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Persistence properties and blow-up phenomena for a generalized Camassa–Holm equation

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Abstract

In this paper, we investigate a generalized Camassa–Holm equation. Firstly, we establish the persistence properties of strong solutions for the equation in weighted spaces $L^p_\phi = L^p(\mathbb{R}, \phi^p\,dx)$. Then we present some sufficient conditions of blow-up solutions assuming that the initial data satisfy certain conditions, which are more precise than those in the previous work.

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Keywords: Persistence properties; Weighted space; Blow-up; A generalized Camassa-Holm equation

1 Introduction

In 2009, Novikov [26] used the perturbative symmetry approach to deduce a series of generalized Camassa–Holm equations, including both quadratic and cubic nonlinearities, which are integrable and possess an infinite hierarchy of quasi-local higher symmetries. They are of the following structure:

$$\left(1-\partial_x^2\right)u_t = F(u, u_x, u_{xx}, u_{xxx}, \dots), \qquad u = u(t,x), \tag{1}$$

where F is a function of u and its derivatives with respect to x, and the subscript denotes partial derivative. Among them, the most celebrated example is the Camassa–Holm equation (also called the CH equation)

$$u_t - u_{txx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0, \tag{2}$$

derived by Camassa and Holm [2] and Fokas and Fuchssteiner [11]. It describes the motion of shallow water waves and possesses a Lax pair, a bi-Hamiltonian structure, and infinitely many conserved integrals [2]. It can be solved by the inverse scattering method. One of the remarkable features of the CH equation is that it has the single-peakon solutions

$$u(t,x) = ce^{-|x-ct|}, \qquad c \in \mathbb{R},$$

and the multipeakon solutions

$$u(t,x)=\sum_{i=1}^{N}p_i(t)e^{-|x-q_i(t)|},$$

where $p_i(t)$ and $q_i(t)$ satisfy the Hamilton system [2]

$$\begin{cases}\frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}=\sum_{i\neq j}p_ip_j\operatorname{sign}(q_i-q_j)e^{|q_i-q_j|},\\ \frac{dq_i}{dt}=-\frac{\partial H}{\partial p_i}=\sum_j p_je^{|q_i-q_j|},\end{cases}$$

with Hamiltonian $H=\frac{1}{2}\sum_{i,j=1}^{N}p_ip_je^{|q_i|}$. It is shown that those peaked solitons are orbitally stable in the energy space [9]. Another remarkable feature of the CH equation is the so-called wave-breaking phenomenon, that is, the wave profile remains bounded while its slope becomes unbounded in finite time [5–7]. Hence equation (2) has attracted lots of attention since it was born. The dynamic properties related to the equation can be found in [4, 8, 10, 12, 14–20, 23, 31, 33–38] and the references therein.

The other example is the Novikov equation

$$u_t-u_{txx}+4u^2u_x-3uu_xu_{xx}-u^2u_{xxx}=0. \tag{3}$$

It is shown in [26] that equation (3) possesses soliton solutions, infinitely many conserved quantities, a Lax pair in matrix form, and a bi-Hamiltonian structure. The conserved quantities

$$H_1\bigl[u(t)\bigr]=\int_{\mathbb{R}}\bigl(u^2+u_x^2\bigr)\,dx$$

and

$$H_2(t)=\int_{\mathbb{R}}\left(u^4+2u^2u_x^2-\frac{1}{3}u_x^4\right)dx$$

play an important role in the study of the dynamic properties related to equation (3). More information about the Novikov equation can be found in Tiglay [27], Ni and Zhou [25], Wu and Yin [29, 30], Yan, Li, and Zhang [32], Mi and Mu [24] and the references therein.

In this paper, we are interested in the following equation:

$$\begin{cases}u_t-u_{txx}=\frac{1}{2}(3u_x^2-2u_xu_{xxx}-u_{xx}^2),\\ u(0,x)=u_0(x),\end{cases} \tag{4}$$

for $t>0$ and $x\in\mathbb{R}$, and u stands for the unknown function on the line $\mathbb{R}$. Problem (4) admits traveling wave solutions and possesses conserved laws [21]

$$E\bigl(u(t)\bigr)=\int_{\mathbb{R}}\bigl(u_x^2+u_{xx}^2\bigr)\,dx=\int_{\mathbb{R}}\bigl(u_{0x}^2+u_{0xx}^2\bigr)\,dx. \tag{5}$$

Tu and Yin [28] established the local well-posedness for the Cauchy problem in the critical Besov spaces $B^{\frac{1}{2}}_{2,1}$ relying on the Littlewood–Paley decomposition, transport equations theory, logarithmic inequalities, and Osgood's lemma. The global existence of a strong solution and some blow-up results are also presented. It is shown in [21] that the solutions of problem (4) are velocity potentials of the classical Camassa–Holm equation and also are locally well posed in the other Besov spaces $B^s_{p,r}$, $s > \max\{\frac{1}{p}, \frac{1}{2}\}$. To our best knowledge, the asymptotic behaviors for the Cauchy problem (4) have not been studied yet. In this paper, we first investigate the asymptotic behaviors of the strong solutions for problem (4) in weighted spaces $L^p_\phi := L^P(\mathbb{R}, \phi^p\,dx)$, extending the result in [22]. Then we present some blow-up results, provided that the initial data satisfy certain conditions, which are more precise than those in [28].

Notations The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $[0,+\infty)\times\mathbb{R}$ is denoted by C_0^∞. Let $L^p = L^p(\mathbb{R})(1 \le p < +\infty)$ be the space of all measurable functions h such that $\|h\|^p_{L^P} = \int_\mathbb{R} |h(t,x)|^p\,dx < \infty$. We define $L^\infty = L^\infty(\mathbb{R})$ with the standard norm $\|h\|_{L^\infty} = \inf_{m(e)=0}\sup_{x\in R\setminus e}|h(t,x)|$. For any real number s, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space with the norm

$$\|h\|_{H^s} = \left(\int_\mathbb{R} \left(1+|\xi|^2\right)^s \left|\hat{h}(t,\xi)\right|^2 d\xi\right)^{\frac{1}{2}} < \infty,$$

where $\hat{h}(t,\xi) = \int_\mathbb{R} e^{-ix\xi} h(t,x)\,dx$.

We denote by $*$ the convolution. Note that if $G(x) := \frac{1}{2}e^{-|x|}$, $x\in\mathbb{R}$, then $(1-\partial_x^2)^{-1}f = G*f$ for all $f \in L^2(\mathbb{R})$, and $G*(u-u_{xx}) = u$. Using this identity, we rewrite problem (4) as follows:

$$\begin{cases} u_t - \frac{1}{2}u_x^2 = G*[u_x^2 + \frac{1}{2}u_{xx}^2], \\ u(0,x) = u_0(x), \end{cases} \tag{6}$$

for $t > 0$ and $x \in \mathbb{R}$, which is equivalent to

$$\begin{cases} y_t - u_x y_x = -\frac{1}{2}y^2 + uy + \frac{1}{2}u_x^2 - \frac{1}{2}u^2, \\ y = u - u_{xx}, \\ u(0,x) = u_0(x), \qquad y_0 = u_0 - u_{0xx}. \end{cases} \tag{7}$$

2 Persistence properties

Motivated by the recent work [22, 35, 36], the aim of this section is to establish the persistence properties for a generalized Camassa–Holm equation in the weighted L^p spaces. Let us first give some standard definitions.

Definition 2.1 An admissible weight function for problem (4) is a locally absolutely continuous function $\phi : \mathbb{R} \to \mathbb{R}$ such that, for some $A > 0$ and almost all $x \in \mathbb{R}$, $|\phi'(x)| \le A|\phi(x)|$, and that is ν-moderate for some submultiplicative weight function ν satisfying $\inf_\mathbb{R} \nu > 0$ and

$$\int_\mathbb{R} \frac{\omega(x)}{e^{|x|}}\,dx < \infty. \tag{8}$$

Definition 2.2 In general, a weight function is simply a nonnegative function $\nu : \mathbb{R}^n \to \mathbb{R}$, which is called submultiplicative if

$$\nu(x,y) \le \nu(x)\nu(y) \quad \text{for all } x,y \in \mathbb{R}^n.$$

Given a submultiplicative function ν, a positive function ϕ is ν-moderate if and only if

$$\exists C_0 > 0 : \phi(x+y) \le C_0 \nu(x)\phi(y) \quad \text{for all } x,y \in \mathbb{R}^n.$$

If ϕ is ν-moderate for some submultiplicative function ν, then we say that ϕ is moderate. This is usually used in the theory of time-frequency analysis [1]. Let us recall the most standard example with such weights. Let

$$\phi(x) = \phi_{a,b,c,d}(x) = e^{a|x|^b}\bigl(1+|x|\bigr)^c \log\bigl(e+|x|\bigr)^d.$$

Then we have the following two properties [22].

(i) For $a,c,d \ge 0$ and $0 \le b \le 1$, such a weight is submultiplicative.

(ii) If $a,c,d \in \mathbb{R}$ and $0 \le b \le 1$, then ϕ is moderate. More precisely, $\phi_{a,b,c,d}$ is $\phi_{\alpha,\beta,\gamma,\delta}$-moderate for $|a| \le \alpha$, $|b| \le \beta$, $|c| \le \gamma$, and $|d| \le \delta$.

The elementary properties of submultiplicative and moderate weights can be found in [22]. Let us collect our results on admissible weights.

Lemma 2.1 ([22]) *Let $\nu : \mathbb{R}^n \to \mathbb{R}^+$ and $C_0 > 0$. Then the following conditions are equivalent*:

(1) $\forall x,y : \nu(x+y) \le C_0 \nu(x)\nu(y)$;

(2) *for all $1 \le p,q,r \le \infty$ and for any measurable functions $f_1, f_2 : \mathbb{R}^n \to C$, we have the weighted Young inequality*

$$\bigl\|(f_1 * f_2)\nu\bigr\|_r \le C_0 \|f_1\nu\|_p \|f_2\nu\|_q, \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Lemma 2.2 ([22]) *Let $1 \le p \le \infty$, and let ν be a submultiplicative weight on $\mathbb{R}^n$. The following two conditions are equivalent*:

(1) *ϕ is a ν-moderate weight function* (*with constant C_0*);

(2) *for all measurable functions f_1 and f_2, we have the weighted Young estimate*

$$\bigl\|(f_1 * f_2)\phi\bigr\|_p \le C_0 \|f_1\nu\|_1 \|f_2\phi\|_p.$$

Theorem 2.1 *Let $T > 0$, $s > \frac{5}{2}$, and $2 \le p \le \infty$. Assume that $u \in C([0,T], H^s(\mathbb{R}))$ is a strong solution of problem* (4) *such that $u(0,x) = u_0$ satisfies*

$$u_0\phi, u_{0,x}\phi, u_{0,xx}\phi \in L^p(\mathbb{R}),$$

where ϕ is an admissible weight function for problem (4). *Then, for all $t \in [0,T]$, we have the estimate*

$$\begin{aligned}&\|u\phi\|_{L^p} + \|u_x\phi\|_{L^p} + \|u_{xx}\phi\|_{L^p}\\ &\quad \le \bigl(\|u_0\phi\|_{L^p} + \|u_{0,x}\phi\|_{L^p} + \|u_{0,xx}\phi\|_{L^p}\bigr)e^{CMt}\end{aligned}$$

for some constant $C > 0$ depending only on ν, ϕ (through the constants A, C_0, $\inf_{x\in\mathbb{R}} \nu$, and $\int_{\mathbb{R}} \frac{\nu(x)}{e^{|x|}}\,dx < \infty$), and

$$M = \sup_{t\in[0,T]} \left(\left\|u(t)\right\|_{L^\infty} + \left\|\partial_x u(t)\right\|_{L^\infty} + \left\|\partial_{xx} u(t)\right\|_{L^\infty}\right).$$

Proof Assume that ϕ is ν-moderate and satisfies the above conditions. From the assumption $u \in C([0,T],H^s)$, $s > 5/2$, we get

$$M = \sup_{t\in[0,T]} \left(\left\|u(t)\right\|_{L^\infty} + \left\|\partial_x u(t)\right\|_{L^\infty} + \left\|\partial_{xx} u(t)\right\|_{L^\infty}\right) < \infty.$$

For any $N \in Z^+$, let us consider the N-truncations of ϕ: $f(x) = f_N(x) = \min\{\phi, N\}$. Then $f : \mathbb{R} \to \mathbb{R}$ is a locally absolutely continuous function such that

$$\|f\|_{L^\infty} \le N, \left|f'(x)\right| \le A\left|f(x)\right| \quad \text{a.e. on } \mathbb{R}.$$

On the other hand, if $C_1 = \max\{C_0, \alpha^{-1}\}$, where $\alpha = \inf_{x\in\mathbb{R}} \nu(x) > 0$, then

$$f(x+y) \le C_1 \nu(x) f(x) \quad \forall x, y \in \mathbb{R}.$$

In addition, as shown in [22], the N-truncations f of a ν-moderate weight ϕ are uniformly ν-moderate with respect to N. We begin to consider the case $2 \le p < \infty$. Multiply the first equation of problem (6) by $f|uf|^{p-2}(uf)$ and integrate to obtain

$$\begin{aligned} &\int_{\mathbb{R}} |uf|^{p-2}(uf)\partial_t(uf)\,dx - \int_{\mathbb{R}} |uf|^{p-2}(uf) f u_x^2\,dx \\ &\quad - \int_{\mathbb{R}} |uf|^{p-2}(uf) f G * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right] dx = 0. \end{aligned} \tag{9}$$

For the first term on the left-hand side of (9), we have

$$\int_{\mathbb{R}} |uf|^{p-2}(uf)\partial_t(uf)\,dx = \frac{1}{p}\frac{d}{dt}\|uf\|_{L^p}^p = \|uf\|_{L^p}^{p-1}\frac{d}{dt}\|uf\|_{L^p}, \tag{10}$$

and the third term on the left-hand side of (9) reads

$$\begin{aligned} \left|\int_{\mathbb{R}} |uf|^{p-2}(uf) f G * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right] dx\right| &\le \|uf\|_{L^p}^{p-1} \left\| f G * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^p} \\ &\le \|uf\|_{L^p}^{p-1} \|G\nu\|_{L^1} \left(\left\|u_x^2 f\right\|_{L^p} + \left\|u_{xx}^2 f\right\|_{L^p}\right) \\ &\le CM\|uf\|_{L^p}^{p-1}\left(\|u_x f\|_{L^p} + \|u_{xx} f\|_{L^p}\right), \end{aligned} \tag{11}$$

where we used the Hölder inequality, Lemmas 3.1 and 3.2, and (8). For the second term, we have

$$\begin{aligned} \left|\int_{\mathbb{R}} |uf|^{p-2}(uf) f u_x^2\,dx\right| &= \left|\int_{\mathbb{R}} u_x |uf|^{p-2}(uf)\left[\partial_x(fu) - uf_x\right] dx\right| \\ &\le \left|\int_{\mathbb{R}} u_x |uf|^{p-2}(uf)\partial_x(fu)\,dx\right| + \left|\int_{\mathbb{R}} u_x |uf|^{p-2}(uf) u f_x\,dx\right| \\ &\le \frac{M}{p}\|uf\|_{L^p}^p + AM\|uf\|_{L^p}^p. \end{aligned} \tag{12}$$

From (9) we obtain

$$\frac{d}{dt}\|uf\|_{L^p} \leq \left(\frac{M}{p} + AM\right)\|uf\|_{L^p} + CM\big(\|u_x f\|_{L^p} + \|u_{xx} f\|_{L^p}\big). \tag{13}$$

Now we will give estimates on $u_x f$. Differentiating the first equation of problem (6) with respect to x, we get

$$u_{xt} - u_x u_{xx} - G_x * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right] = 0, \tag{14}$$

which multiplied by f results in the equation

$$\partial_t(u_x f) - f u_x u_{xx} - f G_x * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right] = 0. \tag{15}$$

Multiplying (15) by $|u_x f|^{p-2}(u_x f)$ with $p \in Z^+$ and integrating give the equation

$$\begin{aligned} &\int_{\mathbb{R}} |u_x f|^{p-2}(u_x f)\partial_t(u_x f)\,dx - \int_{\mathbb{R}} |u_x f|^{p-2}(u_x f) f u_x u_{xx}\,dx \\ &\quad - \int_{\mathbb{R}} |u_x f|^{p-2}(u_x f) f G_x * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right] dx = 0. \end{aligned} \tag{16}$$

Using as similar procedure, we obtain the estimates

$$\int_{\mathbb{R}} |u_x f|^{p-2}(u_x f)\partial_t(u_x f)\,dx = \frac{1}{p}\frac{d}{dt}\|u_x f\|_{L^p}^p = \|u_x f\|_{L^p}^{p-1}\frac{d}{dt}\|u_x f\|_{L^p}, \tag{17}$$

$$\left|\int_{\mathbb{R}} |u_x f|^{p-2}(u_x f) f u_x u_{xx}\,dx\right| \leq M\|u_x f\|_{L^p}^p, \tag{18}$$

and

$$\begin{aligned} \left|\int_{\mathbb{R}} |u_x f|^{p-2}(u_x f) f G_x * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right] dx\right| &\leq \|u_x f\|_{L^p}^{p-1}\left\| f G_x * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^p} \\ &\leq C\|u_x f\|_{L^p}^{p-1}\|G_x \nu\|_{L^1}\left(\left\|u_x^2 f\right\|_{L^p} + \left\|u_{xx}^2 f\right\|_{L^p}\right) \\ &\leq CM\|u_x f\|_{L^p}^{p-1}\big(\|u_x f\|_{L^p} + \|u_{xx} f\|_{L^p}\big), \end{aligned} \tag{19}$$

where we used $|\partial_x G(x)| < \frac{1}{2}e^{-|x|}$. Therefore from (16)–(19) it follows that

$$\frac{d}{dt}\|u_x f\|_{L^p} \leq (C+1)M\|u_x f\|_{L^p} + CM\|u_{xx} f\|_{L^p}). \tag{20}$$

Next, we focus on estimates of $u_{xx} f$. Differentiating (14) with respect to x and multiplying by f result in the equation

$$\partial_t(u_{xx} f) - f u_{xx}^2 - f u_x u_{xxx} - f G_{xx} * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right] = 0. \tag{21}$$

Multiply by $|u_{xx}f|^{p-2}(u_{xx}f)$ with $p \in Z^+$ and integrate to obtain the equation

$$\begin{aligned}&\int_{\mathbb{R}} |u_{xx}f|^{p-2}(u_{xx}f)\partial_t(u_{xx}f)\,dx - \int_{\mathbb{R}} |u_{xx}f|^{p-2}(u_{xx}f)fu_{xx}^2\,dx - \int_{\mathbb{R}} |u_{xx}f|^{p-2}(u_{xx}f)fu_x u_{xxx}\,dx \\ &\quad - \int_{\mathbb{R}} |u_{xx}f|^{p-2}(u_{xx}f)fG_{xx} * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right]dx = 0. \end{aligned} \tag{22}$$

Notice that the estimates

$$\int_{\mathbb{R}} |u_{xx}f|^{p-2}(u_{xx}f)\partial_t(u_{xx}f)\,dx = \frac{1}{p}\frac{d}{dt}\|u_{xx}f\|_{L^p}^p = \|u_{xx}f\|_{L^p}^{p-1}\frac{d}{dt}\|u_{xx}f\|_{L^p}, \tag{23}$$

$$\left|\int_{\mathbb{R}} |u_{xx}f|^{p-2}(u_{xx}f)fu_{xx}^2\,dx\right| \le M\|u_{xx}f\|_{L^p}^p, \tag{24}$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}} |u_{xx}f|^{p-2}(u_{xx}f)fG_{xx} * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right]dx\right| &\le \|u_{xx}f\|_{L^p}^{p-1}\left\|fG_{xx} * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^p} \\ &\le CM\|u_{xx}f\|_{L^p}^{p-1}\big(\|u_xf\|_{L^p} + \|u_{xx}f\|_{L^p}\big) \end{aligned} \tag{25}$$

hold, where we used the equality $\partial_x^2 G = G - 1$.

For the third-order derivative term, we have

$$\begin{aligned}\left|\int_{\mathbb{R}} |u_{xx}f|^{p-2}(u_{xx}f)fu_x u_{xxx}\,dx\right| &= \left|\int_{\mathbb{R}} u_x|u_{xx}f|^{p-2}(u_{xx}f)\big[\partial_x(fu_{xx}) - u_{xx}f_x\big]\,dx\right| \\ &\le \left|\int_{\mathbb{R}} u_x|u_{xx}f|^{p-2}(u_{xx}f)\partial_x(fu_{xx})\,dx\right| \\ &\quad + \left|\int_{\mathbb{R}} u_x|u_{xx}f|^{p-2}(u_{xx}f)u_{xx}f_x\,dx\right| \\ &\le \frac{M}{p}\|u_{xx}f\|_{L^p}^p + AM\|u_{xx}f\|_{L^p}^p. \end{aligned} \tag{26}$$

Therefore from (22)–(26) it follows that

$$\frac{d}{dt}\|u_{xx}f\|_{L^p} \le CM\|u_xf\|_{L^p} + M\left(1 + C + A + \frac{1}{p}\right)\|u_{xx}f\|_{L^p}. \tag{27}$$

Now, combining (13), (20), and (27), we deduce

$$\begin{aligned}&\frac{d}{dt}\big(\|uf\|_{L^p} + \|u_xf\|_{L^p} + \|u_{xx}f\|_{L^p}\big) \\ &\quad \le \left(\frac{M}{p} + AM\right)\|uf\|_{L^p} + (3C+1)M\|u_xf\|_{L^p} + M\left(1 + 3C + A + \frac{1}{p}\right)\|u_{xx}f\|_{L^p} \\ &\quad \le CM\big(\|uf\|_{L^p} + \|u_xf\|_{L^p} + \|u_{xx}f\|_{L^p}\big). \end{aligned} \tag{28}$$

Integrating (28) gives

$$\begin{aligned}&\|uf\|_{L^p} + \|u_xf\|_{L^p} + \|u_{xx}f\|_{L^p} \\ &\quad \le \big(\|u_0f\|_{L^p} + \|u_{0x}f\|_{L^p} + \|u_{0xx}f\|_{L^p}\big)e^{CMt} \quad \forall t \in [0,T]. \end{aligned} \tag{29}$$

Since $f(x) = f_N(x) \to \phi(x)$ as $N \to \infty$ for a.e. $x \in \mathbb{R}$, recalling that $u_0\phi$, $u_{0x}\phi$, $u_{0xx}\phi \in L^p$, we deduce

$$\begin{aligned} &\|u\phi\|_{L^p} + \|u_x\phi\|_{L^p} + \|u_{xx}\phi\|_{L^p} \\ &\quad \le \left(\|u_0\phi\|_{L^p} + \|u_{0x}\phi\|_{L^p} + \|u_{0xx}\phi\|_{L^p}\right)e^{CMt} \quad \forall t \in [0,T]. \end{aligned} \tag{30}$$

Finally, we will treat the case $p = \infty$. We have $u_0, u_{0x}, u_{0xx} \in L^2 \cap L^\infty$ and $f(x) = f_N(x) \in L^\infty$. So we obtain

$$\begin{aligned} &\|uf\|_{L^q} + \|u_xf\|_{L^q} + \|u_{xx}f\|_{L^q} \\ &\quad \le \left(\|u_0f\|_{L^q} + \|u_{0x}f\|_{L^q} + \|u_{0xx}f\|_{L^q}\right)e^{CMt} \quad \forall t \in [0,T] \end{aligned} \tag{31}$$

for $q \in [2,\infty)$, where the last factor on the right-hand side is independent of q. Since $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$ for any $f \in L^2 \cap L^\infty$, we get

$$\begin{aligned} &\|uf\|_{L^\infty} + \|u_xf\|_{L^\infty} + \|u_{xx}f\|_{L^\infty} \\ &\quad \le \left(\|u_0f\|_{L^\infty} + \|u_{0x}f\|_{L^\infty} + \|u_{0xx}f\|_{L^\infty}\right)e^{CMt} \quad \forall t \in [0,T], \end{aligned} \tag{32}$$

where the last factor on the right-hand side is independent of N. Now taking the limit as $N \to \infty$ implies that estimate (31) remains valid for $p = \infty$. □

Remark 1 (1) Let $\phi = \phi_{0,0,c,0}$ with $c > 0$, and let $p = \infty$. Then Theorem 2.1 states that the condition

$$\left|u_0(x)\right| + \left|u_{0,x}(x)\right| + \left|u_{0,xx}(x)\right| \le C\left(1 + |x|\right)^{-c}$$

implies the uniform algebraic decay in $[0,T]$:

$$\left|u(t,x)\right| + \left|u_x(t,x)\right| + \left|u_{xx}(t,x)\right| \le C\left(1 + |x|\right)^{-c}.$$

It is shown that the algebraic decay rates of a strong solution to problem (4) are obtained.

(2) Let $\phi = \phi_{a,1,0,0}$ if $x \ge 0$ and $\phi(x) = 1$ if $x \le 0$ with $0 \le a < 1$. It is easy to see that such a weight satisfies the admissibility conditions of Definition 2.1. Moreover, let $p = \infty$ in Theorem 2.1. Then problem (4) preserves the pointwise decay $O(e^{-ax})$ as $x \to +\infty$ for each $t > 0$. Similarly, we have persistence of the decay $O(e^{-ax})$ as $x \to -\infty$.

Clearly, the limit case $\phi = \phi_{1,1,c,d}$ is not covered in Theorem 2.1. Furthermore, in the following theorem, we may choose the weight $\phi = \phi_{1,1,c,d}$ with $c < 0$, $d \in \mathbb{R}$, and $\frac{1}{|c|} < p \le \infty$. More generally, when $(1 + |\cdot|)^c \log(e + |\cdot|)^d \in L^p(\mathbb{R})$, Theorem 2.2 covers the case of fast growing weights, which means that when a v-moderate weight ϕ does not satisfy condition (8), we may establish a variant of Theorem 2.1, putting instead of assumption (8), the following weaker condition:

$$ve^{-|\cdot|} \in L^p(\mathbb{R}), \tag{33}$$

where $2 \le p \le \infty$.

Theorem 2.2 *Let $2 \le p \le \infty$, let ϕ be a ν-moderate weight function as in Definition 2.1 satisfying condition (33), and let the initial data $u_0 = u(0,x)$ satisfy*

$$u_0\phi, u_{0,x}\phi, u_{0,xx}\phi \in L^p(\mathbb{R}) \quad \textit{and} \quad u_0\phi^{\frac{1}{2}}, u_{0,x}\phi^{\frac{1}{2}}, u_{0,xx}\phi^{\frac{1}{2}} \in L^2(\mathbb{R}).$$

Then the strong solution u of the Cauchy problem for (4), emanating from u_0, satisfy

$$\sup_{t\in[0,T]} \left(\left\|u(t)\phi\right\|_{L^p} + \left\|u_x(t)\phi\right\|_{L^p} + \left\|u_{xx}(t)\phi\right\|_{L^p}\right) < \infty$$

and

$$\sup_{t\in[0,T]} \left(\left\|u(t)\phi^{\frac{1}{2}}\right\|_{L^2} + \left\|u_x(t)\phi^{\frac{1}{2}}\right\|_{L^2} + \left\|u_{xx}(t)\phi^{\frac{1}{2}}\right\|_{L^2}\right) < \infty.$$

Remark 2 Let $\phi = \phi_{1,1,0,0}(x) = e^{|x|}$ and $p = \infty$ in Theorem 2.2. If $|u_0(x)|$, $|u_{0,x}(x)|$, and $|u_{0,xx}|$ are bounded by $Ce^{-|x|}$, then the strong solution satisfies

$$\left|u(t,x)\right| + \left|u_x(t,x)\right| + \left|u_{xx}(t,x)\right| \le Ce^{-|x|}$$

uniformly in $[0,T]$.

Proof The assumption that ϕ is a ν-moderate weight function implies

$$\exists C_0 > 0, \quad \text{s.t.,} \quad \phi(x+y) \le C_0\nu(x)\nu(y) \quad \forall x, y \in \mathbb{R},$$

which, combined with $\inf_R \nu > 0$, gives

$$\phi^{\frac{1}{2}}(x+y) \le C_0^{\frac{1}{2}}\nu^{\frac{1}{2}}(x)\nu^{\frac{1}{2}}(y) \quad \forall x, y \in \mathbb{R},$$

that is, $\phi^{\frac{1}{2}}$ is a $\nu^{\frac{1}{2}}$-moderate weight function. The inequality $|\phi'(x)| \le A|\phi(x)|$ reads

$$\left|\left(\phi^{\frac{1}{2}}\right)'(x)\right| = \frac{1}{2}\phi^{-\frac{1}{2}}\left|\phi'(x)\right| \le \frac{A}{2}\phi^{\frac{1}{2}}.$$

By condition (33), $\nu e^{-|x|} \in L^p$. So the Hölder inequality yields

$$\left\|\nu^{\frac{1}{2}}e^{-|x|}\right\|_{L^1} \le \left\|\nu^{\frac{1}{2}}e^{\frac{-|x|}{2}}\right\|_{L^{2p}}\left\|e^{\frac{-|x|}{2}}\right\|_{L^q} \le c\left\|\nu e^{-|x|}\right\|_{L^p} < \infty, \quad q = \frac{2p}{2p-1}.$$

Thus Theorem 2.1 with $p = 2$ applied to the weight $\phi^{\frac{1}{2}}$ results in

$$\begin{aligned} &\left\|u\phi^{\frac{1}{2}}\right\|_{L^2} + \left\|u_x\phi^{\frac{1}{2}}\right\|_{L^2} + \left\|u_{xx}\phi^{\frac{1}{2}}\right\|_{L^2} \\ &\quad \le \left(\left\|u_0\phi^{\frac{1}{2}}\right\|_{L^2} + \left\|u_{0x}\phi^{\frac{1}{2}}\right\|_{L^2} + \left\|u_{0xx}\phi^{\frac{1}{2}}\right\|_{L^2}\right)e^{CMt} \quad \forall t \in [0,T]. \end{aligned} \tag{34}$$

From Lemma 2.2 and $f(x) = f_N(x) = \min\{\phi(x), N\}$, applying (33), we have

$$\begin{aligned} \left\|fG * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^p} &\lesssim \|fG\|_{L^p}\left\|f\left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^1} \\ &\lesssim c\left(\left\|f^{\frac{1}{2}}u_x\right\|_{L^2}^2 + \left\|f^{\frac{1}{2}}u_{xx}\right\|_{L^2}^2\right) \\ &\le Ce^{CMt}. \end{aligned} \tag{35}$$

Similarly, we have the estimates

$$\begin{aligned}\left\|fG_x * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^p} &\lesssim \|fG_x\|_{L^p}\left\|f\left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^1} \\ &\lesssim c\left(\left\|f^{\frac{1}{2}}u_x\right\|_{L^2}^2 + \left\|f^{\frac{1}{2}}u_{xx}\right\|_{L^2}^2\right) \\ &\le Ce^{CMt}\end{aligned} \tag{36}$$

and

$$\begin{aligned}\left\|fG_{xx} * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^p} &= \left\|fG * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^p} + \left\|f\left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^p} \\ &\lesssim \|fG\|_{L^p}\left\|f\left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^1} + \left\|f\left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^p} \\ &\lesssim c\left(\left\|f^{\frac{1}{2}}u_x\right\|_{L^2}^2 + \left\|f^{\frac{1}{2}}u_{xx}\right\|_{L^2}^2\right) + CM\left(\|u_xf\|_{L^p} + \|u_{xx}f\|_{L^p}\right) \\ &\le Ce^{CMt} + CM\left(\|u_xf\|_{L^p} + \|u_{xx}f\|_{L^p}\right).\end{aligned} \tag{37}$$

Here the constants on the right-hand side of (35)–(37) are independent of N. By using the procedure as in the proof of Theorem 2.1, we readily get

$$\frac{d}{dt}\|uf\|_{L^p} \le \left(\frac{M}{p} + AM\right)\|uf\|_{L^p} + \left\|fG * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^p}, \tag{38}$$

$$\frac{d}{dt}\|u_xf\|_{L^p} \le M\|u_xf\|_{L^p} + \left\|fG_x * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^p}, \tag{39}$$

and

$$\frac{d}{dt}\|u_{xx}f\|_{L^p} \le M\left(1 + A + \frac{1}{p}\right)\|u_{xx}f\|_{L^p} + \left\|fG_{xx} * \left[u_x^2 + \frac{1}{2}u_{xx}^2\right]\right\|_{L^p}. \tag{40}$$

Substituting (35), (36), and (37) into (38), (39), and (40), respectively, and summing up them, we have

$$\begin{aligned}&\frac{d}{dt}\left(\|uf\|_{L^p} + \|u_xf\|_{L^p} + \|u_{xx}f\|_{L^p}\right) \\ &\quad\le KM\left(\|uf\|_{L^p} + \|u_xf\|_{L^p} + \|u_{xx}f\|_{L^p}\right) + Ce^{CMt}.\end{aligned} \tag{41}$$

From Gronwall's inequality it follows that

$$\begin{aligned}&\|uf\|_{L^p} + \|u_xf\|_{L^p} + \|u_{xx}f\|_{L^p} \\ &\quad\le e^{KMt}\left(\|u_0f\|_{L^p} + \|u_{0x}f\|_{L^p} + \|u_{0xx}f\|_{L^p}\right) + Ce^{(C+K)Mt}.\end{aligned} \tag{42}$$

We obtain desired result by letting $N \to \infty$ in the case $2 \le p < \infty$. The constants throughout the proof are independent of p. So for $p = \infty$, we can obtain the result from that established for the finite exponents q by letting $q \to \infty$. The rest of the proof is fully similar to that of Theorem 2.1. □

3 Blow-up

3.1 Several lemmas

In this section, we study the sufficient conditions of blow-up solutions for problem (4) by using some classical methods. Firstly, we need several lemmas.

Lemma 3.1 ([3]) *Let $f \in C^1(\mathbb{R})$, $a > 0$, $b > 0$, and $f(0) > \sqrt{\frac{b}{a}}$. If $f'(t) \geq af^2(t) - b$, then*

$$f(t) \to +\infty \quad as\ t \to T = \frac{1}{2\sqrt{ab}} \log\left(\frac{f(0) + \sqrt{\frac{b}{a}}}{f(0) - \sqrt{\frac{b}{a}}}\right). \tag{43}$$

Lemma 3.2 ([13]) *Let $u_0 \in H^s$, $s \geq 5/2$. Then the corresponding solution u has the constant energy integral*

$$\int_R \left(u_x^2 + u_{xx}^2\right) dx = \int_R \left(u_{0x}^2 + u_{0xx}^2\right) dx = \|u_{0x}\|_{H^1}^2.$$

Lemma 3.3 ([13]) *Let $u_0 \in H^s$, $s \geq 5/2$. Let T be the lifespan of the solution to problem (4). Then the corresponding solution blows up in finite time if and only if*

$$\liminf_{t \to T} \inf_{x \in R} m = -\infty.$$

Remark 3 From Lemma 3.2 we see that $u(t,x)$ is bounded. This implies that the solution to problem (4) blows up if and only if

$$\lim_{t \to T} \|u_{xx}\|_{L^\infty} = +\infty.$$

3.2 Blow-up phenomenon

Theorem 3.1 *Let $u_0 \in H^s(\mathbb{R})$ for $s > \frac{3}{2}$. Let $u(t,x)$ be the corresponding solution of with the initial datum u_0. Suppose that the slope of u_0' satisfies*

$$\int_{\mathbb{R}} \left(u_0'\right)^3 < -\frac{K}{K_1}, \tag{44}$$

where $K^2 = \frac{(24+3\sqrt{2})^2}{16}\|u_0'\|_{H^1}^4$ and $K_1^2 = \frac{1}{4\|u_0'\|_{H^1}^2}$. Then there exists the lifespan $T < \infty$ such that the corresponding solution $u(t,x)$ blows up in finite time T with

$$T = \frac{1}{2KK_1} \log\left(\frac{K_1 h(0) + K}{K_1 h(0) - K}\right). \tag{45}$$

Proof Define $g(t) = u_x(t,x)$ and $h(t) = \int_{\mathbb{R}} g_x^3\, dx$. Then it follows that

$$g_t - gg_x = Q, \tag{46}$$

where $Q = \partial_x(1 - \partial_x^2)^{-1}(u_x^2 + \frac{1}{2}u_{xx}^2)$.

Differentiating equation (46) with respect to x yields

$$g_{tx} - gg_{xx} = \frac{1}{2}g_x^2 - g^2 + \left(1 - \partial_x^2\right)^{-1}\left(u_x^2 + \frac{1}{2}u_{xx}^2\right). \tag{47}$$

Multiplying by $3g_x^2$ both sides of (47) and integrating with respect to x over $\mathbb{R}$, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} g_x^3\,dx &= \frac{1}{2}\int_{\mathbb{R}} g_x^4\,dx - 3\int_{\mathbb{R}} g^2 g_x^2\,dx \\ &\quad + 3\int_{\mathbb{R}} g_x^2\big(1-\partial_x^2\big)^{-1}\big(u_x^2\big)\,dx + \frac{3}{2}\int_{\mathbb{R}} g_x^2\big(1-\partial_x^2\big)^{-1}\big(u_{xx}^2\big)\,dx \\ &= \Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4. \end{aligned}\tag{48}$$

Using Hölder's and Yong's inequalities, (5), and (48), we get

$$\begin{aligned}\Gamma_2 &= 3\int_{\mathbb{R}} g^2 g_x^2\,dx \\ &\le 3\left(\int_{\mathbb{R}} g^4\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}(g_x)^4\,dx\right)^{\frac{1}{2}} \\ &\le 3\|u_0'\|_{H^1}^2\left(\int_{\mathbb{R}}(g_x)^4\,dx\right)^{\frac{1}{2}} \\ &\le 3\left(\frac{\|u_0'\|_{H^1}^4}{2\epsilon}+\frac{\epsilon\int_{\mathbb{R}}(g_x)^4\,dx}{2}\right), \end{aligned}\tag{49}$$

$$\begin{aligned}\Gamma_3 &= 3\int_{\mathbb{R}} g_x^2\big(1-\partial_x^2\big)^{-1}\big(u_x^2\big)\,dx \\ &\le 3\big\|\big(1-\partial_x^2\big)^{-1}\big(u_x^2\big)\big\|_{L^2}\left(\int_{\mathbb{R}}(g_x)^4\,dx\right)^{\frac{1}{2}} \\ &\le 3\|u_0'\|_{H^1}^2\left(\int_{\mathbb{R}}(g_x)^4\,dx\right)^{\frac{1}{2}} \\ &\le 3\left(\frac{\|u_0'\|_{H^1}^4}{2\epsilon}+\frac{\epsilon\int_{\mathbb{R}}(g_x)^4\,dx}{2}\right), \end{aligned}\tag{50}$$

and

$$\begin{aligned}\Gamma_4 &= \frac{3}{2}\int_{\mathbb{R}} g_x^2\big(1-\partial_x^2\big)^{-1}\big(u_{xx}^2\big)\,dx \\ &\le \frac{3}{2}\big\|\big(1-\partial_x^2\big)^{-1}\big(u_{xx}^2\big)\big\|_{L^2}\left(\int_{\mathbb{R}}(g_x)^4\,dx\right)^{\frac{1}{2}} \\ &\le \frac{3\sqrt{2}}{4}\|u_0'\|_{H^1}^2\left(\int_{\mathbb{R}}(g_x)^4\,dx\right)^{\frac{1}{2}} \\ &\le \frac{3\sqrt{2}}{4}\left(\frac{\|u_0\|_{H^1}^4}{2\epsilon}+\frac{\epsilon\int_{\mathbb{R}}(g_x)^4\,dx}{2}\right). \end{aligned}\tag{51}$$

Combining inequalities (49)–(51), we obtain

$$|\Gamma_2|+|\Gamma_3|+|\Gamma_4| \le \frac{24+3\sqrt{2}}{8\epsilon}\|u_0\|_{H^1}^4 + \frac{(24+3\sqrt{2})\epsilon}{8}\int_{\mathbb{R}}(g_x)^4\,dx. \tag{52}$$

Choosing $\epsilon = \frac{2}{24+3\sqrt{3}}$ yields

$$\Gamma_2 + \Gamma_3 + \Gamma_4 \geq -\left(\frac{(24+3\sqrt{2})^2}{16}\|u_0\|_{H^1}^4 + \frac{1}{4}\int_{\mathbb{R}} (g_x)^4\,dx\right). \tag{53}$$

Therefore, combining (48) and (52), we get

$$\frac{d}{dt}\int_{\mathbb{R}} g_x^3\,dx \geq \frac{1}{4}\int_{\mathbb{R}} g_x^4\,dx - K^2, \tag{54}$$

where $K^2 = \frac{(24+3\sqrt{2})^2}{16}\|u_0'\|_{H^1}^4$. Using Hölder's inequality, we get

$$\left(\int_{\mathbb{R}} g_x^3\,dx\right)^2 \leq \int_{\mathbb{R}} g_x^2\,dx \int_R g_x^4\,dx \leq \|u_0'\|_{H^1}^2 \int_{\mathbb{R}} g_x^4\,dx. \tag{55}$$

Combining (54) and (55), we have

$$\frac{d}{dt}h(t) \geq -K_1^2 h^2(t) + K^2, \tag{56}$$

where $K_1^2 = \frac{1}{4\|u_0'\|_{H^1}^2}$.

From the assumption of the theorem we have that $h(0) > \frac{K}{K_1}$, and the continuity argument ensures that $h(t) > h(0)$. Lemma 3.1 (with $a = K_1^2$ and $b = K^2$) implies that $h(t) \to +\infty$ as $t \to T = \frac{1}{2K_1K}\log\frac{K_1h(0)+K}{K_1h(0)-K}$.

On the other hand, using the fact that

$$\int_{\mathbb{R}} g_x^3\,dx \leq \int_{\mathbb{R}} g_x^3\,dx \leq \|u_{xx}(t,x)\|_{L^\infty}\int_{\mathbb{R}} g_x^2\,dx = \|u_{xx}(t,x)\|_{L^\infty}\|u_0'\|_{H^1}^2, \tag{57}$$

Remark 3 implies the statement of Theorem 3.1.

The characteristics $q(t,x)$ related to problem (4) is governed by

$$q_t(t,x) = -u_x\big(t,q(t,x)\big), \quad t \in [0,T),$$
$$q(0,x) = x, \quad x \in \mathbb{R}.$$

Applying the classical results in the theory of ordinary differential equations, we can obtain that the characteristics $q(t,x) \in C^1([0,T) \times \mathbb{R})$ with $q_x(t,x) = e^{\int_0^t -u_{xx}(\tau,q(\tau,x))\,d\tau} > 0$ for all $(t,x) \in [0,T) \times \mathbb{R}$. Furthermore, it is shown in [28] that the potential $y = u - u_{xx}$ satisfies

$$y_x\big(t,q(t,x)\big)q_x(t,x) = y_0'(x)e^{\int_0^t u_{xx}(\tau,q(\tau,x))\,d\tau}. \tag{58}$$

Therefore we obtain the second blow-up result. □

Theorem 3.2 *Let $u_0 \in H^s(\mathbb{R})$, $s > \frac{5}{2}$. Suppose that there is a point $x_2 \in \mathbb{R}$ such that*

$$\frac{\sqrt{2}}{2}\left(u_{0x}^2(x_2) - \frac{1}{2}u_{0xx}^2(x_2)\right) + \frac{2+\sqrt{2}}{4}\|u_{0x}\|_{H^1}^2 < 0 \quad \text{and} \quad u_{0x} < \frac{\sqrt{2}}{2}u_{0xx}. \tag{59}$$

Then the blow-up occurs in finite time

$$T_0 = \frac{1}{\sqrt{\sqrt{2}+1}\|u_{0x}\|_{H^1}} \log\left(\frac{\sqrt{u_{0xx}^2 - 2u_{0x}^2} + \sqrt{\sqrt{2}+1}\|u_{0x}\|_{H^1}}{\sqrt{u_{0xx}^2 - 2u_{0x}^2} - \sqrt{\sqrt{2}+1}\|u_{0x}\|_{H^1}}\right).$$

Proof We track the dynamics of $P(t) = (u_x - \frac{\sqrt{2}}{2}u_{xx})(t, q(t, x_2))$ and $Q(t) = (u_x + \frac{\sqrt{2}}{2}u_{xx})(t, q(t, x_2))$ along the characteristics

$$\begin{aligned} P'(t) &= (u_{tx} + u_{xx}q_t) - \frac{\sqrt{2}}{2}(u_{txx} + u_{xxx}q_t) \\ &= \frac{\sqrt{2}}{2}PQ + \partial_x(1-\partial_x^2)^{-1}\left(u_x^2 + \frac{1}{2}u_{xx}^2\right) - \frac{\sqrt{2}}{2}(1-\partial_x^2)^{-1}\left(u_x^2 + \frac{1}{2}u_{xx}^2\right) \\ &\le \frac{\sqrt{2}}{2}PQ + \frac{2+\sqrt{2}}{4}\|u_0'\|_{H^1}^2 \end{aligned} \tag{60}$$

and

$$\begin{aligned} Q'(t) &= (u_{tx} + u_{xx}q_t) + \frac{\sqrt{2}}{2}(u_{txx} + u_{xxx}q_t) \\ &= -\frac{\sqrt{2}}{2}PQ + \partial_x(1-\partial_x^2)^{-1}\left(u_x^2 + \frac{1}{2}u_{xx}^2\right) + \frac{\sqrt{2}}{2}(1-\partial_x^2)^{-1}\left(u_x^2 + \frac{1}{2}u_{xx}^2\right) \\ &\ge -\frac{\sqrt{2}}{2}PQ - \frac{2+\sqrt{2}}{4}\|u_0'\|_{H^1}^2. \end{aligned} \tag{61}$$

From (59) we see that the right-hand side of (60) is positive and the right-hand side of (61) is negative initially. Hence P increases, and Q decreases. Then we obtain

$$P(t) < P(0) = u_{0x} - \frac{\sqrt{2}}{2}u_{0xx} < 0, \tag{62}$$

$$Q(t) > Q(0) = u_{0x} + \frac{\sqrt{2}}{2}u_{0xx} > 0. \tag{63}$$

Letting $h(t) = \sqrt{-PQ(t)}$ and using the estimate $\frac{Q-P}{2} \ge h(t)$, we have

$$\begin{aligned} h'(t) &= -\frac{P'Q + PQ'}{2\sqrt{-PQ}} \\ &\ge \frac{-(\frac{\sqrt{2}}{2}PQ + \frac{2+\sqrt{2}}{4}\|u_0'\|_{H^1}^2)Q + P(\frac{\sqrt{2}}{2}PQ + \frac{2+\sqrt{2}}{4}\|u_0'\|_{H^1}^2)}{2\sqrt{-PQ}} \\ &\ge \frac{-(\frac{\sqrt{2}}{2}PQ + \frac{2+\sqrt{2}}{4}\|u_0'\|_{H^1}^2)(Q-P)}{2\sqrt{-PQ}} \\ &\ge -\frac{\sqrt{2}}{2}PQ - \frac{2+\sqrt{2}}{4}\|u_0'\|_{H^1}^2 \\ &\ge \frac{\sqrt{2}}{2}h^2(t) - \frac{2+\sqrt{2}}{4}\|u_0'\|_{H^1}^2. \end{aligned} \tag{64}$$

In view of Lemma 3.1, we obtain that $h \to +\infty$ as $t \to T_0$ with

$$T_0 = \frac{1}{\sqrt{\sqrt{2}+1}\|u_{0x}\|_{H^1}} \log\left(\frac{\sqrt{2}h_0 + (\sqrt{\sqrt{2}+1})\|u_{0x}\|_{H^1}}{\sqrt{2}h_0 - (\sqrt{\sqrt{2}+1})\|u_{0x}\|_{H^1}}\right), \tag{65}$$

Observe that $h(t) = \sqrt{\frac{1}{2}u_{xx}^2 - u_x^2} < |\frac{\sqrt{2}}{2}u_{xx}(t,q(t,x_2))|$. Therefore $h \to +\infty$ as $t \to T_0$ implies that $|u_{xx}(t,q(t,x_2))| \to +\infty$ as $t \to T_0$.

The proof of Theorem 3.2 is completed. □

Theorem 3.3 *Let $u_0 \in H^s(\mathbb{R})$ for $s > \frac{5}{2}$. Suppose that there exists $x_3 \in \mathbb{R}$ such that $u_{0xx}(x_3) > \|u_0'\|_{H^1}$. Then the wave breaking occurs in finite time*

$$T^* = \frac{1}{\|u_0'\|_{H^1}} \log\left(\frac{u_{0xx}(x_2) + \|u_0'\|_{H^1}}{u_{0xx}(x_2) - \|u_0'\|_{H^1}}\right). \tag{66}$$

Proof Now we prove the wave-breaking phenomenon along the characteristics $q(t,x_3)$. It follows from (6) that

$$u_x'(t) = \partial_x\left(1-\partial_x^2\right)^{-1}\left(u_x^2 + \frac{1}{2}u_{xx}^2\right) \tag{67}$$

and

$$u_{xx}'(t) = \frac{1}{2}u_{xx}^2 - u_x^2 + \left(1-\partial_x^2\right)^{-1}\left(u_x^2 + \frac{1}{2}u_{xx}^2\right). \tag{68}$$

Since $(1-\partial_x^2)^{-1}(u_x^2 + \frac{1}{2}u_{xx}^2) \geq \frac{1}{2}u_x^2$, we get

$$u_{xx}'(t) \geq \frac{1}{2}u_{xx}^2 - \frac{1}{2}u_x^2. \tag{69}$$

Setting $M(t) = u_{xx}(t,q(t,x_3))$ and using Young's inequality, Lemmas 3.1 and 3.2, and (69), we get

$$M'(t) \geq \frac{1}{2}M^2 - K_2, \tag{70}$$

where $K_2 = \frac{1}{2}\|u_0'\|_{H^1}^2$.

Since by the assumption of Theorem 3.2, $u_{0xx}(x_3) > \|u_0'\|_{H^1}$, solving (70) results in

$$M \to +\infty \quad \text{as } t \to T^*, \tag{71}$$

where $T^* = \frac{1}{\|u_0'\|_{H^1}} \log(\frac{u_{0xx}(x_3)+\|u_0'\|_{H^1}}{u_{0xx}(x_3)-\|u_0'\|_{H^1}})$. □

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