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# Global weak solutions to a nonlinear equation with fourth order nonlinearities

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**Abstract**
Global weak solution for a nonlinear model with forth order nonlinearities and conserved quantities is considered. Assuming its initial value satisfies certain assumptions that are weaker than the sign condition, we derive a higher integrability estimate and an upper bound estimate about the space derivatives of its solutions and then prove that the equation has global weak solutions.

**MSC:** 35G25; 35L05

**Keywords:** Global weak solutions; Nonlinearities; Viscous approximation technique

## 1 Introduction

Consider the nonlinear equation

$$V_t - V_{txx} + \frac{\partial g(V)}{\partial x} = 4V^2 V_x V_{xx} + V^3 V_{xxx}, \tag{1.1}$$

where $g(V) = \sum_{j=2}^{n} a_j V^j$ and each $a_j$ is constant. Equation (1.1) is regarded as a Camassa–Holm type equation in Grayshan and Himonas [1]. Equation (1.1) possesses fourth order nonlinearities. In fact, the standard Camassa–Holm (CH) model [2] and the Degasperis–Procesi (DP) equation [3] have quadratic nonlinearities, while the Novikov equation [4] possesses cubic nonlinearities.

As CH, DP, Novikov equations and the generalized Camassa–Holm type equations have the same dynamic characteristics such as an infinite hierarchy of higher symmetries, conserved quantities, and a bi-Hamiltonian formulation [5–8], we recall several works relating to the nonlinear CH, DP, and Novikov equations. A nonlinear nonlocal shallow water equation including the CH equation is studied in [6], in which the wave breaking of solutions is discovered. Silva and Freire [7, 8] discuss the persistence and continuation of generalized 0-Holm Staley model with higher order nonlinearities. Guo et al. [9, 10] investigate several dynamical properties of the CH type equation with higher order nonlinearities (also see [11–14]). Constantin and Ivanov [15] employ the dressing method to study the DP equation. The blow up structures, global strong and weak solutions for the DP equation are considered in [16, 17]. Mi and Mu [18] make use of detailed derivations



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to find the property of peakon solutions for a modified Novikov model. Himonas et al. [19] construct two peakon solutions and illustrate the ill-posedness of the Novikov equation. For the models relating to the CH, DP, and Novikov equations and their dynamical properties, we refer the reader to [20–28].

Motivated by the works made in [1, 7, 9, 10] to probe the dynamic properties of nonlinear equations with higher order nonlinearities, we aim to investigate the existence of global weak solutions for Eq. (1.1), which possesses fourth order nonlinearities. We utilize the viscous approximation technique to prove the global existence. Here we state that our assumption on the initial value $V_0(x)$ is $V_0(x) \in H^1(\mathbb{R})$ and $\|\frac{\partial V_0}{\partial x}\|_{L^\infty(\mathbb{R})} < \infty$, which is weaker than the sign condition. The key contributions in this job include a higher integrability estimate and an upper bound estimate on the factor $|\frac{\partial V(t,x)}{\partial x}|$.

The structure of this work is illustrated as follows. Section 2 states the main conclusion. Several lemmas are given in Sect. 3. In Sect. 4, a strong convergent property of the solution for viscous approximations of Eq. (1.1) is derived and the main result is proved.

## 2 Main conclusion

Consider the problem

$$\begin{cases} V_t - V_{txx} + \frac{\partial g(V)}{\partial x} = 4V^2 V_x V_{xx} + V^3 V_{xxx}, \\ V(0,x) = V_0(x). \end{cases} \tag{2.1}$$

Using the inverse operator $\Lambda^{-2} = (1 - \frac{\partial^2}{\partial x^2})^{-1}$, we derive that problem (2.1) is equivalent to

$$\begin{cases} V_t + V^3 V_x = G, \\ V(0,x) = V_0(x), \end{cases} \tag{2.2}$$

where $G = \Lambda^{-2}[\frac{1}{4}(V^4)_x - \frac{\partial g(V)}{\partial x} + V^2 V_x V_{xx} - 3(V^2 V_x^2)_x]$.

The definition of global weak solutions follows that in [21, 22].

**Definition 2.1** Suppose that the solution $V(t,x)$ satisfies

(a) $V \in C([0,\infty) \times \mathbb{R}) \cap L^\infty([0,\infty); H^1(\mathbb{R}))$;
(b) $\|V(t,\cdot)\|_{H^1(\mathbb{R})} \le \|V_0\|_{H^1(\mathbb{R})}$;
(c) $V = V(t,x)$ satisfies (2.1) or (2.2) in the sense of distributions. Then $V(t,x)$ is called a global weak solution to problem (2.1) or (2.2).

In this work, we utilize $c$ to represent any positive constants that do not depend on parameter $\varepsilon$. Our main result is described as follows.

**Theorem 2.2** *Let $\|V_{0x}\|_{L^\infty(\mathbb{R})} < \infty$ and $V_0(x) \in H^1(\mathbb{R})$. Then at least one global weak solution $V(t,x)$ for problem (2.1) or (2.2) exists. Moreover, for any $T > 0$, $(t,x) \in [0,T) \times \mathbb{R}$, the inequalities*

$$\frac{\partial V(t,x)}{\partial x} \le c(1+t) \tag{2.3}$$

*and*

$$\int_{\mathbb{R}} \left| \frac{\partial V(t,x)}{\partial x} \right|^6 dx \le c(1+T)e^{cT} \tag{2.4}$$

*hold.*

## 3 Several lemmas

Assume

$$\phi(x) = \begin{cases} e^{\frac{1}{x^2-1}}, & |x| < 1, \\ 0, & |x| \ge 1, \end{cases}$$

which has a compact set $[-1,1]$. Set the smooth function $\phi_\varepsilon(x) = \varepsilon^{-\frac{1}{4}}\phi(\varepsilon^{-\frac{1}{4}}x)$ associated with $0 < \varepsilon < \frac{1}{4}$ and

$$V_{\varepsilon,0} = \int_{\mathbb{R}} \phi_\varepsilon(x-\varsigma)V_0(\varsigma)\,d\varsigma = \phi_\varepsilon \star V_0.$$

We have $V_{\varepsilon,0} \in C^\infty$ for $V_0 \in H^1(\mathbb{R})$ and

$$\begin{cases} V_{\varepsilon,0} \to V_0 \quad \text{in } H^1(\mathbb{R}), \text{as } \varepsilon \to 0, \\ \|V_{\varepsilon,0}\|_{H^1(\mathbb{R})} \le c\|V_0\|_{H^1(\mathbb{R})}. \end{cases} \tag{3.1}$$

Consider the following viscous approximation problem:

$$\begin{cases} \frac{\partial V_\varepsilon}{\partial t} + V_\varepsilon^3 \frac{\partial V_\varepsilon}{\partial x} - G_\varepsilon = \varepsilon \frac{\partial^2 V_\varepsilon}{\partial x^2}, \\ V_\varepsilon(0,x) = V_{\varepsilon,0}(x), \end{cases} \tag{3.2}$$

in which

$$G_\varepsilon = \Lambda^{-2}\left[\frac{1}{4}\left(V_\varepsilon^4\right)_x - \frac{\partial g(V_\varepsilon)}{\partial x} + V_\varepsilon^2 \frac{V_\varepsilon}{\partial x}\frac{\partial V_\varepsilon}{\partial x^2} - 3\left(V_\varepsilon^2\left(\frac{\partial V_\varepsilon}{\partial x}\right)^2\right)_x\right].$$

Letting $h_\varepsilon(t,x) = \frac{\partial V_\varepsilon(t,x)}{\partial x}$, from (3.2), we obtain

$$\frac{\partial h_\varepsilon}{\partial t} + V_\varepsilon^3 \frac{\partial h_\varepsilon}{\partial x} + \frac{1}{2}V_\varepsilon^2 h_\varepsilon^2 - \varepsilon \frac{\partial^2 h_\varepsilon}{\partial x^2} = H_\varepsilon(t,x), \tag{3.3}$$

where

$$H_\varepsilon(t,x) = g(V_\varepsilon) - \frac{1}{4}V_\varepsilon^4 + \Lambda^{-2}\left(\frac{1}{4}V_\varepsilon^4 - g(V_\varepsilon) - \frac{5}{2}V_\varepsilon^2 h_\varepsilon^2 - \left(V_\varepsilon h_\varepsilon^3\right)_x\right). \tag{3.4}$$

For problem (3.2), we have the conclusion.

**Lemma 3.1** *Provided that $V_0 \in H^1(\mathbb{R})$ and arbitrary $\kappa \ge 2$, then problem (3.2) has a unique solution $V_\varepsilon \in C([0,\infty); H^\kappa(\mathbb{R}))$ and*

$$\int_{\mathbb{R}} \left(V_\varepsilon^2 + \left(\frac{\partial V_\varepsilon}{\partial x}\right)^2\right) dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} \left(\left(\frac{\partial V_\varepsilon}{\partial x}\right)^2 + \left(\frac{\partial^2 V_\varepsilon}{\partial x^2}\right)^2\right)(s,x)\,dx\,ds = \|V_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2,$$

*which is equivalent to*

$$\left\| V_\varepsilon(t,\cdot)\right\|^2_{H^1(\mathbb{R})} + 2\varepsilon\int_0^t \left\|\frac{\partial V_\varepsilon}{\partial x}(s,\cdot)\right\|^2_{H^1(\mathbb{R})} ds = \|V_{\varepsilon,0}\|^2_{H^1(\mathbb{R})}.$$

*Proof* For the initial value $V_0 \in H^1(\mathbb{R})$ and arbitrary $\kappa \geq 2$, we obtain $V_{\varepsilon,0} \in H^\kappa(\mathbb{R})$. Utilizing the main result in [23] yields that system (3.2) possesses a unique solution $V_\varepsilon(t,x)$ belonging to $C([0,\infty); H^\kappa(\mathbb{R}))$.

For integer $j \geq 1$, we have $\int_{\mathbb{R}} \frac{\partial V^j}{\partial x} V\,dx = \int_{\mathbb{R}} jV^{j-1}VV_x\,dx = 0$. Using the definition of $g(V)$ yields $\int_{\mathbb{R}} \frac{\partial g(V)}{\partial x} V\,dx = 0$.

Using system (3.2), we have

$$\frac{\partial V_\varepsilon}{\partial t} - \frac{\partial^3 V_\varepsilon}{\partial t x^2} + \frac{\partial g(V_\varepsilon)}{\partial x} = 4V_\varepsilon^2 \frac{\partial V_\varepsilon}{\partial x}\frac{\partial^2 V_\varepsilon}{\partial x^2} + V_\varepsilon^3 \frac{\partial^3 V_\varepsilon}{\partial x^3} + \varepsilon\left(\frac{\partial^2 V_\varepsilon}{\partial x^2} - \frac{\partial^4 V_\varepsilon}{\partial x^4}\right),$$

which leads to

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(V_\varepsilon^2 + \left(\frac{\partial V_\varepsilon}{\partial x}\right)^2\right) dx + \varepsilon\int_{\mathbb{R}}\left(\left(\frac{\partial V_\varepsilon}{\partial x}\right)^2 + \left(\frac{\partial^2 V_\varepsilon}{\partial x^2}\right)^2\right) dx = 0.$$

The proof is finished. □

If $V_0(x) \in H^1(\mathbb{R})$, then using (3.1) and Lemma 3.1 gives rise to

$$\begin{cases} \|V_\varepsilon\|_{L^\infty(\mathbb{R})} \leq c\|V_\varepsilon\|_{H^1(\mathbb{R})} \leq c\|V_{\varepsilon,0}\|_{H^1(\mathbb{R})} \leq c\|V_0\|_{H^1(\mathbb{R})} \leq c, \\ \|g(V_\varepsilon)\|_{L^\infty(\mathbb{R})} < c. \end{cases} \tag{3.5}$$

Assume $K(x) \in L^r(\mathbb{R})$ ($1 \leq r < \infty$ or $r = \infty$). Note that

$$\Lambda^{-2}K(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-\eta|}K(\eta)\,d\eta$$

and

$$\begin{aligned} \Lambda^{-2}\bigl[K(x)\bigr]_x &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-\eta|}\frac{\partial K(\eta)}{\partial \eta}\,d\eta \\ &= \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\eta}\frac{\partial K(\eta)}{\partial \eta}\,d\eta + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\eta}\frac{\partial K(\eta)}{\partial \eta}\,d\eta \\ &= -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\eta}K(\eta)\,d\eta + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\eta}K(\eta)\,d\eta, \end{aligned}$$

from which we have

$$\bigl|\Lambda^{-2}\bigl[K(x)\bigr]_x\bigr| \leq \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-\eta|}\bigl|K(\eta)\bigr|\,d\eta. \tag{3.6}$$

**Lemma 3.2** *Assume* $0 < t < T$, $V_0 \in H^1(\mathbb{R})$, $\|\frac{\partial V_0(x)}{\partial x}\|_{L^\infty(\mathbb{R})} < \infty$. *Then the solution of problem* (3.2) *satisfies*

$$\int_{\mathbb{R}}\left(\frac{\partial V_\varepsilon}{\partial x}\right)^6 dx \leq c(1+T)e^{cT} \tag{3.7}$$

*and*

$$\varepsilon \int_0^T \int_{\mathbb{R}} \left(\frac{\partial V_\varepsilon}{\partial x}\right)^4 \left(\frac{\partial^2 V_\varepsilon}{\partial x^2}\right)^2 dx\,dt \le c(1+T)e^{cT}. \tag{3.8}$$

*Proof* Using (3.3) and writing $V = V_\varepsilon$ for conciseness, we have

$$\begin{aligned} &V_{tx} + \frac{1}{2}V^2V_x^2 + V^3V_{xx} - \varepsilon V_{xxx} \\ &\quad = -\frac{1}{4}V^4 + g(V) + \Lambda^{-2}\left(\frac{1}{4}V^4 - g(V) - \frac{5}{2}V^2V_x^2 - \left(VV_x^3\right)_x\right). \end{aligned} \tag{3.9}$$

Applying the identity

$$\int_{\mathbb{R}} V_x^5 V^3 V_{xx}\,dx = \int_{\mathbb{R}} V_x^5 V^3\,dV_x = -\int_{\mathbb{R}} V_x\left[5V_x^4V_{xx}V^3 + 3V^2V_x^6\right]dx$$

yields

$$6\int_{\mathbb{R}} V^3V_x^5V_{xx}\,dx = -3\int_{\mathbb{R}} V^2V_x^7\,dx,$$

from which we have

$$\int_{\mathbb{R}} \left(\frac{1}{2}V^2V_x^2 + V^3V_{xx}\right)V_x^5\,dx = 0. \tag{3.10}$$

We multiply (3.9) by $V_x^5$ and apply (3.10) to obtain

$$\begin{aligned} &\frac{1}{6}\frac{d}{dt}\int_{\mathbb{R}} V_x^6\,dx + 5\varepsilon\int_{\mathbb{R}} V_x^4V_{xx}^2\,dx \\ &\quad = \int_{\mathbb{R}} V_x^5\left(g(V) - \frac{1}{4}V^4\right)dx + \int_{\mathbb{R}} V_x^5\Lambda^{-2}\left(\frac{1}{4}V^4 - g(V) - \frac{5}{2}V^2V_x^2 - \left(VV_x^3\right)_x\right)dx \\ &\quad < \int_{\mathbb{R}} \left|V_x^5\left(g(V) - \frac{1}{4}V^4\right)\right|dx + \int_{\mathbb{R}} \left|V_x^5\Lambda^{-2}\left(\frac{1}{4}V^4 - g(V)\right)\right|dx \\ &\qquad + \frac{5}{2}\int_{\mathbb{R}} \left|V_x^5\Lambda^{-2}\left(V^2V_x^2\right)\right|dx + \int_{\mathbb{R}} \left|V_x^5\Lambda^{-2}\left(VV_x^3\right)_x\right|dx. \end{aligned} \tag{3.11}$$

Applying the Hölder inequality gives rise to

$$\begin{aligned} &\int_{\mathbb{R}} \left|V_x^5\left(g(V) - \frac{1}{4}V^4\right)\right|dx \\ &\quad \le c\left(\int_{\mathbb{R}} V_x^6\,dx\right)^{\frac{5}{6}}\left(\int_{\mathbb{R}} V^2\,dx\right)^{\frac{1}{6}} + c \le c\left(1 + \int_{\mathbb{R}} V_x^6\,dx\right), \end{aligned} \tag{3.12}$$

in which inequality (3.5) and $\|V^j\|_{L^\infty(\mathbb{R})} \le c$ $(j = 1, 2, \ldots 6n)$ are employed. Utilizing the Hölder inequality again yields

$$\int_{\mathbb{R}} |V_x|^4\,dx \le \left(\int_{\mathbb{R}} V_x^6\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} V_x^2\,dx\right)^{\frac{1}{2}} \le c\left(1 + \int_{\mathbb{R}} |V_x|^6\,dx\right), \tag{3.13}$$

$$\begin{aligned}\int_{\mathbb{R}} |V_x|^3\,dx &\le \left(\int_{\mathbb{R}} V_x^4\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} V_x^2\,dx\right)^{\frac{1}{2}}\\ &\le c\left(\int_{\mathbb{R}} V_x^6\,dx\right)^{\frac{1}{4}}\left(\int_{\mathbb{R}} V_x^2\,dx\right)^{\frac{1}{2}+\frac{1}{4}} \le c\left(\int_{\mathbb{R}} V_x^6\,dx\right)^{\frac{1}{4}},\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\int_{\mathbb{R}} |V_x|^5\,dx &\le \left(\int_{\mathbb{R}} V_x^6\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} V_x^4\,dx\right)^{\frac{1}{2}}\\ &\le c\left(\int_{\mathbb{R}} V_x^6\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} V_x^6\,dx\right)^{\frac{1}{4}} \le c\left(\int_{\mathbb{R}} V_x^6\,dx\right)^{\frac{3}{4}}.\end{aligned} \tag{3.15}$$

We have

$$\left|\Lambda^{-2}\left(g(V) - \frac{1}{4}V^4\right)\right| = \left|\frac{1}{2}\int_{\mathbb{R}} e^{-|x-\eta|}\left(g\big(V(t,\eta)\big) - \frac{1}{4}V^4(t,\eta)\right)d\eta\right| \le c, \tag{3.16}$$

$$\left|\Lambda^{-2}\left(V^2V_x^2\right)\right| = \left|\frac{1}{2}\int_{\mathbb{R}} e^{-|x-\eta|}\left(V^2V_\eta^2\right)d\eta\right| \le c\left(1 + \int_{\mathbb{R}} V_\eta^2\,d\eta\right) \le c. \tag{3.17}$$

Using (3.6) yields

$$\left|\Lambda^{-2}\left[VV_x^3\right]_x\right| \le \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-\eta|}\left|VV_\eta^3\right|d\eta. \tag{3.18}$$

From (3.13)–(3.18), we obtain

$$\begin{aligned}&\int_{\mathbb{R}} \left|V_x^5\Lambda^{-2}\left(VV_x^3\right)_x\right|dx\\ &\quad\le \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \left|V_x^5\right|e^{-|x-\eta|}\left|VV_\eta^3\right|d\eta\,dx\\ &\quad\le c\int_{\mathbb{R}} |V_x|^5\,dx\int_{\mathbb{R}} |V_\eta|^3\,d\eta\\ &\quad\le c\left(\int_{\mathbb{R}} V_x^6\,dx\right)^{\frac{3}{4}}\left(\int_{\mathbb{R}} V_x^6\,dx\right)^{\frac{1}{4}} = c\int_{\mathbb{R}} V_x^6\,dx.\end{aligned} \tag{3.19}$$

Using (3.11)–(3.19) results in

$$\frac{d}{dt}\int_{\mathbb{R}} V_x^6\,dx \le c\left(1 + \int_{\mathbb{R}} V_x^6\,dx\right),$$

which combining with (3.11) gives (3.7) and (3.8). □

**Lemma 3.3** *For any* $0 < t < T$ *and* $x \in \mathbb{R}$, *then*

$$\left\|G_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}, \qquad \left\|G_\varepsilon(t,\cdot)\right\|_{L^1(\mathbb{R})}, \qquad \left\|G_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} < c \tag{3.20}$$

*and*

$$\|H_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})},\qquad \|H_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})},\qquad \|H_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})},\qquad \left\|\frac{\partial H_\varepsilon(t,\cdot)}{\partial x}\right\|_{L^1(\mathbb{R})} < c, \quad (3.21)$$

*where the constant $c > 0$ does not rely on $\varepsilon$, $G_\varepsilon(t,x)$ and $H_\varepsilon(t,x)$ are defined in* (3.2) *and* (3.4).

*Proof* To write concisely, we utilize the notation $V = V_\varepsilon(t,x)$ and $h = h_\varepsilon(t,x)$. In the proof of Lemma 3.2, we have proved that

$$\|G_\varepsilon\|_{L^\infty(\mathbb{R})} < c. \quad (3.22)$$

Using (3.6) yields

$$\begin{aligned} &\left|\Lambda^{-2}\Big(\frac{1}{4}\Big(\big(V^4\big)_x - \frac{\partial g(V)}{\partial x}\Big)\right| \\ &\quad \le c\Big(\int_{\mathbb{R}} e^{-|x-\eta|} V^4\,d\eta + \int_{\mathbb{R}} e^{-|x-\eta|}\big|g(V)\big|\,d\eta\Big) \le c, \quad (3.23)\\ &\big|\Lambda^{-2}\big[\big(V^2V_x^2\big)_x\big]\big| \le c\int_{\mathbb{R}} e^{-|x-\eta|}\big|V^2V_\eta^2\big|\,d\eta, \quad (3.24)\end{aligned}$$

and

$$\begin{aligned} \big|\Lambda^{-2}\big(V^2V_xV_{xx}\big)\big| &= \left|\frac{1}{4}e^{-x}\int_{-\infty}^x e^\eta V^2\big(V_\eta^2\big)_\eta\,d\eta + \frac{1}{4}e^x\int_x^\infty e^{-\eta}V^2\big(V_\eta^2\big)_\eta\,d\eta\right| \\ &= \left|\frac{1}{4}e^{-x}\int_{-\infty}^x V_\eta^2\big[e^\eta V^2 + 2VV_\eta e^\eta\big]\,d\eta \right.\\ &\quad \left. - \frac{1}{4}e^x\int_x^\infty V_\eta^2 e^{-\eta}\big[-V^2 + 2VV_\eta\big]\,d\eta\right| \\ &\le c\int_{\mathbb{R}} e^{-|x-\eta|}\big(\big|V^2V_\eta^2\big| + \big|VV_\eta^3\big|\big)\,d\eta \\ &\le c\int_{\mathbb{R}} e^{-|x-\eta|}\big(\big|V_\eta^2\big| + \big|V_\eta^3\big|\big)\,d\eta. \quad (3.25)\end{aligned}$$

From (3.22)–(3.25), using (3.14) and the Tonelli theorem, we derive that (3.20) holds.

Now we prove (3.21). Note that $g(V) = \sum_{j=2}^n a_jV^j$. Using (3.5) and (3.6) gives rise to

$$\begin{aligned} &\big\|g(V) - V^4\big\|_{L^\infty(\mathbb{R})} \le c, \qquad \big\|g(V) - V^4\big\|_{L^1(\mathbb{R})} \le c, \quad (3.26)\\ &\big|\Lambda^{-2}\big(V^2h^2\big)\big| \le \frac{1}{2}\int_{\mathbb{R}} e^{-|x-\eta|}V^2h^2\,d\eta \le c, \quad (3.27)\end{aligned}$$

and

$$\big|\Lambda^{-2}\big[Vh^3\big]_x\big| \le c\int_{-\infty}^\infty e^{-|x-\eta|}|Vh|^3\,d\eta \le c\int_{\mathbb{R}}|h|^3\,dx. \quad (3.28)$$

From (3.26)–(3.28), using the Tonelli theorem, we obtain that the first three inequalities in (3.21) hold.

We have

$$\frac{\partial H_\varepsilon(t,x)}{\partial x} = \left[g(V) - \frac{1}{4}V^4\right]_x + \Lambda^{-2}\left(\frac{1}{4}V_\varepsilon^4 - g(V_\varepsilon) - \frac{5}{2}V_\varepsilon^2 h_\varepsilon^2 - \left(V_\varepsilon h_\varepsilon^3\right)_x\right)_x. \tag{3.29}$$

Using (3.6) and (3.26)–(3.29), we obtain the last inequality in (3.21). □

**Lemma 3.4** *Let* $\|V_{0x}\|_{L^\infty(\mathbb{R})} < \infty$, $V_0(x) \in H^1(\mathbb{R})$. *Then solution* $V_\varepsilon$ *of problem* (3.2) *satisfies*

$$\frac{\partial V_\varepsilon(t,x)}{\partial x} \le c(1+t), \quad 0 < t < T. \tag{3.30}$$

*Proof* From Lemma 3.3, we obtain $|H_\varepsilon| \le c$ and

$$\frac{\partial h_\varepsilon}{\partial t} + V_\varepsilon^3 \frac{\partial h_\varepsilon}{\partial x} + \frac{1}{2}V_\varepsilon^2 h_\varepsilon^2 - \varepsilon \frac{\partial^2 h_\varepsilon}{\partial x^2} = H_\varepsilon(t,x) \le c. \tag{3.31}$$

Assume that $J = J(t)$ is the solution of the following ordinary differential equation(ODE):

$$\frac{dJ}{dt} + \frac{1}{2}\left(V_\varepsilon^*\right)^2 J^2 = c \tag{3.32}$$

with initial value $J(0) = \|\frac{\partial V_{\varepsilon,0}}{\partial x}\|_{L^\infty(\mathbb{R})}$. Let $V_\varepsilon^*$ be the solution $V_\varepsilon(t,x)$ when $\sup_{x\in\mathbb{R}} h_\varepsilon(t,x) = J(t)$, we derive that $J(t)$ is a supersolution of Eq. (3.31) associated with $V_{\varepsilon,0}(x)$. Using the comparison principle of parabolic equations gives rise to

$$h_\varepsilon(t,x) \le J(t).$$

Let $I(t) := ct$. Consider that $\frac{dI(t)}{dt} + \frac{1}{2}(V_\varepsilon^*)^2 I^2(t) - c = \frac{1}{2}(V_\varepsilon^*)^2(ct)^2 > 0$. Applying the comparison principle for ODE (3.32), we know $J(t) \le I(t) = ct + \|\frac{\partial u_{\varepsilon,0}}{\partial x}\|_{L^\infty(\mathbb{R})}$, which leads to (3.30). □

We let $\Omega_+ = [0,\infty) \times \mathbb{R}$.

**Lemma 3.5** *Let* $V_\varepsilon(t,x)$ *satisfy problem* (3.2), $\|V_{0x}\|_{L^\infty(\mathbb{R})} < \infty$, *and* $V_0 \in H^1(\mathbb{R})$. *Then there exists a subsequence* $\varepsilon_i \to 0$ *as* $i \to \infty$ *and* $V \in L^\infty([0,\infty); H^1(\mathbb{R})) \cap H^1([0,T]\times\mathbb{R})$ *to satisfy*

$$V_{\varepsilon_i} \rightharpoonup V \quad \text{in } H^1\left([0,T]\times\mathbb{R}\right), \tag{3.33}$$

$$V_{\varepsilon_i} \to V \quad \text{in } L^\infty_{\text{loc}}(\Omega_+). \tag{3.34}$$

*Proof* Since

$$\frac{\partial V_\varepsilon}{\partial t} + V_\varepsilon^3 \frac{\partial V_\varepsilon}{\partial x} - \frac{\partial G_\varepsilon}{\partial x} = \varepsilon \frac{\partial^2 V_\varepsilon}{\partial x^2} \tag{3.35}$$

for every fixed $T > 0$, applying Lemmas 3.1–3.3 and (3.35) yields

$$\left\|\frac{\partial V_\varepsilon}{\partial t}\right\|_{L^2(\mathbb{R})}, \left\|\frac{\partial V_\varepsilon}{\partial t}\right\|_{L^2([0,T]\times\mathbb{R})} \le C_0\left(1 + \sqrt{\varepsilon}\|V_0\|_{H^1(\mathbb{R})}\right), \tag{3.36}$$

where $C_0$ relies on $\|V_{0x}\|_{L^\infty(\mathbb{R})}$, $\|V_0\|_{H^1(\mathbb{R})}$, and $T$. Consequently, we obtain that $\{V_\varepsilon\}$ is bounded uniformly in the space $L^\infty([0,\infty);H^1(\mathbb{R}))\cap H^1([0,T]\times\mathbb{R})$ and (3.33) holds. Note that, for every $s,t\in[0,T]$,

$$\begin{aligned}\|V_\varepsilon(t,\cdot)-V_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}&=\int_{\mathbb{R}}\left(\int_s^t\frac{\partial V_\varepsilon}{\partial\tau}(\tau,x)\,d\tau\right)^2dx\\&\le|t-s|\int_{\mathbb{R}}\int_0^T\left(\frac{\partial V_\varepsilon}{\partial\tau}(\tau,x)\right)^2d\tau\,dx.\end{aligned}$$

Utilizing $H^1(\mathbb{R})\subset L^\infty_{\mathrm{loc}}(\mathbb{R})\subset L^2_{\mathrm{loc}}(\mathbb{R})$ and applying the conclusions in [22], we obtain (3.34). □

**Lemma 3.6** *Suppose that $\|V_{0x}\|_{L^\infty(\mathbb{R})}<\infty$ and $V_0\in H^1(\mathbb{R})$. Then the sequence $H_\varepsilon(t,x)$ is uniformly bounded in $W^{1,1}_{\mathrm{loc}}(\Omega_+)$. Moreover, there is a sequence $\varepsilon_i\to 0$ if $i\to\infty$ and a function $H\in L^\infty([0,\infty);W^{1,\infty}(\mathbb{R}))$ to satisfy*

$$H_{\varepsilon_i}\to H\quad\text{strongly in } L^r_{\mathrm{loc}}(\Omega_+),1<r<\infty.\tag{3.37}$$

*Proof* Applying notations $V=V_\varepsilon(t,x)$ and $h=h_\varepsilon$ for conciseness, we acquire

$$\begin{aligned}\frac{\partial H_\varepsilon}{\partial t}&=\big(g'(V)-V^3\big)V_t\\&\quad+\Lambda^{-2}\big[\big(V^3-g'(V)\big)V_t-5VV_th^2-5V^2hh_t+\partial_x\big(V_th^3+3Vh^2h_t\big)\big]\\&=\big[\big(g'(V)-V^3\big)V_t-\Lambda^{-2}\big(V^3-g'(V)\big)V_t\big]-5\big[\Lambda^{-2}Vh^2V_t\big]\\&\quad-\big[\Lambda^{-2}\Big[5V^2h\Big(H_\varepsilon-V^3h_x-\frac{1}{2}V^2h^2+\varepsilon h_{xx}\Big)\Big]-\big[\Lambda^{-2}\partial_x\big(V_th^3+3Vh^2h_t\big)\big]\\&=A_0+A_1+A_2+A_3.\end{aligned}\tag{3.38}$$

Applying (3.5) and (3.36) yields

$$|A_0|\le c\|V\|^2_{L^\infty(\mathbb{R})}\left(\int_{\mathbb{R}}V^2\,dx\right)^{\frac12}\left(\int_{\mathbb{R}}V_t^2\,dx\right)^{\frac12}\le c.\tag{3.39}$$

Using (3.13) and (3.36) gives rise to

$$\int_{\mathbb{R}}|A_1|\,dx\le c\left[\int_{\mathbb{R}}e^{-|x-\eta|}\left(\int_{\mathbb{R}}\big(Vh^2\big)^2\,d\eta\right)^{\frac12}\left(\int_{\mathbb{R}}V_t^2\,d\eta\right)^{\frac12}dx\le c.\right.\tag{3.40}$$

We have

$$\begin{aligned}|A_2|&\le c\left(\int_{\mathbb{R}}e^{-|x-\eta|}\left|5V^2hH_\varepsilon-\frac52V^4h^3\right|d\eta\right.\\&\quad\left.+\left|\int_{\mathbb{R}}e^{-|x-\eta|}5V^5hh_\eta\,d\eta\right|+\varepsilon\left|\int_{\mathbb{R}}e^{-|x-\eta|}5V^2hh_{\eta\eta}\,d\eta\right|\right)\\&\le c\left(\int_{\mathbb{R}}e^{-|x-\eta|}\left|5V^2hH_\varepsilon-\frac52V^4h^3\right|d\eta\right.\end{aligned}$$

$$+\frac{1}{2}\left|\int_{\mathbb{R}} e^{-|x-\eta|}h^2\left[25V^4h+\operatorname{sign}(\eta-x)V^5\right]d\eta\right|$$
$$+\varepsilon\left|\int_{\mathbb{R}} e^{-|x-\eta|}h_\eta\left[10Vh^2+5V^2h_\eta+\operatorname{sign}(\eta-x)5V^2h\right]d\eta\right|\Bigg). \tag{3.41}$$

Utilizing the Schwartz inequality, (3.13)–(3.14), and Lemma 3.1 yields

$$\begin{aligned}\int_{\mathbb{R}}|A_2|\,dx&\le c\left(1+\int_{\mathbb{R}}|h|^3\,d\eta\right)+\varepsilon c\int_{\mathbb{R}}\left(|hh_\eta|+\left|h^2h_\eta\right|+|h_\eta|^2\right)d\eta\\&\le c\left(1+\int_{\mathbb{R}}|h|^3\,d\eta\right)+\varepsilon c\int_{\mathbb{R}}\left(h^2+h^4+3|h_\eta|^2\right)d\eta\\&\le c\end{aligned}$$

and

$$\int_0^t\int_{\mathbb{R}}|A_2|\,dx\,dt\le cT. \tag{3.42}$$

For $A_3$, we have

$$\begin{aligned}A_3&=\Lambda^{-2}\left[V_th^3+3Vh^2h_t\right]_x\\&=\Lambda^{-2}\left[V_th^3+3Vh^2\left(H_\varepsilon-V^3h_x-\frac{1}{2}V^2h^2+\varepsilon h_{xx}\right)\right]_x\\&=\Lambda^{-2}\left[V_th^3+3Vh^2H_\varepsilon-\frac{3}{2}V^3h^4\right]_x-3\Lambda^{-2}\left[V^4h^2h_x\right]_x+3\varepsilon\Lambda^{-2}\left[Vh^2h_{xx}\right]_x\\&=I_1+I_2+I_3.\end{aligned} \tag{3.43}$$

From (3.6), we obtain

$$\begin{aligned}|I_1|&=\left|\frac{1}{2}\int_{-\infty}^{\infty}e^{-|x-\eta|}\left[V_th^3+3Vh^2H_\varepsilon-\frac{3}{2}V^3h^4\right]_\eta d\eta\right|\\&\le\int_{-\infty}^{\infty}e^{-|x-\eta|}\left|V_th^3+3Vh^2H_\varepsilon-\frac{3}{2}V^3h^4\right|d\eta.\end{aligned} \tag{3.44}$$

By simple calculation, we derive that

$$\begin{aligned}I_2&=-3\Lambda^{-2}\left[V^4h^2h_x\right]_x=-\Lambda^{-2}\left[\left(V^4h^3\right)_x-\left(V^4\right)_xh^3\right]_x\\&=-\Lambda^{-2}\left(1-\Lambda^2\right)\left(V^4h^3\right)+\Lambda^{-2}\left[4V^3h^4\right]_x\\&=-\Lambda^{-2}\left(V^4h^3\right)+4\Lambda^{-2}\left[V^3h^4\right]_x+V^4h^3\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}I_3&=3\varepsilon\Lambda^{-2}\left[\left(Vh^2h_x\right)_x-\left(Vh^2\right)_xh_x\right]_x\\&=3\varepsilon\Lambda^{-2}\left(1-\Lambda^2\right)\left[Vh^2h_x\right]-3\varepsilon\Lambda^{-2}\left[\left(h^3+2Vhh_x\right)h_x\right]_x\\&=-3\varepsilon\left(Vh^2h_x\right)+3\varepsilon\Lambda^{-2}\left[Vh^2h_x\right]-3\varepsilon\Lambda^{-2}\left[\left(h^3+2Vhh_x\right)h_x\right]_x.\end{aligned} \tag{3.46}$$

Using (3.13), (3.21), (3.36), and (3.44) gives rise to

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} |I_1|\,dx\,dt \\
&\quad\le c\int_0^t \left[1+\left(\int_{-\infty}^{\infty}(V_t)^2\,d\eta\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}h^6\,d\eta\right)^{\frac{1}{2}}+c\left(\int_{-\infty}^{\infty}h^4\,d\eta\right)\right]dt \\
&\quad\le c(1+T).
\end{aligned}\tag{3.47}$$

Applying (3.13), (3.14), (3.45), and the Tonelli theorem yields

$$\int_0^t \int_{\mathbb{R}} |I_2|\,dx\,dt \le c\int_0^t \left[1+\int_{\mathbb{R}}|h|^3\,dx+\int_{\mathbb{R}}|h|^4\,dx\right]dt \le c(1+T).\tag{3.48}$$

Using Lemmas 3.1 and 3.2 derives that

$$\begin{aligned}
&\varepsilon\int_0^t \int_{\mathbb{R}} \left|\left(Vh^2h_x\right)\right|dx\,dt \\
&\quad\le \varepsilon c\int_0^t \int_{\mathbb{R}} \left|h^2h_x\right|dx\,dt \le c\varepsilon\int_0^t \int_{\mathbb{R}} \left(h^4+h_x^2\right)dx\,dt \le c.
\end{aligned}\tag{3.49}$$

From (3.6), we obtain

$$\varepsilon\left|\Lambda^{-2}\left[\left(h^3+2Vhh_x\right)h_x\right]_x\right| \le \varepsilon\int_{\mathbb{R}} e^{-|x-\eta|}\left|\left(h^3+2Vhh_\eta\right)h_\eta\right|d\eta.$$

Note that

$$\left|hh_\eta^2\right| \le c\left[(hh_\eta)^2+h_\eta^2\right] \le c\left[h_\eta^2+\left(h^2h_\eta\right)^2+h_\eta^2\right] \le c\left[2h_\eta^2+h^4h_\eta^2\right].$$

We have

$$\begin{aligned}
&\varepsilon\int_0^t \int_{\mathbb{R}} \left|\Lambda^{-2}\left[\left(h^3+2Vhh_x\right)h_x\right]_x\right|dx\,dt \\
&\quad\le \varepsilon c\int_0^t \left(\int_{\mathbb{R}}\left|\left(h^3+2Vhh_\eta\right)h_\eta\right|d\eta\int_{\mathbb{R}} e^{-|x-\eta|}\,dx\right)dt \\
&\quad\le \varepsilon c\int_0^t \int_{\mathbb{R}}\left|\left(h^3+2Vhh_\eta\right)h_\eta\right|d\eta\,dt \\
&\quad\le \varepsilon c\int_0^t \int_{\mathbb{R}}\left(\left|hh^2h_\eta\right|+\left|hh_\eta^2\right|\right)d\eta\,dt \\
&\quad\le \varepsilon c\int_0^t \int_{\mathbb{R}}\left(h^6+h_\eta^2+h^4h_\eta^2\right)d\eta\,dt \le c,
\end{aligned}\tag{3.50}$$

in which we have used Lemma 3.1 and (3.8). From (3.43), (3.46), (3.47)–(3.50), we obtain $\int_0^t \int_{\mathbb{R}} |I_3|\,dx\,dt \le c(1+T)$ and

$$\int_0^t \int_{\mathbb{R}} |A_3|\,dx\,dt \le c(1+T).\tag{3.51}$$

From (3.38)–(3.41), (3.42), and (3.51), we derive that $\frac{\partial H_\varepsilon}{\partial t}$ is uniformly bounded in $L^1_{\mathrm{loc}}(\Omega_+)$. Making use of Corollary 4 on page 85 in [29] and Lemma 3.3, we conclude that $H_\varepsilon$ is uniformly bounded in $W^{1,1}_{\mathrm{loc}}(\Omega_+)$. Thus, we derive that (3.37) holds. □

We employ the overbars to represent weak supper limits.

**Lemma 3.7** *Let* $1 < r_1 < 6$ *and* $1 < r_2 < 3$. *Then there exists a subsequence* $\varepsilon_i \to 0$ *when* $i \to \infty$, *and two functions* $h \in L^{r_1}_{\mathrm{loc}}(\Omega_+)$ *and* $\overline{h^2} \in L^{r_2}_{\mathrm{loc}}(\Omega_+)$ *such that the following conclusions*

$$h_{\varepsilon_i} \rightharpoonup h \quad \text{in } L^{r_1}_{\mathrm{loc}}(\Omega_+), \qquad h_{\varepsilon_i} \overset{\star}{\rightharpoonup} h \quad \text{in } L^{\infty}_{\mathrm{loc}}\big([0,\infty); L^2(\mathbb{R})\big), \tag{3.52}$$

$$h^2_{\varepsilon_i} \rightharpoonup \overline{h^2} \quad \text{in } L^{r_2}_{\mathrm{loc}}(\Omega_+) \tag{3.53}$$

*hold. Moreover,*

$$h^2(t,x) \le \overline{h^2}(t,x) \quad \text{in the sense of distribution on } \Omega_+, \tag{3.54}$$

$$\frac{\partial V}{\partial x} = h \quad \text{in the sense of distribution on } \Omega_+. \tag{3.55}$$

*Proof* Employing Lemmas 3.1 and 3.2 leads to (3.52) and (3.53). Applying the weak convergence in (3.52), we have inequality (3.54). Applying Lemma 3.6, (3.52) and sending $\varepsilon \to 0$ directly give (3.55). The proof is finished. □

For writing concisely, we use $\{V_\varepsilon\}_{\varepsilon>0}$, $\{h_\varepsilon\}_{\varepsilon>0}$, and $\{H_\varepsilon\}_{\varepsilon>0}$ to replace the sequences $\{V_{\varepsilon_i}\}_{i\in N}$, $\{h_{\varepsilon_i}\}_{i\in N}$, and $\{H_{\varepsilon_i}\}_{i\in N}$ ($N$ denotes all the natural numbers), separately. For every convex function $\psi \in C^1(\mathbb{R})$ associated with $\psi'$ bounded, Lipschitz continuous in $\mathbb{R}$, using Lemma 3.7 yields

$$\psi(h_\varepsilon) \rightharpoonup \overline{\psi(h)} \quad \text{in } L^{r_1}_{\mathrm{loc}}(\Omega_+), 1 < r_1 < 6,$$

$$\psi(h_\varepsilon) \overset{\star}{\rightharpoonup} \overline{\psi(h)} \quad \text{in } L^{\infty}_{\mathrm{loc}}\big([0,\infty); L^2(\mathbb{R})\big).$$

Applying $\psi'(h_\varepsilon)$ to multiply Eq. (3.3), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\psi(h_\varepsilon) + \frac{1}{3}\frac{\partial}{\partial x}\big(V_\varepsilon^3 \psi(h_\varepsilon)\big) - \varepsilon\frac{\partial^2 \psi(h_\varepsilon)}{\partial x^2} + \varepsilon\psi''(h_\varepsilon)\left(\frac{\partial h_\varepsilon}{\partial x}\right)^2 \\
&\quad = V_\varepsilon^2 h_\varepsilon \psi(h_\varepsilon) - \frac{1}{2}V_\varepsilon^2 \psi'(h_\varepsilon) h_\varepsilon^2 - \frac{2}{3}V_\varepsilon^3 \frac{\partial \psi(h_\varepsilon)}{\partial x} + H_\varepsilon \psi'(h_\varepsilon).
\end{aligned} \tag{3.56}$$

**Lemma 3.8** *Assume* $\|V_{0x}\|_{L^\infty(\mathbb{R})} < \infty$ *and* $V_0 \in H^1(\mathbb{R})$. *For every convex* $\psi \in C^1(\mathbb{R})$ *associated with* $\psi'$ *bounded, Lipschitz continuous in* $\mathbb{R}$, *let* $\overline{h\psi(h)}$ *and* $\overline{\psi'(h)h^2}$ *represent the weak limits of* $h_\varepsilon\psi(h_\varepsilon)$ *and* $h_\varepsilon^2\psi'(h_\varepsilon)$ *in* $L^{r_2}_{\mathrm{loc}}(\Omega_+)$, $1 < r_2 < 3$, *respectively. Then, in the sense of distributions on* $\Omega_+$, *the inequality*

$$\begin{aligned}
&\overline{\frac{\partial \psi(h)}{\partial t}} + \frac{1}{3}\frac{\partial}{\partial x}\big(V^3 \overline{\psi(h)}\big) \\
&\quad \le V^2 \overline{h\psi(h)} - \frac{1}{2}V^2 \overline{\psi'(h)h^2} - \frac{2}{3}V^3 \overline{\frac{\partial \psi(h)}{\partial x}} + H\overline{\psi'(h)}
\end{aligned} \tag{3.57}$$

*holds.*

*Proof* Applying the assumptions of function $\psi$, Lemmas 3.5–3.7 and taking $\varepsilon \to 0$ in (3.56), we obtain (3.57). □

From (3.52) and (3.53), almost everywhere in $\Omega_+$, we have

$$\begin{cases} h = h_+ + h_- = \overline{h_+} + \overline{h_-}, \\ h^2 = (h_+)^2 + (h_-)^2, \qquad \overline{h^2} = \overline{(h_+)^2} + \overline{(h_-)^2}, \end{cases}$$

in which $\lambda_+ := \lambda_{\chi[0,+\infty)}(\lambda)$, $\lambda_- := \lambda_{\chi(-\infty,0]}(\lambda)$ for $\lambda \in \mathbb{R}$. Using Lemma 3.4 leads to

$$h_\varepsilon, h \leq c(1+t), \quad t \in [0,T].$$

**Lemma 3.9** *Assume* $\|V_{0x}\|_{L^\infty(\mathbb{R})} < \infty$ *and* $u_0 \in H^1(\mathbb{R})$. *In the sense of distributions on* $\Omega_+$, *then*

$$\frac{\partial h}{\partial t} + \frac{1}{3}\frac{\partial}{\partial x}\left(V^3 h\right) = \frac{1}{2}V^2\overline{h^2} - \frac{2}{3}V^3\frac{\partial h}{\partial x} + H. \tag{3.58}$$

*Proof* Making use of (3.3), Lemmas 3.6–3.7, and sending $\varepsilon \to 0$ in (3.3), we derive that (3.58) holds. □

A generalized formulation of (3.58) is described in the next lemma.

**Lemma 3.10** *Let assumptions in Theorem 2.2 hold. For every convex function* $\psi \in C^1(\mathbb{R})$ *with* $\psi' \in L^\infty(\mathbb{R})$ *and an arbitrary* $T > 0$, *then, in the sense of distributions on* $\Omega_+$, *the following identity*

$$\begin{aligned} &\frac{\partial \psi(h)}{\partial t} + \frac{1}{3}\frac{\partial}{\partial x}\left(V^3\psi(h)\right) \\ &\quad = V^2 h\psi(h) - V^2h^2\psi'(h) + \frac{1}{2}V^2\overline{h^2}\psi'(h) - \frac{2}{3}V^3\frac{\partial \psi(h)}{\partial x} + H\psi'(h) \end{aligned} \tag{3.59}$$

*holds.*

*Proof* Let $\{w_\gamma\}$ be a sequence of mollifiers in $\mathbb{R}$. We utilize notation $\star$ to represent the convolution about variable $x$. Assume

$$h_\gamma(t,x) = \left(h(t,\cdot) \star w_\gamma\right)(x).$$

Using $\psi'(h_\gamma)$ to multiply (3.58), we have

$$\begin{aligned} \frac{\partial \psi(h_\gamma)}{\partial t} &= \psi'(h_\gamma)\frac{\partial h_\gamma}{\partial t} \\ &= \psi'(h_\gamma)\left[-\frac{1}{3}\frac{\partial}{\partial x}\left(V^3 h\right) \star w_\gamma + \frac{1}{2}V^2\overline{h^2} \star w_\gamma \right. \\ &\quad \left. - \frac{2}{3}V^3\frac{\partial \psi(h)}{\partial x} \star w_\gamma + H \star w_\gamma\right]. \end{aligned} \tag{3.60}$$

Employing the assumption of $\psi$, $\psi'$ and taking $\gamma \to 0$ in (3.60), we have

$$\frac{\partial \psi(h)}{\partial t} + \frac{1}{3}V^3\frac{\partial \psi(h)}{\partial x} = -V^2h^2\psi'(h) + \frac{1}{2}V^2\overline{h^2}\psi'(h) - \frac{2}{3}V^3\frac{\partial \psi(h)}{\partial x} + H\psi'(h),$$

which leads to

$$\begin{aligned}&\frac{\partial \psi(h)}{\partial t} + \frac{1}{3}\frac{\partial [V^3\psi(h)]}{\partial x}\\&\quad = V^2h\psi(h) - V^2h^2\psi'(h) + \frac{1}{2}V^2\overline{h^2}\psi'(h) - \frac{2}{3}V^3\frac{\partial \psi(h)}{\partial x} + H\psi'(h).\end{aligned}$$

The desired result is obtained. □

## 4 Proof of the main result

Using the methods in [21] or [22], we will derive that $h_\varepsilon$ in (3.59) is strong convergence. Subsequently, we prove the existence of global weak solutions.

**Lemma 4.1** ([22]) *Let $V_0 \in H^1(\mathbb{R})$. Then*

$$\lim_{t\to 0}\int_{\mathbb{R}} h^2(t,x)\,dx = \lim_{t\to 0}\int_{\mathbb{R}} \overline{h^2}(t,x)\,dx = \int_{\mathbb{R}}\left(\frac{\partial V_0}{\partial x}\right)^2 dx.$$

**Lemma 4.2** ([22]) *For each constant $B > 0$, let $V_0 \in H^1(\mathbb{R})$. Then*

$$\lim_{t\to 0}\int_{\mathbb{R}}\left(\overline{\psi_B^{\pm}(h(t,x))} - \psi_B^{\pm}(h(t,x))\right)dx = 0,$$

*where*

$$\psi_B(\zeta) := \begin{cases}\frac{1}{2}\zeta^2, & \text{if } |\zeta| \le B,\\ B|\zeta| - \frac{1}{2}B^2, & \text{if } |\zeta| > B\end{cases}$$

*and $\psi_B^+(\zeta) := \psi_B(\zeta)\chi_{[0,+\infty)}(\zeta)$, $\psi_B^-(\zeta) := \psi_B(\zeta)\chi_{(-\infty,0]}(\zeta)$, $\zeta \in \mathbb{R}$.*

**Lemma 4.3** ([22]) *Suppose constant $B > 0$. Then, for every $\zeta \in \mathbb{R}$,*

$$\begin{cases}\psi_B(\zeta) = \frac{1}{2}\zeta^2 - \frac{1}{2}(B-|\zeta|)^2\chi_{(-\infty,-B)\cap(B,\infty)}(\zeta),\\ \psi_B'(\zeta)\zeta = \zeta + (B-|\zeta|)\,\mathrm{sign}(\zeta)\chi_{(-\infty,-B)\cap(B,\infty)}(\zeta),\\ \psi_B^+(\zeta) = \frac{1}{2}(\zeta_+)^2 - \frac{1}{2}(B-\zeta)^2\chi_{(B,\infty)}(\zeta),\\ (\psi_B^+)'(\zeta) = \zeta_+ + (B-\zeta)\chi_{(B,\infty)}(\zeta),\\ \psi_B^-(\zeta) = \frac{1}{2}(\zeta_-)^2 - \frac{1}{2}(B+\zeta)^2\chi_{(-\infty,-B)}(\zeta),\\ (\psi_B^-)'(\zeta) = \zeta_- - (B+\zeta)\chi_{(-\infty,-B)}(\zeta).\end{cases}$$

**Lemma 4.4** *Let $V_0 \in H^1(\mathbb{R})$ and $\|V_{0x}\|_{L^\infty(\mathbb{R})} < \infty$. Then the inequality*

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}} \big(\overline{(h_+)^2} - h_+^2\big)(t,x)\,dx \le{}& -\frac{3}{2}\int_0^t\int_{\mathbb{R}} V^3\left(\overline{\frac{\partial \psi_B^+(h)}{\partial x}} - \frac{\partial \psi_B^+(h)}{\partial x}\right)dx\,ds \\ &+ \int_0^t\int_{\mathbb{R}} H(s,x)\big[\overline{h_+}(s,x) - h_+(s,x)\big]\,dx\,ds \end{aligned}\tag{4.1}$$

*holds.*

*Proof* For $t \in (0,T)$, we choose that $B$ is sufficiently large to satisfy $B > c$ (see Lemma 3.4). Employing Lemmas 3.8 and 3.10, and $\psi_B^+$ (see Lemma 4.3) from (3.57) and (3.59) leads to

$$\begin{aligned}&\frac{\partial}{\partial t}\big(\overline{\psi_B^+(h)} - \psi_B^+(h)\big) + \frac{1}{3}\frac{\partial}{\partial x}\big(V^3\big[\overline{\psi_B^+(h)} - \psi_B^+(h)\big]\big) \\ &\quad\le V^2\big(\overline{h\psi_B^+(h)} - h\psi_B^+(h)\big) - \frac{V^2}{2}\big(\overline{h^2(\psi_B^+)'(h)} - h^2(\psi_B^+)'(h)\big) \\ &\qquad - \frac{1}{2}V^2\big(\overline{h^2} - h^2\big)(\psi_B^+)'(h) - \frac{2}{3}V^3\left(\overline{\frac{\partial \psi_B^+(h)}{\partial x}} - \frac{\partial \psi_B^+(h)}{\partial x}\right) \\ &\qquad + H(t,x)\big(\overline{(\psi_B^+)'(h)} - (\psi_B^+)'(h)\big). \end{aligned}\tag{4.2}$$

Since $\psi_B^+$ is increasing, we have

$$-V^2\big(\overline{h^2} - h^2\big)(\psi_B^+)'(h) \le 0. \tag{4.3}$$

Applying Lemma 4.3 results in

$$\begin{aligned}&h\psi_B^+(h) - \frac{1}{2}h^2(\psi_B^+)'(h) = -\frac{B}{2}h(B-h)\chi_{(B,\infty)}(h), \\ &\overline{h\psi_B^+(h)} - \frac{1}{2}\overline{h^2(\psi_B^+)'(h)} = -\frac{B}{2}\overline{h(B-h)\chi_{(B,\infty)}(h)}. \end{aligned}\tag{4.4}$$

Making use of Lemma 3.4, we choose sufficiently large $B > 0$ to ensure $h < c(1+t) < B$. Let $\Upsilon_B = (0, \frac{B}{c} - 1) \times \mathbb{R}$. From (4.4), we have

$$h\psi_B^+(h) - \frac{1}{2}h^2(\psi_B^+)'(h) = \overline{h\psi_B^+(h)} - \frac{1}{2}\overline{h^2(\psi_B^+)'(h)} = 0 \quad \text{in } \Upsilon_B.$$

In $(0, \frac{B}{c} - 1) \times \mathbb{R}$, it holds that

$$\begin{cases} \psi_B^+ = \frac{1}{2}(h_+)^2, & (\psi_B^+)'(h) = h_+, \\ \overline{\psi_B^+(h)} = \frac{1}{2}\overline{(h_+)^2}, & \overline{(\psi_B^+)'(h)} = \overline{h_+}. \end{cases}\tag{4.5}$$

Using (4.2)–(4.5), in the domain $(0, \frac{B}{c} - 1) \times \mathbb{R}$, we have the following inequality:

$$\begin{aligned}&\frac{\partial}{\partial t}\big(\overline{\psi_B^+(h)} - \psi_B^+(h)\big) + \frac{1}{3}\frac{\partial}{\partial x}\big(V^3\big[\overline{\psi_B^+(h)} - \psi_B^+(h)\big]\big) \\ &\quad\le -\frac{2}{3}V^3\left(\overline{\frac{\partial \psi_B^+(h)}{\partial x}} - \frac{\partial \psi_B^+(h)}{\partial x}\right) + H(t,x)\big(\overline{(\psi_B^+)'(h)} - (\psi_B^+)'(h)\big). \end{aligned}\tag{4.6}$$

For almost all $0 < t < \frac{B}{c} - 1$, integrating (4.6) over $(0,t) \times \mathbb{R}$ leads to

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}\left(\overline{(h_+)^2} - h_+^2\right) dx \le \lim_{t\to 0}\int_{\mathbb{R}}\left[\overline{\psi_B^+(h)}(t,x) - \psi_B^+(h)\right] dx \\ - \frac{2}{3}\int_0^t\int_{\mathbb{R}} V^3\left(\overline{\frac{\partial\psi_B^+(h)}{\partial x}} - \frac{\partial\psi_B^+(h)}{\partial x}\right) dx\, ds \\ + \int_0^t\int_{\mathbb{R}} H(s,x)\left[\overline{h_+}(s,x) - h_+(s,x)\right] dx\, ds.\end{aligned}$$

Setting $B \to \infty$ and utilizing Lemma 4.2 yield the desired result. □

**Lemma 4.5** *Suppose* $V_0 \in H^1(\mathbb{R})$ *and* $\|V_{0x}\|_{L^\infty(\mathbb{R})} < \infty$. *Then*

$$\begin{aligned}\int_{\mathbb{R}}\left(\overline{\psi_B^-(h)} - \psi_B^-(h)\right) dx \le \frac{B^2}{2}\int_0^t\int_{\mathbb{R}} V^2\overline{(B+h)\chi_{(-\infty,-B)}(h)}\, dx\, ds \\ - \frac{B^2}{2}\int_0^t\int_{\mathbb{R}} V^2(B+h)\chi_{(-\infty,-B)}(h)\, dx\, ds \\ + B\int_0^t\int_{\mathbb{R}} V^2\left[\overline{\psi_B^-(h)} - \psi_B^-(h)\right] dx\, ds \\ + \frac{B}{2}\int_0^t\int_{\mathbb{R}} V^2\left(\overline{h_+^2} - h_+^2\right) dx\, ds \\ - \frac{2}{3}\int_0^t\int_{\mathbb{R}} V^3\left(\overline{\frac{\partial\psi_B^-(h)}{\partial x}} - \frac{\partial\psi_B^-(h)}{\partial x}\right) dx\, ds \\ + \int_0^t\int_{\mathbb{R}} H(t,x)\left(\overline{(\psi_B^-)'(h)} - (\psi_B^-)'(h)\right) dx\, ds. \qquad (4.7)\end{aligned}$$

*Proof* From Lemmas 3.4 and 4.3, choosing that $B > 0$ is suitably large and using $\psi_B^-$, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\overline{\psi_B^-(h)} - \psi_B^-(h)\right) + \frac{1}{3}\frac{\partial}{\partial x}\left(V^3\left[\overline{\psi_B^-(h)} - \psi_B^-(h)\right]\right) \\ &\quad\le V^2\left(\overline{h\psi_B^-(h)} - h\psi_B^-(h)\right) - \frac{V^2}{2}\left(\overline{h^2(\psi_B^-)'(h)} - h^2(\psi_B^-)'(h)\right) \\ &\qquad - \frac{1}{2}V^2\left(\overline{h^2} - h^2\right)(\psi_B^-)'(h) - \frac{2}{3}V^3\left(\overline{\frac{\partial\psi_B^-(h)}{\partial x}} - \frac{\partial\psi_B^-(h)}{\partial x}\right) \\ &\qquad + H(t,x)\left(\overline{(\psi_B^-)'(h)} - (\psi_B^-)'(h)\right). \qquad (4.8)\end{aligned}$$

As $-B \le (\psi_B^-)' \le 0$ and $V^2 \ge 0$, we have

$$-\frac{V^2}{2}\left(\overline{h^2} - h^2\right)(\psi_B^-)'(h) \le \frac{BV^2}{2}\left(\overline{h^2} - h^2\right). \qquad (4.9)$$

Using (4.8)–(4.9) and Lemma 4.3 yields

$$V^2 h\psi_B^-(h) - \frac{V^2}{2}h^2(\psi_B^-)'(h) = -\frac{BV^2}{2}h(B+h)\chi_{(-\infty,-B)}(h), \qquad (4.10)$$

$$V^2\overline{h\psi_B^-(h)} - \frac{V^2}{2}\overline{h^2(\psi_B^-)'(h)} = -\frac{BV^2}{2}\overline{h(B+h)\chi_{(-\infty,-B)}(h)}. \qquad (4.11)$$

Applying (4.10)–(4.11) leads to

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(\overline{\psi_B^-(h)}-\psi_B^-(h)\big)+\frac{1}{3}\frac{\partial}{\partial x}\big(V^3\big[\overline{\psi_B^-(h)}-\psi_B^-(h)\big]\big)\\
&\quad\le -\frac{1}{2}BV^2\overline{h(B+h)\chi_{(-\infty,-B)}(h)}+\frac{1}{2}BV^2h(B+h)\chi_{(-\infty,-B)}(h)\\
&\qquad+\frac{1}{2}BV^2\big(\overline{h^2}-h^2\big)-\frac{2}{3}V^3\left(\overline{\frac{\partial\psi_B^-(h)}{\partial x}}-\frac{\partial\psi_B^-(h)}{\partial x}\right)\\
&\qquad+H(t,x)\big(\overline{\big(\psi_B^-\big)'(h)}-\big(\psi_B^-\big)'(h)\big).
\end{aligned} \tag{4.12}$$

Integrating (4.12) over $(0,t)\times\mathbb{R}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\big(\overline{\psi_B^-(h)}-\psi_B^-(h)\big)(t,x)\,dx\\
&\quad\le -\frac{B}{2}\int_0^t\int_{\mathbb{R}}V^2\overline{h(B+h)\chi_{(-\infty,-B)}(h)}\,dx\,ds\\
&\qquad+\frac{B}{2}\int_0^t\int_{\mathbb{R}}V^2h(B+h)\chi_{(-\infty,-B)}(h)\,dx\,ds+\frac{B}{2}\int_0^t\int_{\mathbb{R}}V^2\big(\overline{h^2}-h^2\big)\,dx\,ds\\
&\qquad-\frac{2}{3}\int_0^t\int_{\mathbb{R}}V^3\left(\overline{\frac{\partial\psi_B^-(h)}{\partial x}}-\frac{\partial\psi_B^-(h)}{\partial x}\right)dx\,ds\\
&\qquad+\int_0^t\int_{\mathbb{R}}H(t,x)\big(\overline{\big(\psi_B^-\big)'(h)}-\big(\psi_B^-\big)'(h)\big)\,dx\,ds.
\end{aligned} \tag{4.13}$$

Using Lemma 4.3 yields

$$\begin{aligned}
\overline{\psi_B^-(h)}-\psi_B^-(h)&=\frac{1}{2}\big(\overline{(h_-)^2}-(h_-)^2\big)\\
&\quad+\frac{1}{2}(B+h)^2\chi_{(-\infty,-B)}(h)-\frac{1}{2}\overline{(B+h)^2\chi_{(-\infty,-B)}(h)},
\end{aligned}$$

which results in

$$\begin{aligned}
BV^2\big[\overline{\psi_B^-(h)}-\psi_B^-(h)\big]&=\frac{B}{2}V^2\big(\overline{(h_-)^2}-(h_-)^2\big)\\
&\quad+\frac{B}{2}V^2(B+h)^2\chi_{(-\infty,-B)}(h)-\frac{B}{2}V^2\overline{(B+h)^2\chi_{(-\infty,-B)}(h)}.
\end{aligned} \tag{4.14}$$

Making use of (4.8), (4.13), and (4.14), we acquire

$$\begin{aligned}
&\int_{\mathbb{R}}\big(\overline{\psi_B^-(h)}-\psi_B^-(h)\big)(t,x)\,dx\\
&\quad\le -\frac{B}{2}\int_0^t\int_{\mathbb{R}}V^2\overline{h(B+h)\chi_{(-\infty,-B)}(h)}\,dx\,ds\\
&\qquad+\frac{B}{2}\int_0^t\int_{\mathbb{R}}V^2h(B+h)\chi_{(-\infty,-B)}(h)\,dx\,ds+B\int_0^t\int_{\mathbb{R}}V^2\big[\overline{\psi_B^-(h)}-\psi_B^-(h)\big]\,dx\,ds\\
&\qquad+\frac{B}{2}\int_0^t\int_{\mathbb{R}}V^2\overline{(B+h)^2\chi_{(-\infty,-B)}(h)}\,dx\,ds
\end{aligned}$$

$$
\begin{aligned}
&-\frac{B}{2}\int_0^t\int_{\mathbb{R}} V^2(B+h)^2\chi_{(-\infty,-B)}(h)\,dx\,ds \\
&+\frac{B}{2}\int_0^t\int_{\mathbb{R}} V^2\big(\overline{h_+^2}-h_+^2\big)\,dx\,ds-\frac{2}{3}\int_0^t\int_{\mathbb{R}} V^3\left(\overline{\frac{\partial\psi_B^-(h)}{\partial x}}-\frac{\partial\psi_B^-(h)}{\partial x}\right)dx\,ds \\
&+\int_0^t\int_{\mathbb{R}} H(t,x)\big(\overline{(\psi_B^-)'(h)}-(\psi_B^-)'(h)\big)\,dx\,ds. \qquad (4.15)
\end{aligned}
$$

Using the identity $B(B+h)^2-Bh(B+h)=B^2(B+h)$ and (4.15), we obtain (4.7). □

**Lemma 4.6** *Assume that all the assumptions in Theorem 2.2 hold. Then*

$$\overline{h^2}=h^2 \quad \text{almost everywhere in } \Omega_+. \qquad (4.16)$$

*Proof* Applying inequalities (4.1) and (4.7), we have

$$
\begin{aligned}
&\int_{\mathbb{R}}\left(\frac{1}{2}\big[\overline{(h_+)^2}-(h_+)^2\big]+\big[\overline{\psi_B^-}-\psi_B^-\big]\right)(t,x)\,dx \\
&\le\frac{B^2}{2}\int_0^t\int_{\mathbb{R}} V^2\overline{(B+h)\chi_{(-\infty,-B)}(h)}\,dx\,ds \\
&-\frac{B^2}{2}\int_0^t\int_{\mathbb{R}} V^2(B+h)\chi_{(-\infty,-B)}(h)\,dx\,ds+B\int_0^t\int_{\mathbb{R}} V^2\big[\overline{\psi_B^-(h)}-\psi_B^-(h)\big]\,dx\,ds \\
&+\frac{B}{2}\int_0^t\int_{\mathbb{R}} V^2\big(\overline{h_+^2}-h_+^2\big)\,dx\,ds-\frac{2}{3}\int_0^t\int_{\mathbb{R}} V^3\left(\overline{\frac{\partial\psi_B(h)}{\partial x}}-\frac{\partial\psi_B(h)}{\partial x}\right)dx\,ds \\
&+\int_0^t\int_{\mathbb{R}} H(s,x)\big([\overline{h_+}-h_+]+\big[\overline{(\psi_B^-)'(h)}-(\psi_B^-)'(h)\big]\big)\,dx\,ds. \qquad (4.17)
\end{aligned}
$$

Using Lemma 3.6, for $0<t<T$, we can find a constant $K_0>0$ such that

$$\big\|H(t,x)\big\|_{L^\infty([0,T)\times\mathbb{R})}\le K_0. \qquad (4.18)$$

We use (4.18) and Lemma 4.3 to obtain

$$
\begin{aligned}
&h_+ + \big(\psi_B^-\big)'(h)=h-(B+h)\chi_{(-\infty,-B)}(h), \\
&\overline{h_+}+\overline{\big(\psi_B^-\big)'(h)}=h-\overline{(B+h)\chi_{(-\infty,-B)}(h)}. \qquad (4.19)
\end{aligned}
$$

Using the convex property of the map $\zeta\to\zeta_+ + (\psi_B^-)'(\zeta)$ yields

$$
\begin{aligned}
0&\le[\overline{h_+}-h_+]+\big[\overline{\big(\psi_B^-\big)'(h)}-\big(\psi_B^-\big)'(h)\big] \\
&=(B+q)\chi_{(-\infty,-B)}-\overline{(B+h)\chi_{(-\infty,-B)}(h)}\le 0. \qquad (4.20)
\end{aligned}
$$

Utilizing (4.17) and (4.20) yields

$$
\begin{aligned}
&\int_{\mathbb{R}}\left(\frac{1}{2}\big[\overline{(h_+)^2}-(h_+)^2\big]+\big[\overline{\psi_B^-}-\psi_B^-\big]\right)(t,x)\,dx \\
&\le B\int_0^t\int_{\mathbb{R}} V^2\big[\overline{\psi_B^-(h)}-\psi_B^-(h)\big]\,dx\,ds+\frac{B}{2}\int_0^t\int_{\mathbb{R}} V^2\big(\overline{h_+^2}-h_+^2\big)\,dx\,ds
\end{aligned}
$$

$$+\int_0^t\int_{\mathbb{R}} 2V^2h\big(\overline{\psi_B(h)}-\psi_B(h)\big)\,dx\,ds. \tag{4.21}$$

From Lemma 3.4, we choose sufficiently large $B$ satisfying $V^2h\le cB$. Thus, from (4.21) and Lemma 4.3, we acquire

$$\begin{aligned}0&\le\int_{\mathbb{R}}\left(\frac{1}{2}\big[\overline{(h_+)^2}-(h_+)^2\big]+\big[\overline{\psi_B^-}-\psi_B^-\big]\right)(t,x)\,dx\\&\le cB\int_0^t\int_{\mathbb{R}}\left(\frac{1}{2}\big[\overline{(h_+)^2}-(h_+)^2\big]+\big[\overline{\psi_B^-}-\psi_B^-\big]\right)(t,x)\,dx\,ds.\end{aligned} \tag{4.22}$$

For each $t>0$, the Gronwall inequality is used for (4.22) to yield

$$0\le\int_{\mathbb{R}}\left(\frac{1}{2}\big[\overline{(h_+)^2}-(h_+)^2\big]+\big[\overline{\psi_B^-}-\psi_B^-\big]\right)(t,x)\,dx=0. \tag{4.23}$$

Using the Fatou lemma and letting $B\to\infty$ in (4.23) lead to rise to

$$0\le\int_{\mathbb{R}}\big(\overline{h^2}-h^2\big)(t,x)\,dx=0,$$

which results in (4.16). □

*Proof of Theorem* 2.2 (a) and (b) in Definition 2.1 are derived by directly applying Lemmas 3.1 and 3.5. From Lemma 4.6, we have

$$h_\varepsilon\to h\quad\text{in } L^2_{\mathrm{loc}}(\Omega_+). \tag{4.24}$$

From Lemma 3.5 and (4.24), we conclude that $V$ is a global weak solution to system (2.2). From Lemmas 3.2 and 3.4, we obtain that (2.3) and (2.4) hold. The proof is finished. □

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**Competing interests**
The authors declare no competing interests.

**Author contributions**
Tang and Liu give the main derivations of inequalities in this work. Lai gives the methods to establish the structures of this work.

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