Boundary Value Problems
a SpringerOpen Journal

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Conformable fractional versions of Hermite–Hadamard-type inequalities for twice-differentiable functions

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Abstract
In this paper, new inequalities for the left and right sides of the Hermite–Hadamard inequality are acquired for twice-differentiable mappings. Conformable fractional integrals are used to derive these inequalities. Furthermore, we provide our results by using special cases of obtained theorems.

MSC: 26D10; 26D15; 26A51

Keywords: Midpoint type inequality; Trapezoid type inequality; Fractional conformable integrals; Convex function

1 Introduction

Fractional calculus and the theory of inequalities have became the cornerstone of the literature in recent years. Fractional calculus is the answer to the question of whether fractional derivatives and fractional integrals can be taken. Therefore, it has offered solutions to many problems in many disciplines. The most famous of the fractional approaches that are developing day by day are the Riemann–Liouville, Caputo, and Conformable fractional approaches. It is well known that the theory of inequalities is one of the most important topics of recent research. In particular, its use in analysis, applied mathematics, and pure mathematics is very wide. One of the most studied of these inequalities is the Hermite–Hadamard inequality. The trapezoidal inequality, which is the right side of this inequality, and the Midpoint inequality, which is the left side, have pioneered many scientific studies.

Over the last century, several articles have been focused on acquiring trapezoid-type and midpoint-type inequalities that demonstrate the bounds via the right-hand side and left-hand side of the Hermite–Hadamard inequality, respectively. For example, Dragomir first acquired trapezoidal inequalities in [6], while Kırmacı first derived midpoint-type inequalities in [17]. Sarikaya et al. and Iqbal et al. presented some fractional trapezoid and midpoint-type inequalities for convex functions in [21] and [10], respectively.

New investigations have been focused on developing a class of fractional integral operators and their applicability in several scientific disciplines. With the help of only the derivative's fundamental limit formulation, a newly well-behaved straightforward fractional derivative known as the conformable derivative is derived in [15]. Some significant requirements that cannot be fulfilled by the Riemann–Liouville fractional operator and Caputo fractional operator definitions are fulfilled with the aid of the conformable derivative. Furthermore, in [1] the researchers showed that the conformable approach in [15] cannot yield good results when compared to the Caputo definition via specific mappings. This imperfection in the conformable description is avoided by some refinements of the conformable approach [9, 22].

Differentiable functions are an indispensable part of the scientific literature. In addition, many mathematicians have studied twice-differentiable functions and have guided the studies on this subject. In [3] and [4], there are inequalities established via twice-differentiable convex functions associated with Hadamard's inequality. Some generalized fractional integral inequalities of midpoint and trapezoid-type based on twice-differentiable convex mappings are established in [18]. In [19], the authors acquired some new inequalities of the Simpson and the Hermite–Hadamard type for mappings whose moduli of derivatives are convex. By using generalized fractional integrals in [5], the authors presented some midpoint- and trapezoid-type inequalities via mappings whose second derivatives in modulus are convex.

The purpose of this investigation is to derive some new trapezoid-type and midpoint-type inequalities with the help of the twice-differentiable mappings including conformable fractional integrals. We also show that the newly established outcomes are the generalization of the existing trapezoid-type and midpoint-type inequalities. The ideas and strategies via our outcomes concerning the right-hand side and the left-hand side of Hermite–Hadamard inequality based on conformable fractional integrals may open up new avenues for further research in this area.

This paper contains four sections along with the introduction. In the second section, some basic information that we will use in our outcomes is mentioned. Also, the definitions of the Riemann–Liouville integral and conformable fractional integral are recalled. The third section consists of two subsections. In the first subsection, trapezoid-type inequalities based on conformable fractional integrals are created for twice-differentiable functions, while in the other, midpoint-type inequalities based on conformable fractional integrals are established for twice-differentiable functions. In the last section, the conclusions obtained from the research are presented. In addition, ideas for future research are given.

2 Preliminaries

This section considers the basics for building our outcomes. Here, definitions of Riemann–Liouville integrals and conformable integrals, which are well known in the literature, are given. From fractional calculus theory, mathematical preliminaries will be presented as follows:

For $0 < x, y < \infty$ and $x, y \in \mathbb{R}$, the well-known *gamma function*, *beta function*, and *incomplete beta function* are defined as

$$\Gamma(x) := \int_0^\infty \mu^{x-1} e^{-\mu}\, d\mu,$$

$$\mathcal{B}(x, y) := \int_0^1 \mu^{x-1}(1-\mu)^{y-1}\, d\mu$$

and

$$\mathscr{B}(x,y,r) := \int_0^r \mu^{x-1}(1-\mu)^{y-1}\,d\mu,$$

respectively.

In [16], Kilbas et al. described fractional integrals, also called Riemann–Liouville integrals as follows:

Definition 1 ([16]) For $\mathscr{F} \in L_1[\sigma,\delta]$, the Riemann–Liouville integrals $J_{\sigma+}^{\beta}\mathscr{F}(x)$ and $J_{\delta-}^{\beta}\mathscr{F}(x)$ of order $\beta > 0$ are, respectively, given by

$$J_{\sigma+}^{\beta}\mathscr{F}(x) = \frac{1}{\Gamma(\beta)}\int_{\sigma}^{x}(x-\mu)^{\beta-1}\mathscr{F}(\mu)\,d\mu, \quad x > \sigma \tag{2.1}$$

and

$$J_{\delta-}^{\beta}\mathscr{F}(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\delta}(\mu-x)^{\beta-1}\mathscr{F}(\mu)\,d\mu, \quad x < \delta, \tag{2.2}$$

where Γ denotes the Gamma function. In the case $\beta = 1$, the Riemann–Liouville integrals reduce to the classical integrals.

In [13], Jarad et al. introduced the following fractional conformable integral operators. They also provided certain characteristics and relationships between these operators and several other fractional operators in the literature. The fractional conformable integral operators are defined as follows:

Definition 2 ([13]) For $\mathscr{F} \in L^1[\sigma,\delta]$, the fractional conformable integral operator ${}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathscr{F}(x)$ and ${}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathscr{F}(x)$ of order $\beta \in C$, $\mathrm{Re}(\beta) > 0$ and $\alpha \in (0,1]$ are, respectively, given by

$${}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathscr{F}(x) = \frac{1}{\Gamma(\beta)}\int_{\sigma}^{x}\left(\frac{(x-\sigma)^{\alpha}-(\mu-\sigma)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathscr{F}(\mu)}{(\mu-\sigma)^{1-\alpha}}\,d\mu, \quad \mu > \sigma \tag{2.3}$$

and

$${}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathscr{F}(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\delta}\left(\frac{(\delta-x)^{\alpha}-(\delta-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathscr{F}(\mu)}{(\delta-\mu)^{1-\alpha}}\,d\mu, \quad \mu < \delta. \tag{2.4}$$

Note that the fractional integral in (2.3) coincides with the Riemann–Liouville fractional integral in (2.1) when $\alpha = 1$. Moreover, the fractional integral in (2.4) coincides with the Riemann–Liouville fractional integral in (2.2) when $\alpha = 1$. For some recent results connected with fractional integral inequalities, see [2, 7, 8, 11, 12, 14, 23] and the references cited therein.

3 Principal outcomes

The resulting new Hermite–Hadamard-type inequalities are presented in this section. Conformable fractional integrals are used for doubly differentiable functions when obtaining these inequalities. These inequalities are examined under two separate subheadings. In the first subheading, midpoint-type inequalities, which are the left side of

Hermite–Hadamard inequalities, and trapezoid-type inequalities, which are the right side of Hermite–Hadamard inequalities, will be discussed in the other.

3.1 Inequalities of midpoint type involving conformable fractional integrals

In this subsection, midpoint-type inequalities are created for twice-differentiable functions with the help of conformable fractional integrals. To obtain these inequalities, let us first set up the following identity.

Lemma 1 *Let $\mathscr{F}:[\sigma,\delta]\to\mathbb{R}$ be a twice-differentiable mapping on (σ,δ) such that $\mathscr{F}''\in L_1[\sigma,\delta]$. Then, the following equality holds*:

$$\begin{aligned}
&\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathscr{F}\left(\frac{\sigma+\delta}{2}\right) \qquad (3.1)\\
&=\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left[\int_0^1\left(\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right)\mathscr{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\right.\\
&\quad\left.+\int_0^1\left(\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right)\mathscr{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)d\mu\right].
\end{aligned}$$

Proof With the help of integrating by parts, we obtain

$$\begin{aligned}
A_1&=\int_0^1\left(\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right)\mathscr{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\\
&=\frac{2}{\delta-\sigma}\left(\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right)\mathscr{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\Bigg|_0^1\\
&\quad+\frac{2}{\delta-\sigma}\int_0^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right]\mathscr{F}'\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\\
&=-\frac{2}{\delta-\sigma}\left(\int_0^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right)\mathscr{F}'\left(\frac{\sigma+\delta}{2}\right)\\
&\quad+\frac{2}{\delta-\sigma}\left\{\frac{2}{\delta-\sigma}\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right]\mathscr{F}\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\Bigg|_0^1\right.\\
&\quad\left.+\frac{2\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathscr{F}\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\right\}.
\end{aligned}$$

If we use the change of variables $x=\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta$, then we have

$$\begin{aligned}
A_1&=-\frac{2}{\delta-\sigma}\left(\int_0^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right)\mathscr{F}'\left(\frac{\sigma+\delta}{2}\right) \qquad (3.2)\\
&\quad-\left(\frac{2}{\delta-\sigma}\right)^2\frac{1}{\alpha^\beta}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\\
&\quad+\left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta+2}\frac{\Gamma(\beta+1)}{\Gamma(\beta)}\int_{\frac{\sigma+\delta}{2}}^{\delta}\left(\frac{(\frac{\delta-\sigma}{2})^\alpha-(\delta-x)^\alpha}{\alpha}\right)^{\beta-1}\frac{\mathscr{F}(x)}{(\delta-x)^{1-\alpha}}\mathscr{F}(x)\,dx\\
&=-\frac{2}{\delta-\sigma}\left(\int_0^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right)\mathscr{F}'\left(\frac{\sigma+\delta}{2}\right)
\end{aligned}$$

$$-\left(\frac{2}{\delta-\sigma}\right)^{2}\frac{1}{\alpha^{\beta}}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)$$
$$+\left(\frac{2}{\delta-\sigma}\right)^{2+\alpha\beta}\Gamma(\beta+1)^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right).$$

Then, similar to the foregoing process, we can easily obtain

$$\begin{aligned} A_2 &= \int_0^1\left(\int_\mu^1\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)\mathscr{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)d\mu \\ &= \frac{2}{\delta-\sigma}\left(\int_\mu^1\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right) \\ &\quad\times\mathscr{F}'\left(\frac{\sigma+\delta}{2}\right)-\left(\frac{2}{\delta-\sigma}\right)^{2}\frac{1}{\alpha^{\beta}}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right) \\ &\quad+\left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta+2}\Gamma(\beta+1)^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right). \end{aligned} \tag{3.3}$$

If (3.2) and (3.3) are added together and then multiplied by $\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}$, the proof of Lemma 1 is completed. □

Theorem 1 *Note that* $\mathscr{F}:[\sigma,\delta]\to\mathbb{R}$ *is a twice-differentiable function on* (σ,δ) *so that* $\mathscr{F}''\in L_1[\sigma,\delta]$. *Note also that* $|\mathscr{F}''|$ *is convex on* $[\sigma,\delta]$. *Then, the following inequality holds*:

$$\begin{aligned} &\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right| \\ &\quad\le\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\Upsilon_1(\alpha,\beta)\left(\left|\mathscr{F}''(\sigma)\right|+\left|\mathscr{F}''(\delta)\right|\right). \end{aligned}$$

Here,

$$\begin{aligned} \Upsilon_1(\alpha,\beta) &= \int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|d\mu \\ &= \frac{1}{\alpha^{\beta}}\int_0^1\left|1-\mu-\frac{1}{\alpha}\left(\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathscr{B}\left(\beta+1,\frac{1}{\alpha},1-(1-\mu)^{\alpha}\right)\right)\right|d\mu, \end{aligned} \tag{3.4}$$

where $\mathcal{B}$ *and* $\mathscr{B}$ *denote the beta function and incomplete beta function, respectively.*

Proof Let us start by taking the absolute values of both sides of (3.1), we have

$$\begin{aligned} &\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right| \\ &\quad\le\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left[\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|\left|\mathscr{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\right|d\mu\right. \\ &\quad\quad\left.+\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|\left|\mathscr{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|d\mu\right]. \end{aligned} \tag{3.5}$$

It is known that $|\mathscr{F}''|$ is convex on $[\sigma,\delta]$. Then, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right|\\
&\leq\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left[\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|\right.\\
&\quad\times\left(\frac{1-\mu}{2}\left|\mathscr{F}''(\sigma)\right|+\frac{1+\mu}{2}\left|\mathscr{F}''(\delta)\right|\right)d\mu\\
&\quad\left.+\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|\left(\frac{1+\mu}{2}\left|\mathscr{F}''(\sigma)\right|+\frac{1-\mu}{2}\left|\mathscr{F}''(\delta)\right|\right)d\mu\right]\\
&=\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left(\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|d\mu\right)\left(\left|\mathscr{F}''(\sigma)\right|+\left|\mathscr{F}''(\delta)\right|\right).
\end{aligned}$$

Hence, the proof of Theorem 1 is completed. □

Remark 1 If we set $\alpha=1$ in Theorem 1, then we have [5, Corollary 4.6].

Remark 2 Consider $\alpha=1$ and $\beta=1$ in Theorem 1. Then, Theorem 1 equals [20, Theorem 3].

Theorem 2 *Let $\mathscr{F}:[\sigma,\delta]\to\mathbb{R}$ be a twice-differentiable function on (σ,δ) such that $\mathscr{F}''\in L_1([\sigma,\delta])$ and $|\mathscr{F}''|^q$ is convex on $[\sigma,\delta]$ with $q>1$. Then, the following inequalities hold*:

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right|\\
&\quad\leq\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left(\Upsilon^{\beta}_{\alpha}(p)\right)^{\frac{1}{p}}\left[\left(\frac{|\mathscr{F}''(\sigma)|^q+3|\mathscr{F}''(\delta)|^q}{4}\right)^{1/q}\right.\\
&\quad\quad\left.+\left(\frac{3|\mathscr{F}''(\sigma)|^q+|\mathscr{F}''(\delta)|^q}{4}\right)^{1/q}\right]\\
&\quad\leq\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left(4\Upsilon^{\beta}_{\alpha}(p)\right)^{\frac{1}{p}}\left[\left|\mathscr{F}''(\sigma)\right|+\left|\mathscr{F}''(\delta)\right|\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and

$$\Upsilon^{\beta}_{\alpha}(p)=\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|^p d\mu.$$

Proof Let us start by using the Hölder inequality in (3.5). Then, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right|\\
&\quad\leq\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left[\left(\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|^p d\mu\right)^{\frac{1}{p}}\right.\\
&\quad\quad\times\left(\int_0^1\left|\mathscr{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|^p d\mu\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left|\mathscr{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\Bigg].$$

Since $|\mathscr{F}''|^q$ is convex on $[\sigma,\delta]$, we obtain

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^\beta\mathcal{J}^\alpha_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right|\\
&\leq\frac{(\delta-\sigma)^2\alpha^\beta}{8}\left(\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|^p d\mu\right)^{\frac{1}{p}}\\
&\times\left[\left(\int_0^1\left(\frac{1-\mu}{2}\left|\mathscr{F}''(\sigma)\right|^q+\frac{1+\mu}{2}\left|\mathscr{F}''(\delta)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\left(\frac{1+\mu}{2}\left|\mathscr{F}''(\sigma)\right|^q+\frac{1-\mu}{2}\left|\mathscr{F}''(\delta)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right]\\
&=\frac{(\delta-\sigma)^2\alpha^\beta}{8}\left(\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|^p d\mu\right)^{\frac{1}{p}}\\
&\times\left[\left(\frac{|\mathscr{F}''(\sigma)|^q+3|\mathscr{F}''(\delta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathscr{F}''(\sigma)|^q+|\mathscr{F}''(\delta)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Let us consider $\varpi_1=|\mathscr{F}''(\sigma)|^q$, $\varrho_1=3|\mathscr{F}''(\delta)|^q$, $\varpi_2=3|\mathscr{F}''(\sigma)|^q$, and $\varrho_2=|\mathscr{F}''(\delta)|^q$ and applying the inequality:

$$\sum_{k=1}^n(\varpi_k+\varrho_k)^s\leq\sum_{k=1}^n\varpi_k^s+\sum_{k=1}^n\varrho_k^s,\quad 0\leq s<1. \tag{3.6}$$

This finishes the proof of Theorem 2. □

Corollary 1 *If we assign $\alpha=1$ in Theorem 2, then we derive*

$$\begin{aligned}
&\left|\frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^\beta}\left[J^\beta_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+J^\beta_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right|\\
&\leq\frac{(\delta-\sigma)^2}{8}\left(\psi^\beta(p)\right)^{\frac{1}{p}}\left[\left(\frac{|\mathscr{F}''(\sigma)|^q+3|\mathscr{F}''(\delta)|^q}{4}\right)^{1/q}+\left(\frac{3|\mathscr{F}''(\sigma)|^q+|\mathscr{F}''(\delta)|^q}{4}\right)^{1/q}\right]\\
&\leq\frac{(\delta-\sigma)^2}{8}\left(4\psi^\beta(p)\right)^{\frac{1}{p}}\left[\left|\mathscr{F}''(\sigma)\right|+\left|\mathscr{F}''(\delta)\right|\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and

$$\psi^\beta(p)=\int_0^1\left|1-\mu+\frac{\mu^{\beta+1}-1}{\beta+1}\right|^p d\mu.$$

Remark 3 If we choose $\alpha=1$ and $\beta=1$ in Theorem 2, then Theorem 2 reduces to [5, Corollary 4.8].

Theorem 3 *Suppose that* $\mathscr{F}:[\sigma,\delta]\to\mathbb{R}$ *is a twice-differentiable function on* (σ,δ) *such that* $\mathscr{F}''\in L_1([\sigma,\delta])$. *If* $|\mathscr{F}''|^q$ *is convex on* $[\sigma,\delta]$ *with* $q\geq 1$, *then the following inequality holds*:

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^\beta\mathcal{J}^\alpha_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right|\\
&\quad\leq\frac{(\delta-\sigma)^2\alpha^\beta}{8}\left(\Upsilon_1(\alpha,\beta)\right)^{1-\frac{1}{q}}\left[\left(\frac{\Upsilon_1(\alpha,\beta)-\Upsilon_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\sigma)\right|^q\right.\right.\\
&\qquad\left.+\frac{\Upsilon_1(\alpha,\beta)+\Upsilon_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\delta)\right|^q\right)^{\frac{1}{q}}\\
&\qquad\left.+\left(\frac{\Upsilon_1(\alpha,\beta)+\Upsilon_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\sigma)\right|^q+\frac{\Upsilon_1(\alpha,\beta)-\Upsilon_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\delta)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Here, $\Upsilon_1(\alpha,\beta)$ *is defined as in* (3.4) *and*

$$\begin{aligned}
\Upsilon_2(\alpha,\beta)&=\int_0^1\mu\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|d\mu\\
&=\frac{1}{\alpha^\beta}\int_0^1\mu\left|1-\mu-\frac{1}{\alpha}\left(\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathscr{B}\left(\beta+1,\frac{1}{\alpha},1-(1-\mu)^\alpha\right)\right)\right|d\mu,
\end{aligned}$$

where $\mathcal{B}$ *and* $\mathscr{B}$ *denote the beta function and incomplete beta function, respectively.*

Proof Let us start by applying the power-mean inequality in (3.5). Then, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^\beta\mathcal{J}^\alpha_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right|\\
&\quad\leq\frac{(\delta-\sigma)^2\alpha^\beta}{8}\left[\left(\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|d\mu\right)^{1-\frac{1}{q}}\right.\\
&\qquad\times\left(\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|\left|\mathscr{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\\
&\qquad+\left(\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|d\mu\right)^{1-\frac{1}{q}}\\
&\qquad\left.\times\left(\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|\left|\mathscr{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $|\mathscr{F}''|^q$ is convex on $[\sigma,\delta]$, we obtain

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^\beta\mathcal{J}^\alpha_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right|\\
&\quad\leq\frac{(\delta-\sigma)^2\alpha^\beta}{8}\left(\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|d\mu\right)^{1-\frac{1}{q}}
\end{aligned}$$

$$\times\left[\left(\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|\frac{1-\mu}{2}\left|\mathscr{F}''(\sigma)\right|^q+\frac{1+\mu}{2}\left|\mathscr{F}''(\delta)\right|^q d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|\int_\mu^1\left[\frac{1}{\alpha^\beta}-\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|\frac{1+\mu}{2}\left|\mathscr{F}''(\sigma)\right|^q+\frac{1-\mu}{2}\left|\mathscr{F}''(\delta)\right|^q d\mu\right)^{\frac{1}{q}}\right].$$

It is clearly seen that

$$\left|\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^\beta\mathcal{J}^\alpha_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right|$$
$$\leq\frac{(\delta-\sigma)^2\alpha^\beta}{8}\left(\Upsilon_1(\alpha,\beta)\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\frac{\Upsilon_1(\alpha,\beta)-\Upsilon_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\sigma)\right|^q+\frac{\Upsilon_1(\alpha,\beta)+\Upsilon_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\delta)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\Upsilon_1(\alpha,\beta)+\Upsilon_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\sigma)\right|^q+\frac{\Upsilon_1(\alpha,\beta)-\Upsilon_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\delta)\right|^q\right)^{\frac{1}{q}}\right].\qquad\square$$

Corollary 2 *Let us consider* $\alpha=1$ *in Theorem* 3. *Then, the following inequality holds*:

$$\left|\frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^\beta}\left[J^\beta_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+J^\beta_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right|$$
$$\leq\frac{(\delta-\sigma)^2}{8}\left(\frac{1}{2}-\frac{1}{\beta+2}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\left(\frac{1}{6}-\frac{\beta+4}{4(\beta+2)(\beta+3)}\right)\left|\mathscr{F}''(\sigma)\right|^q+\left(\frac{1}{3}-\frac{3\beta+8}{4(\beta+2)(\beta+3)}\right)\left|\mathscr{F}''(\delta)\right|^q d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(\frac{1}{3}-\frac{3\beta+8}{4(\beta+2)(\beta+3)}\right)\left|\mathscr{F}''(\sigma)\right|^q+\left(\frac{1}{6}-\frac{\beta+4}{4(\beta+2)(\beta+3)}\right)\left|\mathscr{F}''(\delta)\right|^q d\mu\right)^{\frac{1}{q}}\right].$$

Remark 4 If we take $\alpha=1$ and $\beta=1$, then Theorem 3 becomes [19, Proposition 5].

3.2 Inequalities of trapezoid type involving conformable fractional integrals

In this subsection, inequalities of trapezoid-type are obtained for twice-differentiable functions. We use the conformable fractional integral operators to obtain these inequalities.

Lemma 2 *If* $\mathscr{F}:[\sigma,\delta]\to\mathbb{R}$ *is a twice-differentiable mapping on* (σ,δ) *such that* $\mathscr{F}''\in L_1[\sigma,\delta]$, *then the following equality holds*:

$$\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^\beta\mathcal{J}^\alpha_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]\tag{3.7}$$
$$=\frac{(\delta-\sigma)^2\alpha^\beta}{8}\left\{\int_0^1\left(\int_\mu^1\left[\frac{1-(1-s)^\alpha}{\alpha}\right]^\beta ds\right)\mathscr{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)d\mu\right.$$
$$\left.+\int_0^1\left(\int_\mu^1\left[\frac{1-(1-s)^\alpha}{\alpha}\right]^\beta ds\right)\mathscr{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)d\mu\right\}.$$

Proof By employing integration by parts, we have

$$\begin{aligned}
A_3 &= \int_0^1 \left(\int_\mu^1 \left[\frac{1-(1-s)^\alpha}{\alpha} \right]^\beta ds \right) \mathscr{F}''\left(\frac{1-\mu}{2}\sigma + \frac{1+\mu}{2}\delta \right) d\mu \\
&= \frac{2}{\delta-\sigma} \left(\int_\mu^1 \left[\frac{1-(1-s)^\alpha}{\alpha} \right]^\beta ds \right) \mathscr{F}'\left(\frac{1-\mu}{2}\sigma + \frac{1+\mu}{2}\delta \right) \Big|_0^1 \\
&\quad + \frac{2}{\delta-\sigma} \int_0^1 \left[\frac{1-(1-\mu)^\alpha}{\alpha} \right]^\beta \mathscr{F}'\left(\frac{1-\mu}{2}\sigma + \frac{1+\mu}{2}\delta \right) d\mu \\
&= -\frac{2}{\delta-\sigma} \left(\int_0^1 \left[\frac{1-(1-s)^\alpha}{\alpha} \right]^\beta ds \right) \mathscr{F}'\left(\frac{\sigma+\delta}{2} \right) \\
&\quad + \frac{2}{\delta-\sigma} \left\{ \frac{2}{\delta-\sigma} \left[\frac{1-(1-\mu)^\alpha}{\alpha} \right]^\beta \mathscr{F}\left(\frac{1-\mu}{2}\sigma + \frac{1+\mu}{2}\delta \right) \Big|_0^1 \right. \\
&\quad \left. - \frac{2\beta}{\delta-\sigma} \int_0^1 \left[\frac{1-(1-\mu)^\alpha}{\alpha} \right]^{\beta-1} (1-\mu)^{\alpha-1} \mathscr{F}\left(\frac{1-\mu}{2}\sigma + \frac{1+\mu}{2}\delta \right) d\mu \right\} \\
&= -\frac{2}{\delta-\sigma} \left(\int_0^1 \left[\frac{1-(1-s)^\alpha}{\alpha} \right]^\beta ds \right) \mathscr{F}'\left(\frac{\sigma+\delta}{2} \right) + \left(\frac{2}{\delta-\sigma} \right)^2 \frac{1}{\alpha^\beta} \mathscr{F}(\delta) \\
&\quad - \left(\frac{2}{\delta-\sigma} \right)^2 \frac{\Gamma(\beta+1)}{\Gamma(\beta)} \int_{\frac{\sigma+\delta}{2}}^\delta \left(\frac{1-(\frac{2}{\delta-\sigma}(\delta-x))^\alpha}{\alpha} \right)^{\beta-1} \\
&\quad \times \left(\frac{2}{\delta-\sigma}(\delta-x) \right)^{\alpha-1} \frac{2}{\delta-\sigma} \mathscr{F}(x)\, dx \\
&= -\frac{2}{\delta-\sigma} \left(\int_0^1 \left[\frac{1-(1-s)^\alpha}{\alpha} \right]^\beta ds \right) \mathscr{F}'\left(\frac{\sigma+\delta}{2} \right) + \left(\frac{2}{\delta-\sigma} \right)^2 \frac{1}{\alpha^\beta} \mathscr{F}(\delta) \\
&\quad - \left(\frac{2}{\delta-\sigma} \right)^{\alpha\beta+2} \frac{\Gamma(\beta+1)}{\Gamma(\beta)} \int_{\frac{\sigma+\delta}{2}}^\delta \left(\frac{(\frac{\delta-\sigma}{2})^\alpha - (\delta-x)^\alpha}{\alpha} \right)^{\beta-1} \frac{\mathscr{F}(x)}{(\delta-x)^{1-\alpha}} \mathscr{F}(x)\, dx \\
&= -\frac{2}{\delta-\sigma} \left(\int_0^1 \left[\frac{1-(1-s)^\alpha}{\alpha} \right]^\beta ds \right) \mathscr{F}'\left(\frac{\sigma+\delta}{2} \right) + \left(\frac{2}{\delta-\sigma} \right)^2 \frac{1}{\alpha^\beta} \mathscr{F}(\delta) \\
&\quad - \left(\frac{2}{\delta-\sigma} \right)^{\alpha\beta+2} \Gamma(\beta+1)^\beta \mathcal{J}_{\delta-}^\alpha \mathscr{F}\left(\frac{\sigma+\delta}{2} \right).
\end{aligned}$$

In a similar manner,

$$\begin{aligned}
A_4 &= \int_0^1 \left(\int_\mu^1 \left[\frac{1-(1-s)^\alpha}{\alpha} \right]^\beta ds \right) \mathscr{F}''\left(\frac{1+\mu}{2}\sigma + \frac{1-\mu}{2}\delta \right) d\mu \\
&= \frac{2}{\delta-\sigma} \left(\int_0^1 \left[\frac{1-(1-s)^\alpha}{\alpha} \right]^\beta ds \right) \mathscr{F}'\left(\frac{\sigma+\delta}{2} \right) + \left(\frac{2}{\delta-\sigma} \right)^2 \frac{1}{\alpha^\beta} \mathscr{F}(\delta) \\
&\quad - \left(\frac{2}{\delta-\sigma} \right)^{\alpha\beta+2} \Gamma(\beta+1)^\beta \mathcal{J}_{\sigma+}^\alpha \mathscr{F}\left(\frac{\sigma+\delta}{2} \right).
\end{aligned}$$

Then, it follows that

$$\frac{(\delta-\sigma)^2\alpha^\beta}{8}\{A_3 + A_4\}$$

$$= \frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2} - \frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right].$$

Thus, the proof of Lemma 2 is accomplished. □

Theorem 4 *Let $\mathscr{F}:[\sigma,\delta]\to\mathbb{R}$ be a twice-differentiable mapping on (σ,δ) such that $\mathscr{F}''\in L_1([\sigma,\delta])$. If $|\mathscr{F}''|$ is convex on $[\sigma,\delta]$, then the following expression holds*:

$$\begin{aligned}&\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2} - \frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\ &\quad\le \frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\Phi_1(\alpha,\beta)\big(\left|\mathscr{F}''(\sigma)\right|+\left|\mathscr{F}''(\delta)\right|\big).\end{aligned} \tag{3.8}$$

Here,

$$\begin{aligned}\Phi_1(\alpha,\beta) &= \int_0^1\left|\int_\mu^1\left[\frac{1-(1-s)^{\alpha}}{\alpha}\right]^{\beta}ds\right|d\mu \\ &= \frac{1}{\alpha^{\beta+1}}\int_0^1\left|\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathscr{B}\left(\beta+1,\frac{1}{\alpha},1-(1-\mu)^{\alpha}\right)\right|d\mu,\end{aligned}$$

where $\mathcal{B}$ and $\mathscr{B}$ denote the beta function and incomplete beta function, respectively.

Proof Taking the absolute values of both sides of (3.7), we derive

$$\begin{aligned}&\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2} - \frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\ &\quad\le \frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left\{\int_0^1\left|\int_\mu^1\left[\frac{1-(1-s)^{\alpha}}{\alpha}\right]^{\beta}ds\right|\left|\mathscr{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\right|d\mu\right. \\ &\qquad\left.+\int_0^1\left|\int_\mu^1\left[\frac{1-(1-s)^{\alpha}}{\alpha}\right]^{\beta}ds\right|\left|\mathscr{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|d\mu\right\}.\end{aligned} \tag{3.9}$$

If we use the convexity of the $|\mathscr{F}''|$ on $[\sigma,\delta]$, then we establish

$$\begin{aligned}&\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2} - \frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\ &\quad\le \frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left\{\int_0^1\left|\int_\mu^1\left[\frac{1-(1-s)^{\alpha}}{\alpha}\right]^{\beta}ds\right|\left[\frac{1-\mu}{2}\left|\mathscr{F}''(\sigma)\right|+\frac{1+\mu}{2}\left|\mathscr{F}''(\delta)\right|\right]d\mu\right. \\ &\qquad\left.+\int_0^1\left|\int_\mu^1\left[\frac{1-(1-s)^{\alpha}}{\alpha}\right]^{\beta}ds\right|\left[\frac{1+\mu}{2}\left|\mathscr{F}''(\sigma)\right|+\frac{1-\mu}{2}\left|\mathscr{F}''(\delta)\right|\right]d\mu\right\} \\ &\quad= \frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left(\int_0^1\left|\int_\mu^1\left[\frac{1-(1-s)^{\alpha}}{\alpha}\right]^{\beta}ds\right|d\mu\right)\big(\left|\mathscr{F}''(\sigma)\right|+\left|\mathscr{F}''(\delta)\right|\big).\end{aligned}$$

□

Remark 5 If $\alpha=1$ is chosen in (3.8), then Theorem 4 reduces to [5, Corolalry 3.6].

Remark 6 Let us consider $\alpha=1$ and $\beta=1$ in (3.8). Then, Theorem 4 is equal to [19, Proposition 2].

Theorem 5 *Let $\mathscr{F}:[\sigma,\delta]\to\mathbb{R}$ be a twice-differentiable function on (σ,δ) such that $\mathscr{F}''\in L_1[\sigma,\delta]$ with $\sigma<\delta$. If $|\mathscr{F}''|^q$ is convex on $[\sigma,\delta]$ with $q>1$, then the following double inequality holds*:

$$\begin{aligned}
&\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \qquad (3.10)\\
&\quad\le\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left(\Theta^{\beta}_{\alpha}(p)\right)^{\frac{1}{p}}\left[\left(\frac{|\mathscr{F}''(\sigma)|^q+3|\mathscr{F}''(\delta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathscr{F}''(\sigma)|^q+|\mathscr{F}''(\delta)|^q}{4}\right)^{\frac{1}{q}}\right]\\
&\quad\le\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left(4\Theta^{\beta}_{\alpha}(p)\right)^{\frac{1}{p}}\left[\left|\mathscr{F}''(\sigma)\right|^q+\left|\mathscr{F}''(\delta)\right|^q\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\Theta^{\beta}_{\alpha}(p)=\int_0^1\left|\int_{\mu}^1\left[\frac{1-(1-s)^{\alpha}}{\alpha}\right]^{\beta}ds\right|^p d\mu.$$

Proof Using the Hölder inequality in (3.9), we have

$$\begin{aligned}
&\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\quad\le\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left\{\left(\int_0^1\left|\int_{\mu}^1\left[\frac{1-(1-s)^{\alpha}}{\alpha}\right]^{\beta}ds\right|^p d\mu\right)^{\frac{1}{p}}\right.\\
&\qquad\times\left(\int_0^1\left|\mathscr{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\\
&\qquad\left.+\left(\int_0^1\left|\int_{\mu}^1\left[\frac{1-(1-s)^{\alpha}}{\alpha}\right]^{\beta}ds\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathscr{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $|\mathscr{F}''|^q$ is convex on $[\sigma,\delta]$, we obtain

$$\begin{aligned}
&\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\quad\le\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left(\int_0^1\left|\int_{\mu}^1\left[\frac{1-(1-s)^{\alpha}}{\alpha}\right]^{\beta}ds\right|^p d\mu\right)^{\frac{1}{p}}\\
&\qquad\times\left[\left(\int_0^1\left(\frac{1-\mu}{2}\left|\mathscr{F}''(\sigma)\right|^q+\frac{1+\mu}{2}\left|\mathscr{F}''(\delta)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\int_0^1\left(\frac{1+\mu}{2}\left|\mathscr{F}''(\sigma)\right|^q+\frac{1-\mu}{2}\left|\mathscr{F}''(\delta)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right]\\
&\quad=\frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left(\int_0^1\left|\int_{\mu}^1\left[\frac{1-(1-s)^{\alpha}}{\alpha}\right]^{\beta}ds\right|^p d\mu\right)^{\frac{1}{p}}\\
&\qquad\times\left[\left(\frac{|\mathscr{F}''(\sigma)|^q+3|\mathscr{F}''(\delta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathscr{F}''(\sigma)|^q+|\mathscr{F}''(\delta)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Let us consider $\varpi_1=|\mathscr{F}''(\sigma)|^q$, $\varrho_1=3|\mathscr{F}''(\delta)|^q$, $\varpi_2=3|\mathscr{F}''(\sigma)|^q$ and $\varrho_2=|\mathscr{F}''(\delta)|^q$ and with the help of the inequality (3.6). Finally, the proof of Theorem 5 is completed. □

Remark 7 If we choose $\alpha = 1$ in Theorem 5, then we derive

$$\begin{aligned}
&\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta}}\left[J_{\delta-}^{\beta}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+J_{\sigma+}^{\beta}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\
&\quad\leq \frac{(\delta-\sigma)^2}{8(\beta+1)}\left(\frac{p(\beta+1)}{p(\beta+1)+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\mathscr{F}''(\sigma)|^q+3|\mathscr{F}''(\delta)|^q}{4}\right)^{\frac{1}{q}}\right. \\
&\quad\quad\left.+\left(\frac{3|\mathscr{F}''(\sigma)|^q+|\mathscr{F}''(\delta)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\quad\leq \frac{(\delta-\sigma)^2}{8(\beta+1)}\left(\frac{4p(\beta+1)}{p(\beta+1)+1}\right)^{\frac{1}{p}}\left[\left|\mathscr{F}''(\sigma)\right|^q+\left|\mathscr{F}''(\delta)\right|^q\right],
\end{aligned}$$

which is given in [5, Corollary 3.9].

Proof It will be sufficient to write down the solution of the integral below,

$$\Theta_1^{\beta}(p)=\int_0^1\left|\frac{1}{\beta+1}-\frac{\mu^{\beta+1}}{\beta+1}\right|^p d\mu.$$

Under conditions $A > B > 0$ and $p > 1$, the following inequality is satisfied

$$|A-B|^p \leq A^p - B^p. \tag{3.11}$$

From the inequality (3.11), we have

$$\begin{aligned}
\Theta_1^{\beta}(p)&=\frac{1}{(\beta+1)^p}\int_0^1\left|1-\mu^{\beta+1}\right|^p d\mu \leq \frac{1}{(\beta+1)^p}\left(\int_0^1\left(1-\mu^{p(\beta+1)}\right)d\mu\right) \\
&=\frac{1}{(\beta+1)^p}\left(\frac{p(\beta+1)}{p(\beta+1)+1}\right).
\end{aligned}$$

When the solution of $\Theta_{\alpha}^{\beta}(p)$ is substituted for (3.10), the proof is finished. □

Corollary 3 *If we take $\alpha = 1$ and $\beta = 1$ in Theorem 5, then we obtain*

$$\begin{aligned}
&\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathscr{F}(x)\,dx\right| \\
&\quad\leq \frac{(\delta-\sigma)^2}{16}\left(\frac{2p}{2p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\mathscr{F}''(\sigma)|^q+3|\mathscr{F}''(\delta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathscr{F}''(\sigma)|^q+|\mathscr{F}''(\delta)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\quad\leq \frac{(\delta-\sigma)^2}{16}\left(\frac{8p}{2p+1}\right)^{\frac{1}{p}}\left[\left|\mathscr{F}''(\sigma)\right|^q+\left|\mathscr{F}''(\delta)\right|^q\right].
\end{aligned}$$

Theorem 6 *Assume that $\mathscr{F}:[\sigma,\delta]\to\mathbb{R}$ is a twice-differentiable mapping on (σ,δ) such that $\mathscr{F}''\in L_1[\sigma,\delta]$ and $|\mathscr{F}''|^q$ is convex on $[\sigma,\delta]$ with $q \geq 1$. Then, the following inequality holds*:

$$\begin{aligned}
&\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\
&\quad\leq \frac{(\delta-\sigma)^2\alpha^{\beta}}{8}\left(\Phi_1(\alpha,\beta)\right)^{1-\frac{1}{q}}\left[\left(\frac{\Phi_1(\alpha,\beta)-\Phi_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\sigma)\right|^q\right.\right.
\end{aligned}$$

$$+\frac{\Phi_1(\alpha,\beta)+\Phi_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\delta)\right|^q\Bigg)^{\frac{1}{q}}$$

$$+\left(\frac{\Phi_1(\alpha,\beta)+\Phi_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\sigma)\right|^q+\frac{\Phi_1(\alpha,\beta)-\Phi_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\delta)\right|^q\right)^{\frac{1}{q}}\Bigg].$$

Here,

$$\begin{aligned}\Phi_2(\alpha,\beta)&=\int_0^1\mu\left|\int_\mu^1\left[\frac{1-(1-s)^\alpha}{\alpha}\right]^\beta ds\right|d\mu\\&=\frac{1}{\alpha^{\beta+1}}\int_0^1\mu\left|\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathscr{B}\left(\beta+1,\frac{1}{\alpha},1-(1-\mu)^\alpha\right)\right|d\mu,\end{aligned}$$

where $\mathcal{B}$ and $\mathscr{B}$ denote the beta function and incomplete beta function, respectively.

Proof With the help of the power-mean inequality in (3.9), we have

$$\begin{aligned}&\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^\beta\mathcal{J}^\alpha_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\&\le\frac{(\delta-\sigma)^2\alpha^\beta}{8}\left[\left(\int_0^1\left|\int_\mu^1\left[\frac{1-(1-s)^\alpha}{\alpha}\right]^\beta ds\right|d\mu\right)^{1-\frac{1}{q}}\right.\\&\quad\times\left(\int_0^1\left|\int_\mu^1\left[\frac{1-(1-s)^\alpha}{\alpha}\right]^\beta ds\right|\left|\mathscr{F}''\left(\frac{1-\mu}{2}\sigma+\frac{1+\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\\&\quad+\left(\int_0^1\left|\int_\mu^1\left[\frac{1-(1-s)^\alpha}{\alpha}\right]^\beta ds\right|d\mu\right)^{1-\frac{1}{q}}\\&\quad\left.\times\left(\int_0^1\left|\int_\mu^1\left[\frac{1-(1-s)^\alpha}{\alpha}\right]^\beta ds\right|\left|\mathscr{F}''\left(\frac{1+\mu}{2}\sigma+\frac{1-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}}\right].\end{aligned}$$

Since $|\mathscr{F}''|^q$ is convex on $[\sigma,\delta]$, we obtain

$$\begin{aligned}&\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{\delta-}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+{}^\beta\mathcal{J}^\alpha_{\sigma+}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\&\le\frac{(\delta-\sigma)^2\alpha^\beta}{8}\left[\left(\int_0^1\left|\int_\mu^1\left[\frac{1-(1-s)^\alpha}{\alpha}\right]^\beta ds\right|d\mu\right)^{1-\frac{1}{q}}\right.\\&\quad\times\left[\left(\int_0^1\left|\int_\mu^1\left[\frac{1-(1-s)^\alpha}{\alpha}\right]^\beta ds\right|\left(\frac{1-\mu}{2}\left|\mathscr{F}''(\sigma)\right|^q+\frac{1+\mu}{2}\left|\mathscr{F}''(\delta)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right.\\&\quad\left.+\left(\int_0^1\left|\int_\mu^1\left[\frac{1-(1-s)^\alpha}{\alpha}\right]^\beta ds\right|\left(\frac{1+\mu}{2}\left|\mathscr{F}''(\sigma)\right|^q+\frac{1-\mu}{2}\left|\mathscr{F}''(\delta)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right]\\&=\frac{(\delta-\sigma)^2\alpha^\beta}{8}\left(\Phi_1(\alpha,\beta)\right)^{1-\frac{1}{q}}\left[\left(\frac{\Phi_1(\alpha,\beta)-\Phi_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\sigma)\right|^q\right.\right.\\&\quad\left.+\frac{\Phi_1(\alpha,\beta)+\Phi_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\delta)\right|^q\right)^{\frac{1}{q}}\\&\quad\left.+\left(\frac{\Phi_1(\alpha,\beta)+\Phi_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\sigma)\right|^q+\frac{\Phi_1(\alpha,\beta)-\Phi_2(\alpha,\beta)}{2}\left|\mathscr{F}''(\delta)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Finally, we obtain the required result. □

Corollary 4 *Let us consider* $\alpha = 1$ *in Theorem* 6. *Then, we derive*

$$\begin{aligned} &\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta}}\left[J_{\delta-}^{\beta}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)+J_{\sigma+}^{\beta}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\ &\quad\leq \frac{(\delta-\sigma)^2}{8}\left(\frac{1}{\beta+2}\right)^{1-\frac{1}{q}}\left[\left(\frac{\beta+4}{4(\beta+2)(\beta+3)}\left|\mathscr{F}''(\sigma)\right|^q+\frac{3\beta+8}{2(\beta+2)(\beta+3)}\left|\mathscr{F}''(\delta)\right|^q\right)^{\frac{1}{q}}\right. \\ &\qquad\left.+\left(\frac{3\beta+8}{2(\beta+2)(\beta+3)}\left|\mathscr{F}''(\sigma)\right|^q+\frac{\beta+4}{4(\beta+2)(\beta+3)}\left|\mathscr{F}''(\delta)\right|^q\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Remark 8 If we take $\alpha = 1$ and $\beta = 1$ in Theorem 6, then Theorem 6 becomes [19, Proposition 6].

4 Concluding remarks

In this paper, new inequalities are established, including the Conformable fractional integral, a construct that generalizes inequalities obtained with the Riemann integral and inequalities created with the help of Riemann–Liouville fractional integrals. It will appeal to readers that the inequalities produced in the research include both Conformable fractional integrals and twice-differentiable functions. To the best of our knowledge, these results are new in the literature. We hope that the ideas and techniques of this paper will inspire interested readers working in this field. With the techniques used in the obtained inequalities, different types of fractional integrals can be used to obtain new inequalities in the future. In addition, new inequalities can be obtained by considering different order derivatives of the functions.

Funding
There is no funding.

Availability of data and materials
Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Conceptualization, F.H. and H.B.; investigation, H.K. and H.B.; methodology, F.H.; validation, H.K. and F.H.; visualization, H.B. and F.H.; writing-original draft, H.K. and F.H.; writing-review and editing, H.B. All authors read and approved the final manuscript.

Received: 10 November 2022 Accepted: 16 April 2023 Published online: 24 April 2023

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