Li and Hong *Boundary Value Problems* (2023) 2023:76
https://doi.org/10.1186/s13661-023-01768-5

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Global regularity of 2D MHD equations with almost Laplacian velocity dissipation

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Abstract
We obtain the global existence and global regularity for the 2D MHD equations with almost Laplacian velocity dissipation, which require the dissipative operators weaker than any power of the fractional Laplacian. The result can be regarded as a further improvement and generalization of previous works.

Mathematics Subject Classification: 35Q35; 76B03; 76D03; 76D09

Keywords: Generalized magnetohydrodynamic equations; Global regularity; Viscous flow; Sobolev imbedding

1 Introduction and main results

This paper focuses on the two-dimensional magnetohydrodynamic (MHD) system

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p + B \cdot \nabla B - \nu £_1 u, \\ \partial_t B + (u \cdot \nabla)B = B \cdot \nabla u - \kappa £_2 B, \\ divu = divB = 0, \end{cases} \tag{1.1}$$

where $u = (u_1(x,t), u_2(x,t))$ denotes the velocity, $p = p(x,t)$ the scalar pressure, and $B = (B_1(x,t), B_2(x,t))$ the magnetic field of the fluid. $u_0(x)$ and $B_0(x)$ are the given initial data satisfying $\nabla \cdot u_0 = 0$ and $\nabla \cdot B_0 = 0$. The operators $£_1$, $£_2$ are general negative-definite multipliers with symbol $h(\xi)$, namely

$$\widehat{£_1 u(\xi)} = |\xi|^2 h(\xi)\widehat{u(\xi)}, \widehat{£_2 B(\xi)} = |\xi|^2 h(\xi)\widehat{B(\xi)}. \tag{1.2}$$

We note the convention that by $\nu = 0$ we mean that there is no velocity dissipation in $(1.1)_1$ and similarly $\kappa = 0$ represents that there is no magnetic diffusion in $(1.1)_2$. It is well known that the 2D MHD equations with both Laplacian dissipation Δu and magnetic diffusion ΔB have the global smooth solution [1]. In the case without velocity dissipation and magnetic diffusion ($\nu = \kappa = 0$), the question of whether a smooth solution of the 2D MHD equations develops a singularity in finite time is open [2, 3]. Therefore, more researches focus on the MHD equations with fractional dissipation and partial dissipation (see e.g., [4–7]) and the references cited therein, the issue of global regularity for 2D MHD ($\nu = 0$,

$\kappa = 1$ or $\nu = 1$, $\kappa = 0$) has attracted much interest from many mathematicians and has motivated a large number of research papers concerning various generalizations and improvements (see [8–12]). Recent important progress has been obtained by Ren et al. [13], who prove the global existence and the decay estimates of a small smooth solution for the 2D MHD equations without magnetic diffusion, and these results confirm the numerical observation that the energy of the MHD equations is dissipated at a rate independent of the ohmic resistivity, the main tools used in [13] are the anisotropic Littlewood–Paley decomposition and the anisotropic Besov spaces. More recently, Yuan and Ye [11, 14] studied separately the global regularity of solutions to the 2D incompressible MHD equations with almost Laplacian magnetic diffusion in the whole space. Furthermore, Lei [15] studied the global regularity for the axially symmetric MHD equations with nontrivial magnetic fields.

Inspired by the previous works, the aim of this paper is to weaken the operator $ as possible as one can, then the global H^1-bound of the solution can be achieved as long as the fractional dissipation power of the velocity field is positive without losing the global regularity of the system (1.1), we obtained the global regularity of solutions requiring the velocity dissipative operators to be weaker than any power of the fractional Laplacian, the result can be regarded as a further improvement and generalization of previous works [3]. More precisely, the main result in this article reads as follows.

Theorem 1.1 *Let $u_0, B_0 \in H^s(R^2) \times H^s(R^2)$ for any $s > 2$ and satisfy $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$, where $\nu > 0$ and $\kappa > 0$, assume that $h(\xi) = h(|\xi|)$ is a radial nondecreasing smooth function satisfying the following conditions*:

(1) *h is a nonnegative function for all $\xi \neq 0$*;

(2) *there exists a constant $\widetilde{C} > 0$ such that*

$$\left|\partial_\xi^k h(\xi)\right| \leq \widetilde{C}|\xi|^{-k}\left|h(\xi)\right|, \quad k \in \{1,2,3\}, \forall \xi \neq 0;$$

(3) *for any given $T \in (0,\infty)$ such that*

$$\int_T^\infty \frac{ds}{sh(s)} = C_T < \infty,$$

where $B_T > 0$ is the unique solution of the following equation

$$x^2 h(x) = \frac{1}{T},$$

then for the MHD equations (1.1) *there exists a unique global solution such that*

$$u, B \in L^\infty\big([0,T]; H^s\big(R^2\big)\big), \qquad u \in L^2\big([0,T]; H^{s+1}\big(R^2\big)\big).$$

Remark 1.2 By virtue of the expression in (1.2), the dissipative operator £ in Theorem 1.1 is weaker than any power of the fractional Laplacian. Thus, we improve the results in [3] for system (1.1) in which $\nu > 0$ and $\kappa > 0$.

Remark 1.3 For the 2D generalized MHD equations, it remains an open problem whether there exists a global smooth solution without the dissipative operator £.

2 The proof of Theorem 1.1

In this section, we will give the global existence and uniqueness of the smooth solution to the system (1.1). By the classical hyperbolic method, there exists a finite time T_0 such that the system (1.1) is local well-posed in the interval $[0, T_0]$ in H^s with $s > 2$. Therefore, it is sufficient to establish a priori estimates in the interval $[0, T]$ for the given $T > T_0$. We shall prove Theorem 1.1 by adapting the elaborate nonlinear energy method. Furthermore, throughout this paper, we use C to denote the positive constants that may vary from line to line.

2.1 L^2 estimate for (u, B)

Lemma 2.1 *Assume that the conditions in Theorem 1.1 hold, then it holds that*

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 + 2\int_0^t \|£^{\frac{1}{2}} u(\tau)\|_{L^2}^2 \, d\tau = \|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2. \tag{2.1}$$

Proof Taking the inner products of $(1.1)_1$ with u and $(1.1)_2$ with B, adding the results and utilizing the following cancelation identity:

$$\int_{R^2} (B\cdot\nabla)B\cdot u \, dx + \int_{R^2} (B\cdot\nabla)u\cdot B \, dx = 0, \tag{2.2}$$

we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2\right) + \|£^{\frac{1}{2}} u\|_{L^2}^2 = 0. \tag{2.3}$$

Thanks to the Plancherel theorem

$$\int_{R^2} £u\cdot u \, dx = \int_{R^2} \widehat{£u(\xi)}\widehat{u(\xi)} \, d\xi = \int_{R^2} |\xi|^2 g(\xi)\left|\widehat{u(\xi)}\right|^2 d\xi \tag{2.4}$$

$$= \int_{R^2} \left|\widehat{£^{\frac{1}{2}} u(\xi)}\right|^2 d\xi = \|£^{\frac{1}{2}} u\|_{L^2}^2, \tag{2.5}$$

then integrating the above inequality we obtain

$$\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 + 2\int_0^t \|£^{\frac{1}{2}} u(\tau)\|_{L^2}^2 \, d\tau = \|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2. \tag{2.6}$$

□

2.2 H^1 estimate for (u, B)

Lemma 2.2 *Assume that the conditions in Theorem 1.1 hold, then there exists a positive constant C dependent on T, u_0 and B_0, such that*

$$\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \int_0^t \|£^{\frac{1}{2}} \omega\|_{L^2}^2 \, d\tau \le C(T, u_0, B_0). \tag{2.7}$$

Proof First, we obtain the vorticity equation by applying $\nabla\times$ to the MHD equations (1.1), where the vorticity $\omega = \nabla\times u = \partial_{x_1} u_2 - \partial_{x_2} u_1$ and the current $j = \nabla\times B = \partial_{x_1} B_2 - \partial_{x_2} B_1$, the corresponding vorticity equation:

$$\partial_t\omega + (u\cdot\nabla)\omega + \nu£\omega = (B\cdot\nabla)j, \tag{2.8}$$

$$\partial_t j + (u \cdot \nabla) j = (B \cdot \nabla \omega) + T(\nabla u, \nabla B), \tag{2.9}$$

where $T(\nabla u, \nabla B) = 2\partial_x B_1(\partial_{x_2} u_1 + \partial_{x_2} u_1) - 2\partial_{x_1} u_1(\partial_{x_2} b_1 + \partial_{x_1} b_2)$.

Taking the inner products of the first equation in (2.8) with ω, the second equation in (2.8) with j, adding them up and using the incompressible condition as well as the following fact

$$\int_{R^2} (B \cdot \nabla j)\omega \, dx + \int_{R^2} (B \cdot \nabla \omega) j \, dx = 0, \tag{2.10}$$

then we easily obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_{L^2}^2 + \left\|j(t)\right\|_{L^2}^2\right) + \left\|£^{\frac{1}{2}}\omega\right\|_{L^2}^2 = \int_{R^2} T(\nabla u, \nabla B) j \, dx. \tag{2.11}$$

For the arbitrary function

$$\begin{aligned}
\|\nabla u\|_{L^2}^2 &= C\left\|\xi \widehat{u(\xi)}\right\|_{L^2}^2 \\
&= C\int_{R^2} |\xi|^2 \left|\widehat{u(\xi)}\right|^2 d\xi \\
&= C\int_{|\xi|\le 1} |\xi|^2 \left|\widehat{u(\xi)}\right|^2 d\xi + C\int_{|\xi|\ge 1} |\xi|^2 \left|\widehat{u(\xi)}\right|^2 d\xi \\
&= C\int_{|\xi|\le 1} |\widehat{u(\xi)}|^2 d\xi + C\int_{|\xi|\ge 1} \frac{1}{m(\xi)} |\xi|^2 m(\xi) \left|\widehat{u(\xi)}\right|^2 d\xi \\
&\le C\int_{R^2} \left|\widehat{u(\xi)}\right|^2 d\xi + C\int_{|\xi|\ge 1} \frac{1}{m(1)} |\xi|^2 m(\xi) \left|\widehat{u(\xi)}\right|^2 d\xi \\
&\le C_1\|u\|_{L^2}^2 + C_2\left\|£^{\frac{1}{2}} u\right\|_{L^2}^2,
\end{aligned} \tag{2.12}$$

we obtain finally

$$\|\nabla u\|_{L^2} \le C_1\|u\|_{L^2} + C_2\left\|£^{\frac{1}{2}} u\right\|_{L^2}. \tag{2.13}$$

Utilizing the estimate of $\|\nabla u\|_{L^2}$ in (2.13) and the interpolation inequality, we obtain

$$\begin{aligned}
&\int_{R^2} T(\nabla u, \nabla B) j \, dx \\
&\quad \le C\|\nabla u\|_{L^2}\|\nabla B\|_{L^4}\|j\|_{L^4} \\
&\quad \le C\|\omega\|_{L^2}\|j\|_{L^4}^2 \\
&\quad \le C\|\omega\|_{L^2}\|j\|_{L^2}\|\nabla j\|_{L^2} \\
&\quad \le C\|\omega\|_{L^2}\|\nabla B\|_{L^2}\|\nabla j\|_{L^2} \\
&\quad \le C\|\omega\|_{L^2}\left(\|B\|_{L^2} + \left\|£^{\frac{1}{2}} B\right\|_{L^2}\right)\left(\|j\|_{L^2} + \left\|£^{\frac{1}{2}} j\right\|_{L^2}\right) \\
&\quad \le \frac{1}{2}\left\|£^{\frac{1}{2}}\omega\right\|_{L^2}^2 + C\left(1 + \|B\|_{L^2}^2 + \left\|£^{\frac{1}{2}} B\right\|_{L^2}^2\right)\left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\right).
\end{aligned} \tag{2.14}$$

Hence, combined with (2.14) and (2.11) this implies that

$$\begin{aligned}&\frac{d}{dt}\big(\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\big)+\|\pounds^{\frac{1}{2}}\omega\|_{L^2}^2\\&\quad\le C\big(1+\|B\|_{L^2}^2+\|\pounds^{\frac{1}{2}}B\|_{L^2}^2\big)\big(\|\omega\|_{L^2}^2+\|j\|_{L^2}^2\big).\end{aligned}\tag{2.15}$$

The Gronwall lemma provides us with

$$\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2+\int_0^t\|\pounds^{\frac{1}{2}}\omega\|_{L^2}^2\,d\tau\le C(T,u_0,B_0).\tag{2.16}$$

□

2.3 $L_t^\infty L^\infty$ estimate for u

Lemma 2.3 *Assume that the conditions in Theorem 1.1 hold, then it holds that*

$$\|u\|_{L_t^\infty L^\infty}\le C(t,u_0,B_0),\quad \textit{for any } t\in[0,T].\tag{2.17}$$

Proof By the classical hyperbolic theory, the system (1.1) is local well-posed in the interval $[0,T_0]$ in H^s with $s>2$, we have the estimate

$$\|u\|_{L_{T_0}^\infty H^s}+\|B\|_{L_{T_0}^\infty H^s}\le C(u_0,B_0).\tag{2.18}$$

Hence, by the embedding theorem,

$$\|u\|_{L_{T_0}^\infty L^\infty}\le C(u_0,B_0).\tag{2.19}$$

Thus, it is sufficient to establish that an a priori estimate (2.19) is valid for $t>T_0$. We rewrite the first equation in system (1.1) as

$$\partial_t u+\nu\pounds u=\nabla\cdot(B\otimes B)-\nabla\cdot(u\otimes u)-\nabla p.\tag{2.20}$$

According to the solution of the linear inhomogeneous equation, the solution of (2.20) can be explicitly given by

$$u(t)=K(t)*u_0+\int_0^t K(t-\tau)*\big[\nabla\cdot(B\otimes B)-\nabla\cdot(u\otimes u)-\nabla p\big]\,d\tau.\tag{2.21}$$

Taking L^∞ norm in terms of space variables and using the Young inequality, we obtain

$$\begin{aligned}\|u(t)\|_{L^\infty}&=\|K(t)*u_0\|_{L^\infty}+\int_0^t\big\|\nabla K(t-\tau)*\big[(B\otimes B)-(u\otimes u)\big](\tau)\big\|_{L^\infty}\,d\tau\\&\quad+\int_0^t\big\|div\big(K(t-\tau)\big)p\big\|_{L^\infty}\,d\tau\\&\le C\|K(t)\|_{L^1}\|u_0\|_{L^\infty}+C\int_0^t\|\nabla K(t-\tau)\|_{L^2}\big\|\big[(B\otimes B)\\&\quad-(u\otimes B)\big](\tau)\big\|_{L^2}\,d\tau+\int_0^t\big\|div\big(K(t-\tau)\big)\big\|_{L^2}\|p\|_{L^2}\,d\tau\\&\le C\|K(t)\|_{L^1}\|u_0\|_{L^\infty}+C\int_0^t\|\nabla K(t-\tau)\|_{L^2}\|B(\tau)\|_{L^4}\|B(\tau)\|_{L^4}\,d\tau\end{aligned}\tag{2.22}$$

$$\begin{aligned} &\le C\|K(t)\|_{L^1}\|u_0\|_{L^\infty} + C\|\nabla K(\tau)\|_{L_t^1L^2}\|u(\tau)\|_{L_t^\infty L^4}\|B(\tau)\|_{L_t^\infty L^4} \\ &\le C(1+t^{-1}+t^3)\|u_0\|_{L^\infty} + C(t,u_0,B_0) \\ &\le C(1+t^3)\|B_0\|_{L^\infty} + C(t,u_0,B_0), \end{aligned} \tag{2.23}$$

where we have used the inequality $\|K(t)\|_{L^1} \le C(1+t^{-1}+t^3)$, which is the property of the operator £ proved in Lemma 2.5 [14], hence we obtain

$$\|u\|_{L_t^\infty L^\infty} \le C(t,u_0,B_0), \quad \text{for } t > T_0. \tag{2.24}$$

Combining (2.19) and (2.24), we have

$$\|u\|_{L_t^\infty L^\infty} \le C(t,u_0,B_0), \quad \text{for any } t \in [0,T]. \tag{2.25}$$

2.4 $L_t^2L^\infty$ estimate for ω

Lemma 2.4 *Assume that the conditions in Theorem 1.1 hold, then it holds that*

$$\int_0^t \|\omega(\tau)\|_{L^\infty}^2 \, d\tau \le C(T,u_0,B_0). \tag{2.26}$$

Proof We write the first equation of (2.8) as

$$\partial_t\omega + \nu£\omega = (B\cdot\nabla)j - (u\cdot\nabla)\omega = \nabla\cdot(B\otimes j) - \nabla\cdot(u\otimes\omega). \tag{2.27}$$

According to the solution of the linear inhomogeneous equation, the solution of (2.27) can be explicitly written as

$$\omega = K(t)*\omega_0 + \int_0^t K(t-\tau)*\big[\nabla\cdot(B\otimes j) - \nabla\cdot(u\otimes\omega)\big](\tau)\,d\tau. \tag{2.28}$$

Taking L^∞ norm in terms of space variables and utilizing the Young inequality, we have

$$\begin{aligned} \|\omega\|_{L^\infty} &= \|K(t)*\omega_0\|_{L^\infty} + \int_0^t \|\nabla K(t-\tau)*\big[(B\otimes j)-(u\otimes j)\big](\tau)\|_{L^\infty}\,d\tau \\ &\le C\|K(t)\|_{L^2}\|\omega_0\|_{L^2} + C\int_0^t \|\nabla K(t-\tau)\|_{L^2}(\|(B\otimes j)(\tau)\|_{L^2} \\ &\quad + \|(u\otimes j)(\tau)\|_{L^2}\,d\tau. \end{aligned} \tag{2.29}$$

In order to obtain the $L_t^2L^\infty$ estimate on ω, we take the L^2 norm on the time variable and use the convolution Young inequality as well as the estimate (2.25) to obtain

$$\begin{aligned} \|\omega\|_{L_t^2L^\infty} &\le C\|K(t)\|_{L_t^2L^2}\|\omega_0\|_{L^2} + C\|\nabla K(t)\|_{L_t^1L^2}\big(\|B\otimes j\|_{L_t^2L^2} + \|(u\otimes j)\|_{L_t^2L^2}\big) \\ &\le C\|K(t)\|_{L_t^2L^2}\|\omega_0\|_{L^2} + C\|\nabla K(t)\|_{L_t^1L^2}\big(\|B\|_{L_t^2L^\infty}\|j\|_{L_t^\infty L^2} \\ &\quad + \|u\|_{L_t^4L^4}\|j\|_{L_t^4L^4}\big) \\ &\le C(T,u_0,B_0), \end{aligned} \tag{2.30}$$

where we have used the following known estimates due to the linear inhomogeneous equation:

$$\int_0^t \|K(\tau)\|_{L^2}^2 \, d\tau \le C(T, u_0, B_0), \qquad \int_0^t \|K(\tau)\|_{L^\infty} \, d\tau \le C(T, u_0, B_0). \tag{2.31}$$

On the other hand,we also used the fact that

$$\|u\|_{L_t^4 L^4} \le C\|u\|_{L_t^\infty L^2}^{\frac{1}{2}} \|\omega\|_{L_t^2 L^2}^{\frac{1}{2}} \le C(T, u_0, B_0) \tag{2.32}$$

and

$$\begin{aligned} \|\omega\|_{L_t^4 L^4} &\le C\|\omega\|_{L_t^\infty L^2}^{\frac{1}{2}} \|\nabla\omega\|_{L_t^2 L^2}^{\frac{1}{2}} \\ &\le C\|\omega\|_{L_t^\infty L^2}^{\frac{1}{2}} \big(\|\omega\|_{L_t^2 L^2}^{\frac{1}{2}} + \big\|£^{\frac{1}{2}}\omega\big\|_{L_t^2 L^2}^{\frac{1}{2}} \big) \\ &\le C(T, u_0, B_0). \end{aligned} \tag{2.33}$$

As a result, one has

$$\int_0^t \|\omega(\tau)\|_{L^\infty}^2 \, d\tau \le C(T, u_0, B_0). \tag{2.34}$$

□

2.5 The key estimates of $\int_0^t \|\nabla j(\tau)\|_{L^\infty} \, d\tau$

Lemma 2.5 *Assume that the conditions in Theorem 1.1 hold, then it holds that*

$$\|\omega(t)\|_{L^\infty} \le C(T, u_0, B_0), \tag{2.35}$$

$$\|\nabla u\|_{L^\infty} \le C\|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla\omega\|_{L^\infty}^{\frac{1}{2}}, \tag{2.36}$$

$$\int_0^t \|\nabla u\|_{L^\infty}^2 \, d\tau \le C(T, u_0, B_0). \tag{2.37}$$

Proof By (2.28), we can check that

$$\omega = K(t) * \omega_0 + \int_0^t K(t-\tau) * \big[\nabla \cdot (B \otimes j) - \nabla \cdot (u \otimes \omega)\big](\tau) \, d\tau. \tag{2.38}$$

Taking the operation ∇ to both sides of the above equality, and then taking the L^∞ norm in terms of space variables and using the Young inequality, we have

$$\begin{aligned} &\|\nabla\omega\|_{L_\infty} \\ &\quad\le \|\nabla K(t) * \omega_0\|_{L_\infty} + \int_0^t \big\|\nabla^2 K(t-\tau) * \big[(B \otimes j) - (u \otimes \omega)\big](\tau)\big\|_{L^\infty} \, d\tau \\ &\quad\le C\|\nabla K(t)\|_{L^2} \|\omega_0\|_{L^2} + C\int_0^t \|\nabla^2 K(t-\tau)\|_{L^1} \big(\|(B \otimes j)(\tau)\|_{L^\infty} \\ &\qquad + \|(u \otimes \omega)(\tau)\|_{L^\infty} \big) \, d\tau. \end{aligned} \tag{2.39}$$

Then, we obtain by taking the L^1 norm in terms of time variables and utilizing the convolution Young inequality

$$\begin{aligned}
&\left\|\nabla\omega(t)\right\|_{L_t^1L^\infty}\\
&\quad\leq C\left\|\nabla K(t)\right\|_{L_t^1L^2}\|\omega_0\|_{L^2}+C\left\|\nabla^2K(t)\right\|_{L_t^1L^1}\left(\|B\otimes j\|_{L_t^1L^\infty}+\|u\otimes\omega\|_{L_t^1L^\infty}\right)\\
&\quad\leq C\left\|\nabla K(t)\right\|_{L_t^1L^2}\|\omega_0\|_{L^2}+C\left\|\nabla^2K(t)\right\|_{L_t^1L^1}\left(\|Bj\|_{L_t^1L^\infty}+\|u\|_{L_t^2L^\infty}\|\omega\|_{L_t^2L^\infty}\right)\\
&\quad\leq C\left\|\nabla K(t)\right\|_{L_t^1L^2}\|\omega_0\|_{L^2}+C\left\|\nabla^2K(t)\right\|_{L_t^1L^1}\\
&\qquad\left(\|B\|_{L_t^\infty L^\infty}\|j\|_{L_t^1L^\infty}+\|u\|_{L_t^\infty L^2}^{\frac12}\|\omega\|_{L_t^1L^\infty}^{\frac12}\|\omega\|_{L_t^2L^\infty}\right).
\end{aligned}\tag{2.40}$$

By the previous estimates on $\|\nabla K\|_{L_t^1L^2}$, it is easy to show that $\|\nabla K\|_{L_t^1L^2}\|j_0\|_{L^2}$ are nondecreasing functions satisfying

$$\left\|\nabla K(t)\right\|_{L_t^1L^2}\|\omega_0\|_{L^2}\leq C(T,u_0,B_0),\tag{2.41}$$

$$\left\|\nabla^2K(t)\right\|_{L_t^1L^1}\|B\|_{L_t^\infty L^\infty}\leq C(T,u_0,B_0).\tag{2.42}$$

Therefore, it follows from (2.40) that

$$\begin{aligned}
&\int_0^t\left\|\nabla\omega(\tau)\right\|_{L^\infty}d\tau\\
&\quad\leq C\left\|\nabla K(t)\right\|_{L_t^1L^2}\|\omega_0\|_{L^2}+\left(C\left\|\nabla^2K(t)\right\|_{L_t^1L^1}+\|u\|_{L_t^\infty L^2}^{\frac12}\|\omega\|_{L_t^2L^\infty}\right.\\
&\qquad\left.+C\|B\|_{L_t^\infty L^\infty}\left\|\nabla^2K(t)\right\|_{L_t^1L^1}\right)\int_0^t\left\|\omega(\tau)\right\|_{L^\infty}d\tau.
\end{aligned}\tag{2.43}$$

Multiplying both sides of the vorticity equation by $|\omega|^{p-2}\omega$ and integrating over R^2, we obtain

$$\begin{aligned}
\frac1p\frac{d}{dt}\left\|\omega(t)\right\|_{L^p}^p&=\int_{R^2}(B\cdot\nabla j)\omega|\omega|^{p-2}\,dx-\nu\int_{R^2}£\omega|\omega|^{p-2}\omega\,dx\\
&\leq\|B\|_{L^\infty}\|\nabla j\|_{L^p}\|\omega\|_{L^p}^{p-1},
\end{aligned}\tag{2.44}$$

where we have used the nonnegativity of the almost Laplacian velocity dissipation, then we have

$$\frac{d}{dt}\left\|\omega(t)\right\|_{L^p}\leq\|B\|_{L^\infty}\|\nabla j\|_{L^p}.\tag{2.45}$$

Integrating over time, we have

$$\left\|\omega(t)\right\|_{L^p}\leq\left\|\omega(0)\right\|_{L^p}+\int_0^t\left\|B(\tau)\right\|_{L^\infty}\left\|\nabla j(\tau)\right\|_{L^p}d\tau.\tag{2.46}$$

Letting $p\to\infty$, this yields

$$\left\|\omega(t)\right\|_{L^\infty}\leq\left\|\omega(0)\right\|_{L^\infty}+\int_0^t\left\|B(\tau)\right\|_{L^\infty}\left\|\nabla j(\tau)\right\|_{L^\infty}d\tau.\tag{2.47}$$

Due to the previous estimate, we obtain

$$\|\omega(t)\|_{L^\infty} \le \|\omega(0)\|_{L^\infty} + C(T)\int_0^t \|\nabla j(\tau)\|_{L^\infty}\,d\tau. \tag{2.48}$$

Now, suppose that

$$H(t) = h_1(T) + h_2(T)\int_0^t \|\omega(\tau)\|_{L^\infty}\,d\tau, \tag{2.49}$$

it follows from (2.43) that

$$\int_0^t \|\nabla\omega(\tau)\|_{L^\infty}\,d\tau \le H(t). \tag{2.50}$$

Due to (2.48), we easily obtain

$$\begin{aligned}\frac{d}{dt}H(t) &= H_2(T)\|\omega(t)\|_{L^\infty}\\ &\le H_2(T)\left(\|\omega(0)\|_{L^\infty} + C(T)\int_0^t \|\nabla\omega(\tau)\|_{L^\infty}\,d\tau\right)\\ &\le H_2(T)\big(\|\omega(0)\|_{L^\infty} + C(T)H(t)\big).\end{aligned} \tag{2.51}$$

The Gronwall lemma provides us with

$$G(t) \le C(T,u_0,B_0), \tag{2.52}$$

which further implies that

$$\int_0^t \|\nabla j(\tau)\|_{L^\infty}\,d\tau \le C(T,u_0,B_0). \tag{2.53}$$

Combing (2.53) and (2.48), we obtain

$$\|\omega(t)\|_{L^\infty} \le C(T,u_0,B_0). \tag{2.54}$$

According to the interpolation inequality

$$\|\nabla u\|_{L^\infty} \le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^\infty}^{\frac{1}{2}} \tag{2.55}$$

and (2.16), we obtain

$$\int_0^t \|\nabla u\|_{L^\infty}^2\,d\tau \le C(T,u_0,B_0). \tag{2.56}$$ □

2.6 Global $H^s(s>2)$ estimates of (u,B)

Lemma 2.6 *Assume that the conditions in Theorem 1.1 hold, then it holds that*

$$\|u(t)\|_{H^s} + \|B(t)\|_{H^s} + \int_0^t \|\pounds^{\frac{1}{2}}u(t)\|_{H^s}^2\,d\tau \le C(t). \tag{2.57}$$

Proof To obtain the global bound for (u,B) in $H^s(s>2)$, we apply Λ^s with $\Lambda=(I-\Delta)^{\frac{1}{2}}$ to the original equation u and B, and take the L^2 inner product of the resulting equations with $(\Lambda^s u,\Lambda^s B)$ to obtain the energy inequality.

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|u(t)\|_{H^s}^2+\|B(t)\|_{H^s}^2\big)+\|£^{\frac{1}{2}}u\|_{H^s}^2\\
&\quad=-\int_{R^2}\big[\Lambda^s,u\cdot\nabla\big]u\cdot\Lambda^s u\,dx+\int_{R^2}\big[\Lambda^s,B\cdot\nabla\big]B\cdot\Lambda^s u\,dx\\
&\qquad-\int_{R^2}\big[\Lambda^s,u\cdot\nabla\big]B\cdot\Lambda^s B\,dx+\int_{R^2}\big[\Lambda^s,B\cdot\nabla\big]u\cdot\Lambda^s B\\
&\quad=I_1+I_2+I_3+I_4,
\end{aligned}\tag{2.58}$$

where $[m,n]$ is the standard commutator notation, namely $[a,b]=ab-ba$. Utilizing the following Kato–Ponce inequality [16].

$$\big\|\big[\Lambda^s,f\big]g\big\|_{L^p}\le C\big(\|\nabla f\|_{L^\infty}\big\|\Lambda^{s-1}g\big\|_{L^p}+\|g\|_{L^\infty}\big\|\Lambda^s f\big\|_{L^p}\big),\quad 1<p<\infty.\tag{2.59}$$

Then, we estimate the energy terms as follows:

$$I_1\le C\|\nabla u\|_{L^\infty}\|u\|_{H^s}^2,\qquad I_2\le C\|\nabla B\|_{L^\infty}\big(\|u\|_{H^s}^2+\|B\|_{H^s}^2\big),\tag{2.60}$$

$$I_3+I_4\le C\big(\|\nabla u\|_{L^\infty}+\|\nabla B\|_{L^\infty}\big)\big(\|u\|_{H^s}^2+\|B\|_{H^s}^2\big).\tag{2.61}$$

Finally,we obtain

$$\begin{aligned}
&\frac{d}{dt}\big(\|u(t)\|_{H^s}^2+\|B(t)\|_{H^s}^2\big)+\|£^{\frac{1}{2}}u\|_{H^s}^2\\
&\quad\le C\big(\|\nabla u\|_{L^\infty}+\|\nabla B\|_{L^\infty}\big)\big(\|u\|_{H^s}^2+\|B\|_{H^s}^2\big).
\end{aligned}\tag{2.62}$$

We need to estimate the bounds on $\|\nabla u\|_{L^\infty}$ and $\|\omega\|_{L^\infty}$, according to the following Sobolev extrapolation inequality with logarithmic correction [17].

$$\|\nabla u\|_{L^\infty}\le C\big(1+\|u\|_{L^2(R^2)}+\|\omega\|_{L^\infty(R^2)}\big)ln\big(e+\|u\|_{H^s(R^2)}\big),\quad (s>2).\tag{2.63}$$

Collecting the above estimate, we have

$$\begin{aligned}
&\frac{d}{dt}\big(\|u(t)\|_{H^s}^2+\|B(t)\|_{H^s}^2\big)+\|£^{\frac{1}{2}}u\|_{H^s}^2\\
&\quad\le C\big(1+\|\omega\|_{L^\infty}+\|\nabla B\|_{L^\infty}\big)ln\big(e+\|u\|_{H^s}^2+\|B\|_{H^s}^2\big)\\
&\qquad\big(\|u\|_{H^s}^2+\|B\|_{H^s}^2\big).
\end{aligned}\tag{2.64}$$

Utilizing the log-Gronwall-type inequality and estimates (2.34) and (2.41), we obtain

$$\big\|u(t)\big\|_{H^s}+\big\|B(t)\big\|_{H^s}+\int_0^t\big\|£^{\frac{1}{2}}u(t)\big\|_{H^s}^2\,d\tau\le C(t).\tag{2.65}$$

Combining (2.13) we obtain

$$\int_0^t \left\|u(t)\right\|_{H^{s+1}}^2 d\tau \le C(t). \tag{2.66}$$

According to the six steps priori estimates in previous results, we give the proof of Theorem 1.1. We introduce the mollification of $\varrho_N f$ given by

$$(\varrho_N f)(x) = N^2 \int_{R^2} \eta\big(N(x-y)\big) f(y)\,dy, \tag{2.67}$$

where $0 < \eta(|x|) \in C_0^\infty(R^2)$ satisfies $\int_{R^2} \eta(y)\,dy = 1$. We regularize the system (1.1) as follows:

$$\begin{cases} \partial_t u^N + P\varrho_N((\varrho_N u^N \cdot \nabla)\varrho_N u^N) + \nu J\pounds_1 \varrho_N u^N = P\varrho_N(\varrho_N B^N \cdot \nabla)\varrho_N B^N, \\ \partial_t B^N + \varrho_N(\varrho_N u \cdot \nabla)\varrho_N B = \varrho_N((\varrho_N B^N \cdot \nabla)\varrho_N u^N), \\ \nabla \cdot u^N = \nabla \cdot u^N = 0, \\ u^N(x,0) = \varrho_N u_0(x), B^N(x,0) = \rho_N B_0(x), \end{cases}$$

where P denotes the Leray projection operator. For any fixed $N > 0$, using the properties of mollifiers and the priori estimates obtained in proving the global bounds, for any $t \in (0,\infty)$,

$$\left\|u^N(t)\right\|_{H^s} + \left\|B^N(t)\right\|_{H^s} + \int_0^t \left\|\pounds^{\frac{1}{2}} u^N(\tau)\right\|_{H^s}^2 d\tau \le C(t), \tag{2.68}$$

$$\int_0^t \left\|u^N(\tau)\right\|_{H^{s+1}}^2 d\tau \le C(t), \tag{2.69}$$

which are uniform in N, according to the standard Alaogu theorem and compactness arguments, we can extract a subsequence (u^{N_i}, B^{N_i}) and pass to the limit as $N \to \infty$ to obtain the fact that the limit function (u,B) is indeed a global classical solution of the problem (1.1). The uniqueness can also be easily obtained. This completes the proof of Theorem 1.1. □

Acknowledgements
The authors would like to thank Professor Shu Wang for reviewing an early draft of this article and giving valuable comments. The authors thanks the referee for helpful comments.

Funding
The research of L. R. Li was supported by the Scientific Research Plan Projects for Special Fund of Fundamental Scientific Research Business Expense for Higher School of Central Government (Grant no.ZY20215116). The research of M. L. Hong is partially funded by the National Science Foundation of China (Grant no. 12071192).

Availability of data and materials
No data were used to support this study.

Declarations

Ethics approval and consent to participate
This declaration is "not applicable".

Competing interests
The authors declare no competing interests.

Author contributions
All authorship contributed equally to this paper. All authors read and approved the final manuscript.

Received: 27 December 2022 Accepted: 18 July 2023 Published online: 03 August 2023

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