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Nonexistence results of solutions of parabolic-type equations and systems on the Heisenberg group

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Abstract
In this paper, we present necessary conditions for the existence of weak solution of the parabolic-type equations and systems on the Heisenberg group. The main technique for proving the results relies on the method of test functions.

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1 Introduction

In this paper, we are interested in the nonexistence of nontrivial solutions of pseudo parabolic-type equations and systems in the Heisenberg group. Besides their intrinsic interest, nonexistence results are a useful tool for proving related existence theorems for the corresponding Dirichlet problem on bounded domains.

To better describe the setting, we recall the seminal work of Fujita. In his paper [6], Fujita studied the following nonlinear heat equation:

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = u^{1+p}(x,t), & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^N. \end{cases} \tag{1}$$

He showed the following results. If $0 < p < \frac{2}{N}$, then a solution of problem (1) blows up in finite time for $N > 2$ while being globally well-posed for $p > \frac{2}{N}$.

One of the further generalizations of problem (1) is considering the fractional Laplacian $(-\Delta)^s$ instead of the classical one $(-\Delta)$. For example, in [9], the authors considered the Cauchy problem

$$\begin{cases} u_t(x,t) + (-\Delta)^s u(x,t) = a(x,t)|u(x,t)|^{1+p}(x,t), & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^N, \end{cases} \tag{2}$$

for $s > 0$.

As a generalization to the Heisenberg group, Ahmad and Alsaedi in [1] considered the following problem:

$$\frac{\partial u}{\partial t} + (-\Delta_H)^{\frac{\alpha}{2}} |u|^m = |u|^p, \tag{3}$$

where $p > 1$, supplemented with the initial data

$$u(\eta, 0) = u_0(\eta), \qquad \frac{\partial u}{\partial t}(\eta, 0) = u_1(\eta),$$

where $(-\Delta_H)^s$ is the fractional Kohn–Laplacian, $s \in (0,1)$, $p > 1$.

In this paper, we study nonexistence of all nontrivial weak solutions of pseudo parabolic-type equation of the type

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta_H)^{\frac{\alpha}{2}} u - (-\Delta_H)^{\frac{\alpha}{2}} u_t = |\eta|_H^{\gamma} |u|^p, \\ u(\eta, 0) = u_0(\eta) \geq 0, \quad \eta \in H^n, \end{cases} \tag{4}$$

where $p > 1$, and γ is a real number, $|\eta|_H$ is defined in (8), the operator $(-\Delta_H)^{\frac{\alpha}{2}}$ $(0 < \alpha < 2)$ accounts for anomalous diffusion (see below for the definition).

We also present the critical exponent for the pseudo parabolic-type system

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta_H)^{\frac{\alpha}{2}} u - (-\Delta_H)^{\frac{\alpha}{2}} u_t = |\eta|_H^{\gamma_1} |v|^q, \\ \frac{\partial v}{\partial t} + (-\Delta_H)^{\frac{\alpha}{2}} v - (-\Delta_H)^{\frac{\alpha}{2}} v_t = |\eta|_H^{\gamma_2} |u|^p, \\ u(\eta, 0) = u_0(\eta) \geq 0, \quad \eta \in H^n, \\ v(\eta, 0) = v_0(\eta) \geq 0, \quad \eta \in H^n, \end{cases} \tag{5}$$

where $p > 1$, $q > 1$.

In recent years, increasing attention has been given to the analysis and PDEs, An important aspect most directly related to the present work is the fine analysis of blow-up solutions of the nonlinear elliptic equations, see [14, 16]. In particular, equations on the Heisenberg have been widely studied, see [2, 3, 5, 7, 8, 10, 11, 13, 15, 18–20] and the references therein. In [12], the authors studied the Kirchhoff elliptic, parabolic, and hyperbolic-type equations on the Heisenberg group, In addition, the analogous results have been transferred to the cases of systems. Later, Zheng [21] established Liouville theorems for the following system of differential inequalities:

$$\begin{cases} \Delta_H u^{m_1} + |\eta|_H^{\gamma_1} |v|^p \leq 0, \\ \Delta_H v^{m_2} + |\eta|_H^{\gamma_2} |u|^q \leq 0 \end{cases} \tag{6}$$

on different unbounded open domains of Heisenberg group H^n, including the whole space and the half space of H^n.

2 Preliminaries

In this section we recall some basic facts regarding the Heisenberg vector fields and fractional powers of subelliptic Laplacian in the Heisenberg group, which will be used in the sequel.

The Heisenberg group H^n can be identified with $(\mathbb{R}^{2n+1}, \circ)$, where $2n+1$ stands for the topological dimension and the group multiplication "$\circ$" is defined by

$$\hat{\xi} \circ \xi := \left(x + \hat{x}, y + \hat{y}, t + \hat{t} + 2 \sum_{i=1}^{n} x_i \hat{y}_i - y_i \hat{x}_i \right)$$

for any $\xi = (x, y, t)$, $\hat{\xi} = (\hat{x}, \hat{y}, \hat{t})$ in H^n with $x = (x_1, \dots, x_n)$, $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$, $y = (y_1, \dots, y_n)$, and $\hat{y} = (\hat{y}_1, \dots, \hat{y}_n)$ denoting the elements of $\mathbb{R}^n$. Moreover, they are homogeneous of degree one with respect to the dilations

$$\delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t) \quad (\lambda > 0). \tag{7}$$

We consider the norm on H^n defined by

$$|\xi|_H := \rho(\xi) = \left[\left(\sum_{i=1}^{n} x_i^2 + y_i^2 \right)^2 + t^2 \right]^{\frac{1}{4}} \tag{8}$$

and the associated Heisenberg distance

$$d_H(\xi, \hat{\xi}) = \rho(\hat{\xi}^{-1} \circ \xi),$$

where $\hat{\xi}^{-1}$ is the inverse of $\hat{\xi}$ with respect to "$\circ$", i.e., $\hat{\xi}^{-1} = -\hat{\xi}$. Let $D_R(\xi)$ denote the Koranyi ball with center at ξ and radius R associated with the gauge distance $d_H(\xi, \hat{\xi}) = \rho(\hat{\xi}^{-1} \circ \xi)$, and we will refer to it as Heisenberg ball. For every $\xi \in H^n$ and $R > 0$, we will use the notation

$$D_R(\xi) := \{ \eta \in H^n | d_H(\xi, \eta) < R \},$$

it follows that

$$|D_R(\xi)| = |D_R(0)| = |D_1(0)| R^Q,$$

where $|D_1(0)|$ is the volume of the unit Heisenberg ball under Haar measure, which is equivalent to $2n+1$ dimensional Lebesgue measure of $\mathbb{R}^{2n+1}$. The n-dimensional Heisenberg algebra is the Lie algebra spanned by the left-invariant vector fields

$$\begin{aligned} X_i &:= \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad i = 1, \dots, n, \\ Y_i &:= \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, \dots, n, \\ T &:= \frac{\partial}{\partial t}. \end{aligned} \tag{9}$$

The Heisenberg gradient, or the horizontal gradient of a regular function u, is then defined by

$$\nabla_H u := (X_1 u, \dots, X_n u, Y_1 u, \dots, Y_n u).$$

While its Heisenberg Hessian matrix is

$$\nabla_H^2 u = \left[\begin{array}{ccc|ccc} X_1X_1u & \cdots & X_nX_1u & Y_1X_1u & \cdots & Y_nX_1u \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_1X_nu & \cdots & X_nX_nu & Y_1X_nu & \cdots & Y_nX_nu \\ \hline X_1Y_1u & \cdots & X_nY_1u & Y_1Y_1u & \cdots & Y_nY_1u \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_1Y_nu & \cdots & X_nY_nu & Y_1Y_nu & \cdots & Y_nY_nu \end{array}\right].$$

Consider the vector fields X_j, Y_j for $j = 1, \dots, n$ in (9), the sub-Laplacian on the Heisenberg group is a linear differential operator of the second order defined by

$$\begin{aligned} \Delta_H u &:= \sum_{j=1}^n X_j^2 u + Y_j^2 u \\ &= \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} + \frac{\partial^2 u}{\partial y_j^2} + 4y_j \frac{\partial^2 u}{\partial x_j \partial t} - 4x_j \frac{\partial^2 u}{\partial y_j \partial t} + 4\left(x_j^2 + y_j^2\right) \frac{\partial^2 u}{\partial t^2}. \end{aligned} \tag{10}$$

A natural group of dilatations on H^n is given by

$$\delta_\lambda(\eta) = (\lambda x, \lambda y, t), \quad \lambda > 0,$$

whose Jacobian determinant is λ^Q, where

$$Q = 2n + 2$$

is the homogeneous dimension of H^n.

Here, we recall a result on fractional powers of sub-Laplacian in the Heisenberg group taken from [4]. Let $\mathcal{N}(t,x)$ be the fundamental solution of $-\Delta_H + \frac{\partial}{\partial t}$. For all $0 < \beta < 4$, the integral

$$R_\beta(x) = \frac{1}{\Gamma(\frac{\beta}{2})} \int_0^{+\infty} t^{\frac{\beta}{2}-1} \mathcal{N}(t,x)\, dt$$

converges absolutely for $x \neq 0$. If $\beta < 0$, $\beta \notin \{0, -2, -4, \dots\}$, then

$$\tilde{R}_\beta(x) = \frac{\frac{\beta}{2}}{\Gamma(\frac{\beta}{2})} \int_0^{+\infty} t^{\frac{\beta}{2}-1} \mathcal{N}(t,x)\, dt$$

defines a smooth function in $H^n \setminus \{0\}$ since $t \mapsto \mathcal{N}(t,x)$ vanishes of infinite order as $t \to 0$ if $x \neq 0$. In addition, $\tilde{R}_\beta$ is positive and homogeneous of degree $\beta - 4$.

Lemma 2.1 *If u belongs to the Schwartz class $S(H^n)$ (see [5]) and $0 < \alpha < 2$, then $(-\Delta_H)^s u \in L^2(H)$ and*

$$\begin{aligned} (-\Delta_H)^s u(x) &= \int_H \left(u(x \circ y) - u(x) - \chi(y)\langle \nabla_H u(x), y\rangle\right) \tilde{R}_{-2s}(y)\, dy \\ &= P.V. \int_H \left(u(y) - u(x)\right) \tilde{R}_{-2s}\left(y^{-1} \circ x\right) dy, \end{aligned} \tag{11}$$

where P.V. is the Cauchy principal value and χ is the characteristic function of the unit ball $B_\rho(0,1)$, ($\rho(x) = R_{2-\alpha}^{\frac{-1}{2+\alpha}}(x)$, $0 < \alpha < 2$, ρ is an H-homogeneous norm in H^n smooth outside the origin).

3 Main results

Definition 3.1 A locally integrable function $u \in L^p_{\mathrm{loc}}(\Omega)$ is called a local weak solution to (4) in $\Omega = H^n \times (0,T)$ with the initial data $0 \le u_0(\eta) \in L^1_{\mathrm{loc}}(H^n)$ if the equality

$$\begin{aligned}
&\int_\Omega |\eta|_H^\gamma |u|^p \psi \, d\eta \, dt + \int_{H^n} u_0(\eta)\psi(\eta,0)\, d\eta \\
&\quad = -\int_\Omega u\psi_t \, d\eta\, dt + \int_\Omega u(-\Delta_H)^{\frac{\alpha}{2}}\psi \, d\eta \, dt \\
&\qquad + \int_\Omega u(-\Delta_H)^{\frac{\alpha}{2}}\psi_t \, d\eta\, dt + \int_{H^n} u_0(\eta)(-\Delta_H)^{\frac{\alpha}{2}}\psi(\eta,0)\, d\eta\, dt
\end{aligned} \tag{12}$$

is satisfied for any test function $0 \le \psi \in C^{2,1}_{\eta,t}(\Omega)$.

Theorem 3.2 *Let $1 < p \le \frac{\gamma+Q+\alpha}{Q}$ and $\gamma > -2$, then (4) does not have a nontrivial weak solution.*

Proof Let u be a global weak solution of (4) and ψ be a smooth nonnegative test function such that

$$\begin{aligned}
D(\psi) &\doteq \int_\Omega |\eta|_H^{\gamma(1-p')} \psi^{1-p'} \left(|\psi_t|^{p'} + \left|(-\Delta_H)^{\frac{\alpha}{2}}\psi\right|^{p'} + \left|(-\Delta_H)^{\frac{\alpha}{2}}\psi_t\right|^{p'} \right) d\eta\, dt \\
&\quad + \int_{H^n} u_0(\eta)(-\Delta_H)^{\frac{\alpha}{2}}\psi(\eta,0)\, d\eta \\
&< \infty,
\end{aligned} \tag{13}$$

where $p' = p/(p-1)$. By using Young's inequality, from identity (12) we obtain

$$\begin{aligned}
&\int_\Omega |\eta|_H^\gamma |u|^p \psi \, d\eta\, dt + \int_{H^n} u_0(\eta)\psi(\eta,0)\, d\eta \\
&\quad \le \int_\Omega |u||\psi_t|\, d\eta\, dt + \int_\Omega |u|\left|(-\Delta_H)^{\frac{\alpha}{2}}\psi\right| d\eta\, dt \\
&\qquad + \int_\Omega |u|\left|(-\Delta_H)^{\frac{\alpha}{2}}\psi_t\right| d\eta\, dt + \int_{H^n} u_0(\eta)(-\Delta_H)^{\frac{\alpha}{2}}\psi(\eta,0)\, d\eta\, dt \\
&\quad \le \frac{1}{2}\int_\Omega |\eta|_H^\gamma |u|^p \psi\, d\eta\, dt \\
&\qquad + C\int_\Omega |\eta|_H^{\gamma(1-p')}\psi^{1-p'}\left(|\psi_t|^{p'} + \left|(-\Delta_H)^{\frac{\alpha}{2}}\psi\right|^{p'} + \left|(-\Delta_H)^{\frac{\alpha}{2}}\psi_t\right|^{p'} \right) d\eta\, dt \\
&\qquad + \int_{H^n} u_0(\eta)(-\Delta_H)^{\frac{\alpha}{2}}\psi(\eta,0)\, d\eta.
\end{aligned} \tag{14}$$

In the sequel C denotes a constant which may vary from line to line but is independent of the terms which will take part in any limit processing. Therefor the inequality

$$\begin{aligned}
&\int_{\Omega} |\eta|_H^{\gamma}|u|^p\psi\,d\eta\,dt + 2\int_{H^n} u_0(\eta)\psi(\eta,0)\,d\eta \\
&\quad\le 2C\int_{\Omega} |\eta|_H^{\gamma(1-p')}\psi^{1-p'}\bigl(|\psi_t|^{p'} + \bigl|(-\Delta_H)^{\frac{\alpha}{2}}\psi\bigr|^{p'} + \bigl|(-\Delta_H)^{\frac{\alpha}{2}}\psi_t\bigr|^{p'}\bigr)\,d\eta\,dt \\
&\qquad + 2\int_{H^n} u_0(\eta)(-\Delta_H)^{\frac{\alpha}{2}}\psi(\eta,0)\,d\eta \qquad (15)
\end{aligned}$$

follows from (14).

Taking

$$\psi(\eta,t) = \Psi\left(\frac{|\xi|^4 + |\tilde{\xi}|^4 + \tau^2}{R^4}\right)\Psi\left(\frac{t}{R^\alpha}\right), \quad \eta = (\xi,\tilde{\xi},\tau) \in H^n, t > 0, R > 0, \qquad (16)$$

with $\Psi \in C_c^\infty(\mathbb{R}^+)$ is the standard cut-off function

$$\Psi(r) = \begin{cases} 1, & \text{if } 0 \le r \le 1, \\ \searrow, & \text{if } 1 < r \le 2, \\ 0, & \text{if } r > 2. \end{cases} \qquad (17)$$

We note that $\operatorname{supp}(\psi)$ is a subset of

$$\Omega_R = \bigl\{(\eta,t) = (\xi,\tilde{\xi},\tau,t) \in H^n \times \mathbb{R}_+ : |\xi|^4 + |\tilde{\xi}|^4 + |\tau|^2 + |t|^2 \le 2R^4\bigr\},$$

and $\operatorname{supp}(\Delta_H\psi)$ is included in

$$\bar{\Omega}_R = \bigl\{(\eta,t) = (\xi,\tilde{\xi},\tau,t) \in H^n \times \mathbb{R}_+ : R^4 \le |\xi|^4 + |\tilde{\xi}|^4 + |\tau|^2 + |t|^2 \le 2R^4\bigr\}.$$

We note from [17] that there is a positive constant $C_1 > 0$, independent of R, such that

$$\bigl|(-\Delta_H)^{\frac{\alpha}{2}}\psi\bigr| \le C_1R^{-\alpha}$$

and

$$|\psi_t| \le C_1R^{-2}, \qquad \bigl|(-\Delta_H)^{\frac{\alpha}{2}}\psi_t\bigr| \le C_1R^{-2\alpha}.$$

We perform the change of variables $R\bar{\xi} = \xi$, $R\hat{\xi} = \tilde{\xi}$, $R^2\tilde{\tau} = \tau$, $R^\alpha\tilde{t} = t$, and from (15), we obtain the estimates

$$\begin{aligned}
&\int_{\Omega} |\eta|_H^{\gamma}|u|^p\psi\,d\eta\,dt + 2\int_{H^n} u_0(\eta)\psi(\eta,0)\,d\eta \\
&\quad\le C\bigl(R^{-2p'} + R^{-\alpha p'} + R^{-2\alpha p'}\bigr)R^{\gamma(1-p')+Q+\alpha} + 2\int_{H^n}|u_0(\eta)|\bigl|(-\Delta)^{\frac{\alpha}{2}}\phi(\eta,0)\bigr|\,d\eta \\
&\quad\le CR^{-\alpha p'+\gamma(1-p')+Q+\alpha} + CR^{-\alpha}. \qquad (18)
\end{aligned}$$

When $1 < p < \frac{\gamma+Q+\alpha}{Q}$, the exponents of R of (18) are negative. Letting R go to infinity and using Fatou's lemma yields

$$\int_\Omega |\eta|_H^\gamma |u|^p \, d\eta \, dt + 2\int_{H^n} u_0(\eta)\, d\eta$$
$$\leq \liminf_{R\to\infty}\left(\int_\Omega |\eta|_H^\gamma |u|^p \psi \, d\eta \, dt + 2\int_{H^n} u_0(\eta)\psi(\eta,0)\, d\eta\right), \tag{19}$$

which is zero from (18), and nonzero u cannot exist since

$$\int_{H^n} u_0(\eta)\, d\eta \geq 0.$$

In the case where $p = \frac{\gamma+Q+\alpha}{Q}$, from (15), we get

$$\int_\Omega |\eta|_H^\gamma |u|^p \psi \, d\eta \, dt < \infty. \tag{20}$$

We recall the domain

$$\bar{\Omega}_R := \left\{x = (\xi, \tilde{\xi}, \tau, t) \in \Omega | R^4 \leq |\xi|^4 + |\tilde{\xi}|^4 + \tau^2 + t^2 \leq 2R^4\right\},$$

then by (20) and the Lebesgue dominated convergence theorem, we have

$$\lim_{R\to\infty} \int_{\bar{\Omega}_R} |\eta|_H^\gamma |u|^p \psi \, d\eta \, dt = 0. \tag{21}$$

By using the Hölder inequality in (14), we get

$$\int_\Omega |\eta|_H^\gamma |u|^p \psi \, d\eta \, dt$$
$$\leq \left(\int_{\bar{\Omega}_R} |\eta|_H^\gamma |u|^p \psi \, d\eta \, dt\right)^{\frac{1}{p}} \left[\left(\int_{\bar{\Omega}_R} |\eta|_H^{\gamma(1-p')} \psi^{1-p'} |\psi_t|^{p'}\right)^{\frac{1}{p'}}\right.$$
$$+ \left(\int_{\bar{\Omega}_R} |\eta|_H^{\gamma(1-p')} \psi^{1-p'} \left|(-\Delta_H)^{\frac{\alpha}{2}} \psi\right|^{p'}\right)^{\frac{1}{p'}}$$
$$\left. + \left(\int_{\bar{\Omega}_R} |\eta|_H^{\gamma(1-p')} \psi^{1-p'} \left|(-\Delta_H)^{\frac{\alpha}{2}} \psi_t\right|^{p'}\right)^{\frac{1}{p'}}\right] + \int_{H^n} u_0(\eta)(-\Delta_H)^{\frac{\alpha}{2}} \psi(\eta,0)\, d\eta$$
$$\leq C \int_{\bar{\Omega}_R} |\eta|_H^\gamma |u|^p \psi \, d\eta \, dt. \tag{22}$$

Finally, we get

$$\int_\Omega |\eta|_H^\gamma |u|^p \psi \, d\eta \, dt \leq C \int_{\bar{\Omega}_R} |\eta|_H^\gamma |u|^p \psi \, d\eta \, dt. \tag{23}$$

By letting $R \to \infty$ and using (21), we get $u \equiv 0$, ending the proof. □

Next, we will consider system (5).

Definition 3.3 A pair (u, v) with $p, q > 1$ is called a weak solution of system (5) in $\Omega = H^n \times (0, T)$ with the Cauchy data $(u_0, v_0) \in L^1_{\text{loc}}(H^n) \times L^1_{\text{loc}}(H^n)$ if the following identities:

$$\begin{aligned}
&\int_\Omega |\eta|_H^{\gamma_1} |v|^q \psi \, d\eta \, dt + \int_{H^n} u_0(\eta)\psi(\eta, 0) \, d\eta \\
&\quad = -\int_\Omega u\psi_t \, d\eta \, dt + \int_\Omega u(-\Delta_H)^{\frac{\alpha}{2}} \psi \, d\eta \, dt \\
&\qquad + \int_\Omega u(-\Delta_H)^{\frac{\alpha}{2}} \psi_t \, d\eta \, dt + \int_{H^n} u_0(\eta)(-\Delta_H)^{\frac{\alpha}{2}} \psi(\eta, 0) \, d\eta \, dt
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
&\int_\Omega |\eta|_H^{\gamma_2} |u|^p \psi \, d\eta \, dt + \int_{H^n} v_0(\eta)\psi(\eta, 0) \, d\eta \\
&\quad = -\int_\Omega v\psi_t \, d\eta \, dt + \int_\Omega v(-\Delta_H)^{\frac{\alpha}{2}} \psi \, d\eta \, dt \\
&\qquad + \int_\Omega v(-\Delta_H)^{\frac{\alpha}{2}} \psi_t \, d\eta \, dt + \int_{H^n} v_0(\eta)(-\Delta_H)^{\frac{\alpha}{2}} \psi(\eta, 0) \, d\eta \, dt
\end{aligned} \tag{25}$$

are satisfied for any regular function $0 \le \psi \in C^{2,1}_{\eta,t}(\Omega)$.

Theorem 3.4 *Assume that*

$$Q \le \max\left\{ \frac{\frac{\gamma_2+\alpha}{p} + \frac{\gamma_1}{pq} + \alpha - \frac{2}{pq'} - \frac{2}{p'}}{\frac{1}{pq'} + \frac{1}{p'}}, \frac{\frac{\gamma_1+\alpha}{q} + \frac{\gamma_2}{pq} + \alpha - \frac{2}{qp'} - \frac{2}{q'}}{\frac{1}{qp'} + \frac{1}{q'}} \right\}.$$

Then there is no nontrivial weak solution (u, v) *of system* (5).

Proof Let ψ_R be a nonnegative function such that

$$\psi_R(\eta, t) = \Psi\left(\frac{|\xi|^4 + |\tilde{\xi}|^4 + \tau^2}{R^4}\right)\Psi\left(\frac{t}{R^\alpha}\right), \quad \eta = (\xi, \tilde{\xi}, \tau) \in H^n, t > 0, R > 0, \tag{26}$$

with $\Psi \in C_c^\infty(\mathbb{R}^+)$ being the standard cut-off function

$$\Psi(r) = \begin{cases} 1, & \text{if } 0 \le r \le 1, \\ \searrow, & \text{if } 1 < r \le 2, \\ 0, & \text{if } r > 2. \end{cases} \tag{27}$$

Let (u, v) be a nontrivial weak solution of (5). We note that for large R,

$$\int_{H^n} u_0(\eta)\psi(\eta, 0) \, d\eta - \int_{H^n} u_0(\eta)(-\Delta_H)^{\frac{\alpha}{2}} \psi(\eta, 0) \, d\eta \, dt \ge 0$$

and

$$\int_{H^n} v_0(\eta)\psi(\eta, 0) \, d\eta - \int_{H^n} v_0(\eta)(-\Delta_H)^{\frac{\alpha}{2}} \psi(\eta, 0) \, d\eta \, dt \ge 0.$$

Using (24) with $\psi = \psi_R$, one has

$$
\begin{aligned}
&\int_\Omega |\eta|_H^{\gamma_1}|v|^q\psi_R\,d\eta\,dt + \int_{H^n} u_0(\eta)\psi_R(\eta,0)\,d\eta - \int_{H^n} u_0(\eta)(-\Delta_H)^{\frac{\alpha}{2}}\psi_R(\eta,0)\,d\eta\,dt \\
&\quad\le \int_\Omega |u|\big|(\psi_R)_t\big|\,d\eta\,dt + \int_\Omega |u|\big|(-\Delta_H)^{\frac{\alpha}{2}}\psi_R\big|\,d\eta\,dt + \int_\Omega |u|\big|(-\Delta)^{\frac{\alpha}{2}}(\psi_R)_t\big|\,d\eta\,dt \\
&\quad\le \left(\int_\Omega |\eta|_H^{\gamma_2}|u|^p\psi_R\,d\eta\,dt\right)^{\frac{1}{p}}\left[\left(\int_\Omega |\eta|_H^{-\frac{p'\gamma_2}{p}}\psi_R^{-\frac{p'}{p}}\big|(\psi_R)_t\big|^{p'}\,d\eta\,dt\right)^{\frac{1}{p'}}\right. \\
&\qquad+\left(\int_\Omega |\eta|_H^{-\frac{p'\gamma_2}{p}}\psi_R^{-\frac{p'}{p}}\big|(-\Delta_H)^{\frac{\alpha}{2}}\psi_R\big|^{p'}\,d\eta\,dt\right)^{\frac{1}{p'}} \\
&\qquad\left.+\left(\int_\Omega |\eta|_H^{-\frac{p'\gamma_2}{p}}\psi_R^{-\frac{p'}{p}}\big|(-\Delta_H)^{\frac{\alpha}{2}}(\psi_R)_t\big|^{p'}\,d\eta\,dt\right)^{\frac{1}{p'}}\right] \\
&\quad= \left(\int_\Omega |\eta|_H^{\gamma_2}|u|^p\psi_R\,d\eta\,dt\right)^{\frac{1}{p}}\mathscr{A}_1(R), \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
&\int_\Omega |\eta|_H^{\gamma_2}|u|^p\psi_R\,d\eta\,dt + \int_{H^n} v_0(\eta)\psi_R(\eta,0)\,d\eta - \int_{H^n} v_0(\eta)(-\Delta_H)^{\frac{\alpha}{2}}\psi_R(\eta,0)\,d\eta\,dt \\
&\quad\le \int_\Omega |v|\big|(\psi_R)_t\big|\,d\eta\,dt + \int_\Omega |v|\big|(-\Delta_H)^{\frac{\alpha}{2}}\psi_R\big|\,d\eta\,dt + \int_\Omega |v|\big|(-\Delta)^{\frac{\alpha}{2}}(\psi_R)_t\big|\,d\eta\,dt \\
&\quad\le \left(\int_\Omega |\eta|_H^{\gamma_1}|v|^q\psi_R\,d\eta\,dt\right)^{\frac{1}{q}}\left[\left(\int_\Omega |\eta|_H^{-\frac{q'\gamma_2}{q}}\psi_R^{-\frac{q'}{q}}\big|(\psi_R)_t\big|^{q'}\,d\eta\,dt\right)^{\frac{1}{q'}}\right. \\
&\qquad+\left(\int_\Omega |\eta|_H^{-\frac{q'\gamma_1}{q}}\psi_R^{-\frac{q'}{q}}\big|(-\Delta_H)^{\frac{\alpha}{2}}\psi_R\big|^{q'}\,d\eta\,dt\right)^{\frac{1}{q'}} \\
&\qquad\left.+\left(\int_\Omega |\eta|_H^{-\frac{q'\gamma_2}{q}}\psi_R^{-\frac{q'}{q}}\big|(-\Delta_H)^{\frac{\alpha}{2}}(\psi_R)_t\big|^{q'}\,d\eta\,dt\right)^{\frac{1}{q'}}\right] \\
&\quad= \left(\int_\Omega |\eta|_H^{\gamma_1}|v|^q\psi_R\,d\eta\,dt\right)^{\frac{1}{q}}\mathscr{A}_2(R). \qquad (29)
\end{aligned}
$$

Setting

$$
I(R) = \int_\Omega |\eta|_H^{\gamma_2}|u|^p\psi_R\,d\eta\,dt, \qquad J(R) = \int_\Omega |\eta|_H^{\gamma_1}|v|^q\psi_R\,d\eta\,dt,
$$

we have

$$
J(R) \le I(R)^{\frac{1}{p}}\mathscr{A}_1(R), \tag{30}
$$

where $\mathscr{A}_1(R) = A_\psi(R) + B_\psi(R) + C_\psi(R)$, and

$$
\begin{aligned}
A_\psi(R) &= \left(\int_\Omega |\eta|_H^{-\frac{p'\gamma_2}{p}}\psi_R^{-\frac{p'}{p}}\big|(\psi_R)_t\big|^{p'}\,d\eta\,dt\right)^{\frac{1}{p'}}, \\
B_\psi(R) &= \left(\int_\Omega |\eta|_H^{-\frac{p'\gamma_2}{p}}\psi_R^{-\frac{p'}{p}}\big|(-\Delta_H)^{\frac{\alpha}{2}}\psi_R\big|^{p'}\,d\eta\,dt\right)^{\frac{1}{p'}},
\end{aligned}
$$

$$C_\psi(R) = \left(\int_\Omega |\eta|_H^{-\frac{p'\gamma_2}{p}} \psi_R^{-\frac{p'}{p}} \left|(-\Delta_H)^{\frac{\alpha}{2}}(\psi_R)_t\right|^{p'} d\eta\, dt\right)^{\frac{1}{p'}}.$$

Similarly, we have

$$I(R) \le J(R)^{\frac{1}{q}} \mathscr{A}_2(R), \tag{31}$$

where $\mathscr{A}_2(R) = A'(\psi) + B'(\psi) + C'(\psi)$ and

$$A'_\psi(R) = \left(\int_\Omega |\eta|_H^{-\frac{q'\gamma_1}{q}} \psi_R^{-\frac{q'}{q}} \left|(\psi_R)_t\right|^{q'} d\eta\, dt\right)^{\frac{1}{q'}},$$

$$B'_\psi(R) = \left(\int_\Omega |\eta|_H^{-\frac{q'\gamma_1}{q}} \psi_R^{-\frac{q'}{q}} \left|(-\Delta_H)^{\frac{\alpha}{2}}\psi_R\right|^{q'} d\eta\, dt\right)^{\frac{1}{q'}},$$

$$C'_\psi(R) = \left(\int_\Omega |\eta|_H^{-\frac{q'\gamma_2}{q}} \psi_R^{-\frac{q'}{q}} \left|(-\Delta_H)^{\frac{\alpha}{2}}(\psi_R)_t\right|^{q'} d\eta\, dt\right)^{\frac{1}{q'}}.$$

To estimate the integrals $\mathscr{A}_1(R)$ and $\mathscr{A}_2(R)$, we introduce the scaled variables $R\bar{\xi} = \xi$, $R\hat{\xi} = \tilde{\xi}$, $R^2\tilde{\tau} = \tau$, $R^\alpha \tilde{t} = t$, and we conclude that

$$\mathscr{A}_2(R) \le C\left(R^{-\frac{\gamma_1}{q} - 2\alpha + \frac{Q+2}{q'}}\right) \tag{32}$$

and

$$\mathscr{A}_1(R) \le C\left(R^{-\frac{\gamma_2}{p} - 2\alpha + \frac{Q+2}{p'}}\right). \tag{33}$$

Using (31) and (33) in (30), we obtain

$$\left(J(R)\right)^{1-\frac{1}{pq}} \le \mathscr{A}_2^{\frac{1}{p}} \mathscr{A}_1(R) \le CR^{\sigma_J},$$

where $\sigma_J = -\frac{\gamma_1}{pq} - \frac{2\alpha}{p} + \frac{Q+2}{pq'} - \frac{\gamma_2}{p} - 2\alpha + \frac{Q+2}{p'}$.

Similarly, we have

$$\left(I(R)\right)^{1-\frac{1}{pq}} \le \mathscr{A}_1^{\frac{1}{q}} \mathscr{A}_2(R) \le CR^{\sigma_I},$$

where $\sigma_I = -\frac{\gamma_2}{pq} - \frac{2\alpha}{q} + \frac{Q+2}{qp'} - \frac{\gamma_1}{q} - 2\alpha + \frac{Q+2}{q'}$.

Now, we require that $\sigma_I \le 0$ or $\sigma_J \le 0$, which is equivalent to

$$Q \le \max\left\{ \frac{\frac{\gamma_2+\alpha}{p} + \frac{\gamma_1}{pq} + \alpha - \frac{2}{pq'} - \frac{2}{p'}}{\frac{1}{pq'} + \frac{1}{p'}}, \frac{\frac{\gamma_1+\alpha}{q} + \frac{\gamma_2}{pq} + \alpha - \frac{2}{qp'} - \frac{2}{q'}}{\frac{1}{qp'} + \frac{1}{q'}} \right\}.$$

In this case, the integrals $I(R)$ and $J(R)$, increasing in R, are bounded uniformly with respect to R. Using the monotone convergence theorem, we deduce that $|\eta|_H^{\gamma_1}|v|^q$ and $|\eta|_H^{\gamma_2}|u|^p$ are in $L^1(H)$. Note that instead of (28), more precisely,

$$\int_\Omega |\eta|_H^{\gamma_1}|v|^q \psi_R\, d\eta\, dt \le C \int_{\bar{\Omega}_R} |\eta|_H^{\gamma_2}|u|^p \psi_R\, d\eta\, dt, \tag{34}$$

where C is a positive constant independent of R. Finally, using the dominated convergence theorem, we obtain that

$$\lim_{R\to+\infty}\int_{\bar{\Omega}_R}|\eta|_H^{\gamma_2}|u|^p\psi_R\,d\eta\,dt=0. \tag{35}$$

Hence,

$$\int_{\Omega}|\eta|_H^{\gamma_1}|v|^q\,d\eta\,dt=0, \tag{36}$$

which implies that $v\equiv 0$ and $u\equiv 0$ via (29). This contradicts the fact that (u,v) is a nontrivial weak solution of (5), which achieves the proof. □

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