b\|_{Y_{\tau,\alpha}} + \|w\|_{Y_{\tau,\alpha}}\big) + \|u\|_{X_{\tau,\alpha}} + \langle t\rangle^{-1/2}\big(\|u\|_{X_{\tau,\alpha}} + \|w\|_{X_{\tau,\alpha}}\big) \\ &\quad - \frac{1}{4}\sum_{m\ge 0} \frac{\|\theta_\alpha \partial_x^m \partial_y u\|^2_{L^2(\mathbb{R}^2_+)} \tau^m M_m}{\|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}^2_+)}}. \end{aligned}$$

Analogously, by choosing $\alpha \le \frac{1}{2}$, we derive

$$\begin{aligned} &\frac{d}{dt}\|w\|_{X_{\tau,\alpha}} + \sum_{m\ge 0} \tau^m M_m \frac{\|\theta_\alpha \partial_x^m \partial_y w\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha \partial_x^m w\|_{L^2(\mathbb{R}^2_+)}} \\ &\le \dot{\tau}(t)\|w\|_{Y_{\tau,\alpha}} + \frac{C}{\tau(t)^{\frac{1}{2}}}\Big(\langle t\rangle^{-1/4}\big(\|u\|_{X_{\tau,\alpha}} + \|b\|_{X_{\tau,\alpha}}\big) + \langle t\rangle^{1/4}\big(\|u\|_{D_{\tau,\alpha}} + \|w\|_{D_{\tau,\alpha}}\big)\Big) \\ &\quad \times \big(\|u\|_{Y_{\tau,\alpha}} + \|b\|_{Y_{\tau,\alpha}} + \|w\|_{Y_{\tau,\alpha}}\big) + \|u\|_{X_{\tau,\alpha}} + \langle t\rangle^{-1/2}\big(\|u\|_{X_{\tau,\alpha}} + \|w\|_{X_{\tau,\alpha}}\big) \\ &\quad - \frac{1}{4}\sum_{m\ge 0} \frac{\|\theta_\alpha \partial_x^m \partial_y u\|^2_{L^2(\mathbb{R}^2_+)} \tau^m M_m}{\|\theta_\alpha \partial_x^m u\|_{L^2(\mathbb{R}^2_+)}}. \end{aligned} \tag{3.50}$$

The proof is now complete. □

Up to now, we have completed all the estimates of solution (u,b,w) to problem (2.17)–(2.19).

Combining the estimates (3.30), (3.40), and (3.50), we conclude

$$\begin{aligned}&\frac{d}{dt}\big\|(u,b,w)\big\|_{X_{\tau,\alpha}}+\frac{3}{4}\sum_{m\geq 0}\tau^m M_m\frac{\|\theta_\alpha\partial_x^m\partial_y u\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha\partial_x^m u\|_{L^2(\mathbb{R}^2_+)}}+\frac{3}{4}\sum_{m\geq 0}\tau^m M_m\frac{\|\theta_\alpha\partial_x^m\partial_y b\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha\partial_x^m b\|_{L^2(\mathbb{R}^2_+)}}\\&\quad+\frac{3}{4}\sum_{m\geq 0}\tau^m M_m\frac{\|\theta_\alpha\partial_x^m\partial_y w\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha\partial_x^m w\|_{L^2(\mathbb{R}^2_+)}}-\frac{\alpha}{2\langle t\rangle}\big\|(u,b,w)\big\|_{X_{\tau,\alpha}}\\&\leq\dot{\tau}(t)\big\|(u,b,w)\big\|_{Y_{\tau,\alpha}}+\frac{C}{\tau(t)^{\frac{1}{2}}}\big(\langle t\rangle^{-1/4}\big\|(u,b,w)\big\|_{X_{\tau,\alpha}}+\langle t\rangle^{1/4}\big\|(u,b,w)\big\|_{D_{\tau,\alpha}}\big)\\&\quad\times\big\|(u,b,w)\big\|_{Y_{\tau,\alpha}}+C\big\|(u,b,w)\big\|_{X_{\tau,\alpha}}.\end{aligned}\tag{3.51}$$

Using Lemma 2.3, we discover

$$\sum_{m\geq 0}\frac{\|\theta_\alpha\partial_y\partial_x^m u\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha\partial_x^m u\|_{L^2(\mathbb{R}^2_+)}}\tau^m M_m\geq\frac{\alpha^{\frac{1}{2}}\beta}{2\langle t\rangle^{\frac{1}{2}}}\|u\|_{D_{\tau,\alpha}}+\frac{\alpha(1-\beta)}{\langle t\rangle}\|u\|_{X_{\tau,\alpha}},\tag{3.52}$$

$$\sum_{m\geq 0}\frac{\|\theta_\alpha\partial_y\partial_x^m b\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha\partial_x^m b\|_{L^2(\mathbb{R}^2_+)}}\tau^m M_m\geq\frac{\alpha^{\frac{1}{2}}\beta}{2\langle t\rangle^{\frac{1}{2}}}\|b\|_{D_{\tau,\alpha}}+\frac{\alpha(1-\beta)}{\langle t\rangle}\|b\|_{X_{\tau,\alpha}}\tag{3.53}$$

and

$$\sum_{m\geq 0}\frac{\|\theta_\alpha\partial_y\partial_x^m w\|^2_{L^2(\mathbb{R}^2_+)}}{\|\theta_\alpha\partial_x^m w\|_{L^2(\mathbb{R}^2_+)}}\tau^m M_m\geq\frac{\alpha^{\frac{1}{2}}\beta}{2\langle t\rangle^{\frac{1}{2}}}\|w\|_{D_{\tau,\alpha}}+\frac{\alpha(1-\beta)}{\langle t\rangle}\|w\|_{X_{\tau,\alpha}},\tag{3.54}$$

for $\beta\in(0,1/3)$.

Substituting (3.52)–(3.54) into (3.51), leads to

$$\begin{aligned}&\frac{d}{dt}\big\|(u,b,w)\big\|_{X_{\tau,\alpha}}+\frac{\alpha(1-3\beta)}{4}\frac{1}{\langle t\rangle}\big\|(u,b,w)\big\|_{X_{\tau,\alpha}}+\frac{3\alpha^{\frac{1}{2}}\beta}{8\langle t\rangle^{\frac{1}{2}}}\big\|(u,b,w)\big\|_{D_{\tau,\alpha}}\\&\leq\left(\dot{\tau}(t)+\frac{C}{\tau(t)^{\frac{1}{2}}}\big(\langle t\rangle^{-1/4}\big\|(u,b,w)\big\|_{X_{\tau,\alpha}}+\langle t\rangle^{1/4}\big\|(u,b,w)\big\|_{D_{\tau,\alpha}}\big)\right)\big\|(u,b,w)\big\|_{Y_{\tau,\alpha}}\\&\quad+C\big\|(u,b,w)\big\|_{X_{\tau,\alpha}}.\end{aligned}\tag{3.55}$$

We will choose a suitable function $\tau(t)$ such that the following ordinary differential equation holds

$$\frac{d}{dt}\big(\tau(t)\big)^{\frac{3}{2}}+\frac{3C}{2}\big(\langle t\rangle^{-1/4}\big\|(u,b,w)\big\|_{X_{\tau,\alpha}}+\langle t\rangle^{1/4}\big\|(u,b,w)\big\|_{D_{\tau,\alpha}}\big)=0.\tag{3.56}$$

Hence, from (3.55) this implies

$$\begin{aligned}&\frac{d}{dt}\big\|(u,b,w)\big\|_{X_{\tau,\alpha}}+\frac{\alpha(1-3\beta)}{4}\frac{1}{\langle t\rangle}\big\|(u,b,w)\big\|_{X_{\tau,\alpha}}+\frac{3\alpha^{\frac{1}{2}}\beta}{8\langle t\rangle^{\frac{1}{2}}}\big\|(u,b,w)\big\|_{D_{\tau,\alpha}}\\&\leq C\big\|(u,b,w)\big\|_{X_{\tau,\alpha}}.\end{aligned}\tag{3.57}$$

Using the classical bootstrap argument [25], we first assume that there exists a $T_* > 0$ such that

$$\begin{aligned}
&\|(u,b,w)\|_{X_{\tau,\alpha}} + \int_0^t \left(\frac{\alpha(1-3\beta)}{4\langle s\rangle}\|(u,b,w)\|_{X_{\tau,\alpha}} + \frac{3\alpha^{\frac{1}{2}}\beta}{8\langle s\rangle^{\frac{1}{2}}}\|(u,b,w)\|_{D_{\tau,\alpha}}\right) ds \\
&\quad \leq 2\|(u_0,b_0,w_0)\|_{X_{\tau_0,\alpha}},
\end{aligned} \tag{3.58}$$

for $t \in [0,T_*]$.

From (3.56), this implies

$$\begin{aligned}
&\tau(t)^{\frac{3}{2}} - \tau(0)^{\frac{3}{2}} \\
&\quad = -\frac{3C}{2}\int_0^t \left(\langle s\rangle^{-1/4}\|(u,b,w)\|_{X_{\tau,\alpha}} + \langle s\rangle^{1/4}\|(u,b,w)\|_{D_{\tau,\alpha}}\right) ds \\
&\quad = -\frac{3C}{2}\int_0^t \left(\frac{4}{\alpha(1-3\beta)}\langle t\rangle^{3/4}\frac{\alpha(1-3\beta)}{4\langle s\rangle}\|(u,b,w)\|_{X_{\tau,\alpha}}\right. \\
&\qquad \left. + \frac{8}{3\alpha^{\frac{1}{2}}\beta}\langle s\rangle^{3/4}\frac{3\alpha^{\frac{1}{2}}\beta}{8\langle t\rangle^{\frac{1}{2}}}\|(u,b,w)\|_{D_{\tau,\alpha}}\right) ds \\
&\quad \geq -3C\|(u_0,b_0,w_0)\|_{X_{\tau_0,\alpha}}\left(\frac{4}{\alpha(1-3\beta)} + \frac{8}{3\alpha^{\frac{1}{2}}\beta}\right)\langle t\rangle^{3/4} \\
&\quad = -C_2\langle t\rangle^{4/3},
\end{aligned} \tag{3.59}$$

where $C_2 = 12C\|(u_0,b_0,w_0)\|_{X_{\tau_0,\alpha}}(\frac{1}{\alpha(1-3\beta)} + \frac{2}{3\alpha^{\frac{1}{2}}\beta})$ and $T = \min\{(\frac{7}{8C_2}\tau_0^{\frac{3}{2}})^{\frac{3}{4}} - 1, T_1\}$ with T_1 to be determined later.

Therefore, taking positive $T_1 = \frac{1}{4C} \leq T_*$, for $\forall t \in [0,T_1]$, from (3.57) gives

$$\begin{aligned}
&\|(u,b,w)\|_{X_{\tau,\alpha}} + \int_0^t \left(\frac{\alpha(1-3\beta)}{4\langle s\rangle}\|(u,b,w)\|_{X_{\tau,\alpha}} + \frac{3\alpha^{\frac{1}{2}}\beta}{8\langle s\rangle^{\frac{1}{2}}}\|(u,b,w)\|_{D_{\tau,\alpha}}\right) ds \\
&\quad \leq C\int_0^t \|(u,b,w)\|_{X_{\tau,\alpha}}\, ds + \|(u_0,b_0,w_0)\|_{X_{\tau_0,\alpha}} \\
&\quad \leq \frac{3}{2}\|(u_0,b_0,w_0)\|_{X_{\tau_0,\alpha}}.
\end{aligned} \tag{3.60}$$

The proof of Theorem 2.1 is thus finished.

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No applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
This article has only one author.

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