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Unconditional superconvergence analysis of an energy-stable finite element scheme for nonlinear Benjamin–Bona–Mahony–Burgers equation

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Abstract

In this paper, an energy-stable Crank–Nicolson fully discrete finite element scheme is proposed for the Benjamin–Bona–Mahony–Burgers equation. Firstly, the stability of energy is proved, which leads to the boundedness of the finite element solution in H^1-norm. Secondly, combining with the above boundedness and the special property of bilinear element, the unconditional superclose and superconvergence results are derived. Finally, numerical examples are provided to illustrate the validity and efficiency of our theoretical analysis and method.

Keywords: Benjamin–Bona–Mahony–Burgers equation; Energy-stable scheme; Unconditional superconvergence analysis

1 Introduction

The nonlinear Benjamin–Bona–Mahony–Burgers (BBMB) equation is often used to describe the propagation of small amplitude long waves in a nonlinear dispersive medium with dissipative effect, which is considered as the following second-order partial differential equation [1]:

$$\begin{cases} u_t - \alpha \Delta u_t - \beta \Delta u = \nabla \cdot \vec{f}(u), & (X,t) \in \Omega \times (0,T], \\ u(X,t) = 0, & (X,t) \in \partial\Omega \times (0,T], \\ u(X,0) = u_0(X), & X \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is an open bounded convex polygonal domain with boundary $\partial\Omega$, $\alpha > 0$, $\beta > 0$, $0 < T < \infty$ are given constants, $X = (x,y)$, $\vec{f}(u) = -(\frac{1}{2}u^2 + u, \frac{1}{2}u^2 + u)$, Δ and $\nabla\cdot$ denote the two-dimensional Laplace and divergence operators, respectively, $u_0(X)$ is a given smooth function. It is remarkable that when $\alpha = 0$, $\beta > 0$, (1) is called Burgers' equation, when $\alpha > 0$, $\beta = 0$, (1) is called Benjamin–Bona–Mahony (BBM) equation. Various analytical and computational methods have been proposed to solve Burgers' and BBM equations, readers with more interests may refer to [2–7] and the references listed.

Due to the nonlinearity of BBMB equation, it is very difficult to find out the true solution. Thus, a lot of numerical methods have been considered, such as the finite difference methods [8–11], collocation method [12], meshless method [13, 14], finite element method (FEM) [15–18], and so on. For the FEM, Kadri [15] proposed semi-discrete and two kinds of fully discrete Galerkin schemes, studied the L^∞-norm error estimates; Kundu [16] established the convergence of unsteady solution to steady state solution; Karakoc [17] obtained the convergence analysis by use of a cubic B-spline FEM; Gao [18] discussed the local discontinuous Galerkin FEM and derived an optimal error estimate. However, [15–18] only focus on the convergence for the one-dimensional (1D) BBMB equation, there are few works for the 2D case till now.

It is well known that the superconvergence analysis is an important approach to improve the precision of FE solution. More precisely, based on the so-called integral identity technique, the order of error in H^1-norm between FE approximation u_h and the interpolation of the exact solution $I_h u$ is much better than that of u and $I_h u$; this fascinating characteristic is called superclose. The global superconvergence will then be investigated by adding a postprocessing without changing the existing FE program. Meanwhile, superconvergence is critical in practical engineering numerical calculation and has always been a research hotspot. To find out more applications, readers may refer to [19–23]. As far as our knowledge is concerned, research on superconvergence for 2D BBMB equation is not yet to be found.

In this work, as a first attempt, we develop an energy-stable conforming FE scheme for problem (1) and study the superclose and superconvergence error estimates. The outline is organized as follows: in Sect. 2, the FE space and the Crank–Nicolson (C-N) fully discrete scheme are provided, then the stability of energy and the boundedness of the numerical solution in H^1-norm are proved; in Sect. 3, the unconditional superclose and superconvergence results are derived without the restriction of the ratio between mesh size parameter h and time step Δt; in the last section, three numerical examples are given to verify the theoretical analysis.

2 The FE space and energy-stable scheme

Assume that $W^{k,p}(\Omega)$ is the standard Sobolev space with the norm $\|\cdot\|_{W^{k,p}(\Omega)}$, $H^k(\Omega) = W^{k,2}(\Omega)$, $H^0(\Omega) = L^2(\Omega)$ with the norm $\|\cdot\|_k$ and $\|\cdot\|_0$, the inner-product in $L^2(\Omega)$ is defined by $(\cdot,\cdot)$.

Denote T_h to be a regular rectangular subdivision of Ω. For $K \in T_h$, $h_K = \operatorname{diam} K$, $h = \max_{K\in T_h} h_K$. The bilinear element space V_h is defined by

$$V_h = \bigl\{v_h \in H_0^1(\Omega) : v_h|_K \in span\{1,x,y,xy\}, v_h|_{\partial\Omega} = 0, \forall K \in T_h\bigr\}.$$

The associated interpolation operator is defined as $I_h : v \in V = H_0^1(\Omega) \to I_h v \in V_h$.

The variational form of (1) is to find $u \in V$ such that

$$\begin{cases} (u_t, v) + \alpha(\nabla u_t, \nabla v) + \beta(\nabla u, \nabla v) = -(\vec{f}(u), \nabla v), \quad \forall v \in V, \\ u(X,0) = u_0(X). \end{cases} \tag{2}$$

Let $\{t_n | t_n = n\Delta t; n = 0,1,2,\ldots,N\}$ be a uniform partition of $[0,T]$ with $\Delta t = T/N$. For a given continuous function u on $[0,T]$, we define that $u^n = u(X,t_n)$, $\bar{\partial}_t u^n = \frac{u^n - u^{n-1}}{\Delta t}$, $t_{n-\frac{1}{2}} =$

$(n-\frac{1}{2})\Delta t$, $u^{n-\frac{1}{2}}=\frac{u^n+u^{n-1}}{2}$. The C-N fully discrete scheme for (2) is to find $U_h^n\in V_h$ such that

$$\begin{cases}(\bar{\partial}_t U_h^n, v_h)+\alpha(\nabla\bar{\partial}_t U_h^n,\nabla v_h)+\beta(\nabla U_h^{n-\frac{1}{2}},\nabla v_h)=-(\vec{f}(U_h^{n-\frac{1}{2}}),\nabla v_h), \quad \forall v_h\in V_h,\\ U_h^n(X,0)=I_h u_0(X).\end{cases} \tag{3}$$

First of all, we achieve the following special properties of bilinear element from [19, 24].

Lemma 2.1 *For all $v_h\in V_h$, there hold*

$$\|u-I_h u\|_{0,p}\le Ch^2\|u\|_{2,p}, \quad u\in W^{2,p}(\Omega), \tag{4}$$

$$\big(\nabla(u-I_h u),\nabla v_h\big)\le Ch^2\|u\|_3\|v_h\|_1, \quad u\in H^3(\Omega). \tag{5}$$

Here and later, we denote by C a generic positive constant which is independent of h and Δt and may stand for different values at different places.

Then the energy stability of (3) and the boundedness of $\|U_h^n\|_1$ are proved as follows.

Theorem 2.1 *Let $E^n=\|U_h^n\|_0^2+\alpha|U_h^n|_1^2$ $(n=0,1,\ldots,N)$ be the discrete energy, U_h^n is the solution of (3), then there holds*

$$E^n\le E^{n-1} \quad (n=1,2,\ldots,N). \tag{6}$$

Furthermore, we obtain

$$\|U_h^n\|_1\le M_1, \tag{7}$$

where $M_1=\sqrt{\frac{\max\{1,\alpha\}}{\min\{1,\alpha\}}}\|U_h^0\|_1$ is a positive constant.

Proof Taking $v_h=U_h^{n-\frac{1}{2}}$ in (3), we can get

$$\big(\bar{\partial}_t U_h^n, U_h^{n-\frac{1}{2}}\big)+\alpha\big(\nabla\bar{\partial}_t U_h^n,\nabla U_h^{n-\frac{1}{2}}\big)+\beta\big(\nabla U_h^{n-\frac{1}{2}},\nabla U_h^{n-\frac{1}{2}}\big)=-\big(\vec{f}(U_h^{n-\frac{1}{2}}),\nabla U_h^{n-\frac{1}{2}}\big). \tag{8}$$

Firstly, the left-hand side of (8) can be rewritten as

$$\frac{1}{2\Delta t}\big(\|U_h^n\|_0^2-\|U_h^{n-1}\|_0^2\big)+\frac{\alpha}{2\Delta t}\big(|U_h^n|_1^2-|U_h^{n-1}|_1^2\big)+\beta\|\nabla U_h^{n-\frac{1}{2}}\|_0^2. \tag{9}$$

Secondly, the right-hand side can be split as

$$\begin{aligned}-\big(\vec{f}(U_h^{n-\frac{1}{2}}),\nabla U_h^{n-\frac{1}{2}}\big)&=\int_\Omega\left(\frac{1}{2}(U_h^{n-\frac{1}{2}})^2+U_h^{n-\frac{1}{2}}\right)\big[(U_h^{n-\frac{1}{2}})_x+(U_h^{n-\frac{1}{2}})_y\big]\,dx\,dy\\ &=\int_\Omega\frac{1}{2}(U_h^{n-\frac{1}{2}})^2\big[(U_h^{n-\frac{1}{2}})_x+(U_h^{n-\frac{1}{2}})_y\big]\,dx\,dy\\ &\quad+\int_\Omega U_h^{n-\frac{1}{2}}\big[(U_h^{n-\frac{1}{2}})_x+(U_h^{n-\frac{1}{2}})_y\big]\,dx\,dy.\end{aligned} \tag{10}$$

By using the Green formula and noting that $U_h^{n-\frac{1}{2}}|_{\partial\Omega} = 0$, we obtain

$$\begin{aligned}
&\int_\Omega (U_h^{n-\frac{1}{2}})^2 \big[(U_h^{n-\frac{1}{2}})_x + (U_h^{n-\frac{1}{2}})_y\big]\,dx\,dy \\
&\quad = -\int_\Omega (U_h^{n-\frac{1}{2}}) \big[2U_h^{n-\frac{1}{2}}(U_h^{n-\frac{1}{2}})_x + 2U_h^{n-\frac{1}{2}}(U_h^{n-\frac{1}{2}})_y\big]\,dx\,dy \\
&\qquad + \int_{\partial\Omega} U_h^{n-\frac{1}{2}}(U_h^{n-\frac{1}{2}})^2 \cdot \vec{n}\,ds \\
&\quad = -\int_\Omega 2(U_h^{n-\frac{1}{2}})^2 \big[(U_h^{n-\frac{1}{2}})_x + (U_h^{n-\frac{1}{2}})_y\big]\,dx\,dy,
\end{aligned}$$

where $\vec{n}$ is the outer normal vector, so we arrive at

$$\int_\Omega (U_h^{n-\frac{1}{2}})^2 \big[(U_h^{n-\frac{1}{2}})_x + (U_h^{n-\frac{1}{2}})_y\big]\,dx\,dy = 0. \tag{11}$$

Similarly, we have

$$\int_\Omega U_h^{n-\frac{1}{2}} \big[(U_h^{n-\frac{1}{2}})_x + (U_h^{n-\frac{1}{2}})_y\big]\,dx\,dy = 0, \tag{12}$$

substituting (11) and (12) into (10), we obtain

$$\big(\vec{f}(U_h^{n-\frac{1}{2}}), \nabla U_h^{n-\frac{1}{2}}\big) = 0. \tag{13}$$

From (8), (9), and (13), there holds

$$\frac{1}{2\Delta t}\big[\big(\|U_h^n\|_0^2 - \|U_h^{n-1}\|_0^2\big) + \alpha\big(|U_h^n|_1^2 - |U_h^{n-1}|_1^2\big)\big] + \beta\|\nabla U_h^{n-\frac{1}{2}}\|_0^2 = 0, \tag{14}$$

therefore, we have

$$\|U_h^n\|_0^2 + \alpha|U_h^n|_1^2 + 2\Delta t\beta\|\nabla U_h^{n-\frac{1}{2}}\|_0^2 = \|U_h^{n-1}\|_0^2 + \alpha|U_h^{n-1}|_1^2,$$

which implies $E^n \le E^{n-1}$, (6) is obtained.

Next we start to demonstrate (7). Multiplying $2\Delta t$ on both sides of (14), replacing n with i, and summing for i from 1 to n, we have

$$\|U_h^n\|_0^2 + \alpha|U_h^n|_1^2 + 2\Delta t\beta\sum_{i=1}^{n}\|\nabla U_h^{i-\frac{1}{2}}\|_0^2 = \|U_h^0\|_0^2 + \alpha|U_h^0|_1^2. \tag{15}$$

From (15) and the triangular inequality, we have

$$\min\{1,\alpha\}\|U_h^n\|_1^2 \le \max\{1,\alpha\}\|U_h^0\|_1^2,$$

which ends the proof. □

3 Superclose and superconvergence analysis

We first demonstrate the following unconditional superclose result.

Theorem 3.1 *Let u^n and U_h^n be solutions of (2) and (3), respectively. Assume that $u \in L^\infty(0,T;H^3(\Omega))$, $u_t \in L^2(0,T;H^3(\Omega))$, $u_{tt}, u_{ttt} \in L^2(0,T;H^1(\Omega))$, there holds*

$$\|I_h u^n - U_h^n\|_1 \le C\big(h^2 + (\Delta t)^2\big), \tag{16}$$

where $\Delta t > 0$ is small enough so that $1 - C\Delta t > 0$.

Proof Let $u^n - U_h^n = (u^n - I_h u^n) + (I_h u^n - U_h^n) := \xi^n + \eta^n$, then the error equation can be derived by (2) and (3):

$$\begin{aligned}
&\big(\bar\partial_t \eta^n, v_h\big) + \alpha\big(\nabla\bar\partial_t\eta^n, \nabla v_h\big) + \beta\big(\nabla\eta^{n-\frac12}, \nabla v_h\big) \\
&\quad= -\big(\bar\partial_t\xi^n, v_h\big) - \alpha\big(\nabla\bar\partial_t\xi^n, \nabla v_h\big) - \beta\big(\nabla\xi^{n-\frac12}, \nabla v_h\big) \\
&\qquad + \big(\vec f\big(U_h^{n-\frac12}\big) - \vec f\big(u(t_{n-\frac12})\big), \nabla v_h\big) - \big(R_1^n, v_h\big) \\
&\qquad - \alpha\big(\nabla R_1^n, \nabla v_h\big) - \beta\big(\nabla R_2^n, \nabla v_h\big),
\end{aligned} \tag{17}$$

where $R_1^n = u_t(t_{n-\frac12}) - \bar\partial_t u^n$, $R_2^n = u(t_{n-\frac12}) - u^{n-\frac12}$.

Taking $v_h = \eta^{n-\frac12}$ in (17), there holds

$$\begin{aligned}
&\big(\bar\partial_t \eta^n, \eta^{n-\frac12}\big) + \alpha\big(\nabla\bar\partial_t\eta^n, \nabla\eta^{n-\frac12}\big) + \beta\big(\nabla\eta^{n-\frac12}, \nabla\eta^{n-\frac12}\big) \\
&\quad= -\big(\bar\partial_t\xi^n, \eta^{n-\frac12}\big) - \alpha\big(\nabla\bar\partial_t\xi^n, \nabla\eta^{n-\frac12}\big) \\
&\qquad - \beta\big(\nabla\xi^{n-\frac12}, \nabla\eta^{n-\frac12}\big) + \big(\vec f\big(U_h^{n-\frac12}\big) - \vec f\big(u(t_{n-\frac12})\big), \nabla\eta^{n-\frac12}\big) \\
&\qquad - \big(R_1^n, \eta^{n-\frac12}\big) - \alpha\big(\nabla R_1^n, \nabla\eta^{n-\frac12}\big) - \beta\big(\nabla R_2^n, \nabla\eta^{n-\frac12}\big).
\end{aligned} \tag{18}$$

The left-hand side of (18) can be rewritten as

$$\frac{1}{2\Delta t}\big[\big(\|\eta^n\|_0^2 - \|\eta^{n-1}\|_0^2\big) + \alpha\big(\|\nabla\eta^n\|_0^2 - \|\nabla\eta^{n-1}\|_0^2\big)\big] + \beta\|\nabla\eta^{n-\frac12}\|_0^2. \tag{19}$$

Now we estimate the right-hand side: By virtue of Lemma 2.1, we arrive at

$$\begin{aligned}
&\big(\bar\partial_t\xi^n, \eta^{n-\frac12}\big) + \alpha\big(\nabla\bar\partial_t\xi^n, \nabla\eta^{n-\frac12}\big) + \beta\big(\nabla\xi^{n-\frac12}, \nabla\eta^{n-\frac12}\big) \\
&\quad\le Ch^4\left(\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\|u_t\|_3^2\,d\tau + \|u^{n-\frac12}\|_3^2\right) + \frac12\|\nabla\eta^{n-\frac12}\|_0^2.
\end{aligned} \tag{20}$$

Using the Taylor expansion, the truncation error can be estimated as

$$\begin{aligned}
&\big(R_1^n, \eta^{n-\frac12}\big) + \alpha\big(\nabla R_1^n, \nabla\eta^{n-\frac12}\big) + \beta\big(\nabla R_2^n, \nabla\eta^{n-\frac12}\big) \\
&\quad\le C\big(\|R_1^n\|_1^2 + \|R_2^n\|_1^2\big) + \frac12\|\nabla\eta^{n-\frac12}\|_0^2 \\
&\quad\le C(\Delta t)^3\int_{t_{n-1}}^{t_n}\big(\|u_{ttt}\|_1^2 + \|u_{tt}\|_1^2\big)\,d\tau + \frac12\|\nabla\eta^{n-\frac12}\|_0^2,
\end{aligned} \tag{21}$$

the nonlinear term can be written as

$$
\begin{aligned}
&\big|\big(\vec{f}(U_h^{n-\frac{1}{2}}) - \vec{f}\big(u(t_{n-\frac{1}{2}})\big), \nabla\eta^{n-\frac{1}{2}}\big)\big| \\
&= \left|\left(\frac{1}{2}\big(\big(u(t_{n-\frac{1}{2}})\big)^2 - \big(U_h^{n-\frac{1}{2}}\big)^2\big), \nabla\eta^{n-\frac{1}{2}}\right) + \big(u(t_{n-\frac{1}{2}}) - U_h^{n-\frac{1}{2}}, \nabla\eta^{n-\frac{1}{2}}\big)\right| \\
&= \left|\left(\frac{1}{2}\big(\big(u(t_{n-\frac{1}{2}})\big)^2 - \big(u^{n-\frac{1}{2}}\big)^2 + \big(u^{n-\frac{1}{2}}\big)^2 - \big(U_h^{n-\frac{1}{2}}\big)^2\big), \nabla\eta^{n-\frac{1}{2}}\right)\right. \\
&\quad \left. + \big(u(t_{n-\frac{1}{2}}) - u^{n-\frac{1}{2}} + u^{n-\frac{1}{2}} - U_h^{n-\frac{1}{2}}, \nabla\eta^{n-\frac{1}{2}}\big)\right| \\
&= \left|\frac{1}{2}\big(u(t_{n-\frac{1}{2}})^2 - \big(u^{n-\frac{1}{2}}\big)^2, \nabla\eta^{n-\frac{1}{2}}\big) + \frac{1}{2}\big(\big(u^{n-\frac{1}{2}}\big)^2 - \big(U_h^{n-\frac{1}{2}}\big)^2, \nabla\eta^{n-\frac{1}{2}}\big)\right. \\
&\quad \left. + \big(u(t_{n-\frac{1}{2}}) - u^{n-\frac{1}{2}} + u^{n-\frac{1}{2}} - U_h^{n-\frac{1}{2}}, \nabla\eta^{n-\frac{1}{2}}\big)\right| \\
&= \left|\frac{1}{2}\big(R_2^n\big(u(t_{n-\frac{1}{2}}) + u^{n-\frac{1}{2}}\big), \nabla\eta^{n-\frac{1}{2}}\big) + \frac{1}{2}\big(\big(\xi^{n-\frac{1}{2}} + \eta^{n-\frac{1}{2}}\big)\big(u^{n-\frac{1}{2}} + U_h^{n-\frac{1}{2}}\big), \nabla\eta^{n-\frac{1}{2}}\big)\right. \\
&\quad \left. + \big(R_2^n, \nabla\eta^{n-\frac{1}{2}}\big) + \big(\xi^{n-\frac{1}{2}} + \eta^{n-\frac{1}{2}}, \nabla\eta^{n-\frac{1}{2}}\big)\right| \\
&= \left|\sum_{i=1}^{4} A_i\right|.
\end{aligned}
\tag{22}
$$

By the estimation of truncation error, there holds

$$
\begin{aligned}
A_1 + A_3 &\le C\|R_2^n\|_0 \|\eta^{n-\frac{1}{2}}\|_1 \big(\|u(t_{n-\frac{1}{2}}) + u^{n-\frac{1}{2}}\|_{0,\infty} + 1\big) \\
&\le C\|R_2^n\|_0 \|\eta^{n-\frac{1}{2}}\|_1 \\
&\le C(\Delta t)^3 \int_{t_{n-1}}^{t_n} \|u_{tt}\|_0^2 \, d\tau + C\|\eta^{n-\frac{1}{2}}\|_1^2.
\end{aligned}
\tag{23}
$$

From the Sobolev imbedding theorem, (4) and (7), we have

$$
\begin{aligned}
A_2 &\le C\|\xi^{n-\frac{1}{2}} + \eta^{n-\frac{1}{2}}\|_{0,4} \|u^{n-\frac{1}{2}} + U_h^{n-\frac{1}{2}}\|_{0,4} \|\eta^{n-\frac{1}{2}}\|_1 \\
&\le C\|\xi^{n-\frac{1}{2}} + \eta^{n-\frac{1}{2}}\|_{0,4} \big(\|u^{n-\frac{1}{2}}\|_{0,4} + \|U_h^{n-\frac{1}{2}}\|_{0,4}\big) \|\eta^{n-\frac{1}{2}}\|_1 \\
&\le C\|\xi^{n-\frac{1}{2}} + \eta^{n-\frac{1}{2}}\|_{0,4} \big(\|u^{n-\frac{1}{2}}\|_{0,\infty} + \|U_h^{n-\frac{1}{2}}\|_1\big) \|\eta^{n-\frac{1}{2}}\|_1 \\
&\le C\|\xi^{n-\frac{1}{2}} + \eta^{n-\frac{1}{2}}\|_{0,4} \|\eta^{n-\frac{1}{2}}\|_1 \\
&\le Ch^4\|u^{n-\frac{1}{2}}\|_{2,4}^2 + C\|\eta^{n-\frac{1}{2}}\|_1^2 \\
&\le Ch^4\|u^{n-\frac{1}{2}}\|_3^2 + C\|\eta^{n-\frac{1}{2}}\|_1^2
\end{aligned}
\tag{24}
$$

and

$$
A_4 \le C\big(\|\xi^{n-\frac{1}{2}}\|_0 + \|\eta^{n-\frac{1}{2}}\|_0\big)\|\eta^{n-\frac{1}{2}}\|_1 \le Ch^4\|u^{n-\frac{1}{2}}\|_2^2 + C\|\eta^{n-\frac{1}{2}}\|_1^2. \tag{25}
$$

Substituting (23)–(25) into (22), we get

$$\begin{aligned}&\left|\left(\vec{f}\left(U_h^{n-\frac{1}{2}}\right)-\vec{f}\left(u(t_{n-\frac{1}{2}})\right),\nabla\eta^{n-\frac{1}{2}}\right)\right|\\&\quad\le C(\Delta t)^3\int_{t_{n-1}}^{t_n}\|u_{tt}\|_0^2\,d\tau+Ch^4\left\|u^{n-\frac{1}{2}}\right\|_3^2+C\left\|\eta^{n-\frac{1}{2}}\right\|_1^2.\end{aligned}\tag{26}$$

Hence, from (18)–(21) and (26), we have

$$\begin{aligned}&\frac{1}{2\Delta t}\left[\left(\left\|\eta^n\right\|_0^2-\left\|\eta^{n-1}\right\|_0^2\right)+\alpha\left(\left|\eta^n\right|_1^2-\left|\eta^{n-1}\right|_1^2\right)\right]\\&\quad\le Ch^4+C(\Delta t)^3\int_{t_{n-1}}^{t_n}\left(\|u_{tt}\|_1^2+\|u_{ttt}\|_1^2\right)d\tau+C\left\|\eta^{n-\frac{1}{2}}\right\|_1^2,\end{aligned}$$

multiplying by $2\Delta t$, then summing up the above inequality and noting that $\eta^0=0$, we can obtain

$$\left\|\eta^n\right\|_1^2\le Ch^4+C(\Delta t)^4+C\Delta t\sum_{i=1}^{n}\left\|\eta^i\right\|_1^2.\tag{27}$$

Choosing Δt small enough so that $1-C\Delta t>0$ and applying discrete Gronwall's lemma, there holds

$$\left\|\eta^n\right\|_1^2\le C\left(h^4+(\Delta t)^4\right),$$

the proof is completed. □

To obtain the global superconvergence estimate, we combine the adjacent four small elements K_1,K_2,K_3,K_4 into a big element $\tilde{K}$, i.e., $\tilde{K}=\bigcup_{i=1}^4 K_i$ (see Fig. 1), the corresponding subdivision is defined by T_{2h}.

As in [19], we define the following interpolation postprocessing operator I_{2h}:

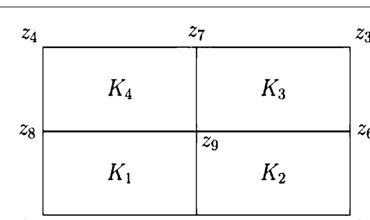
$$I_{2h}u|_{\tilde{K}}\in Q_2(\tilde{K}),\quad\forall u\in C(\tilde{K}),\qquad I_{2h}u(Z_i)=u(Z_i),\quad i=1,2,\ldots,9,\tag{28}$$

where $Q_2(\tilde{K})$ and $C(\tilde{K})$ denote the spaces of biquadratic piecewise polynomial and continuous function on $\tilde{K}$, respectively, Z_i are all vertices of four small elements (see Fig. 1).

Meanwhile, I_{2h} has the following properties (see [19]):

$$I_{2h}I_hu=I_{2h}u,\qquad\|u-I_{2h}u\|_1\le Ch^2\|u\|_3,\qquad\|I_{2h}v_h\|_1\le C\|v_h\|_1,\quad\forall v_h\in V_h.\tag{29}$$

Figure 1 The big element $\tilde{K}$



Theorem 3.2 *Under the assumption of Theorem 3.1, there holds the global superconvergence result as follows*:

$$\|u^n - I_{2h}U_h^n\|_1 \le C(h^2 + (\Delta t)^2).$$

Proof Let $u^n - I_{2h}U_h^n = u^n - I_{2h}I_h u^n + I_{2h}I_h u^n - I_{2h}U_h^n$. By (16), (29), and the triangular inequality, we can get

$$\begin{aligned} \|u^n - I_{2h}U_h^n\|_1 &\le \|u^n - I_{2h}I_h u^n\|_1 + \|I_{2h}I_h u^n - I_{2h}U_h^n\|_1 \\ &\le \|u^n - I_{2h}u^n\|_1 + \|I_{2h}(I_h u^n - U_h^n)\|_1 \\ &\le Ch^2\|u^n\|_3 + C\|I_h u^n - U_h^n\|_1 \\ &\le C(h^2 + (\Delta t)^2), \end{aligned} \tag{30}$$

the proof is completed. □

Remark 3.1 In this paper, the boundedness of $\|U_h^n\|_1$ is crucial to the unconditional superclose and superconvergence results, the technique we used is more simple and direct than those in [22] and [23] ([22] applied an error splitting technique and [23] employed a complicated mathematical inductive hypothesis method to derive the boundedness of numerical solution).

4 Numerical examples

In this section, we give three examples to verify the validity of theoretical analysis, here we divide the domain Ω into $m \times n$ rectangular meshes.

Example 1 We consider the following homogeneous BBMB equation:

$$u_t - \Delta u_t - \Delta u - \nabla \cdot \vec{f}(u) = 0, \quad (x,y,t) \in [0,1] \times [0,1] \times (0,T].$$

Initially, the energy $E^n = \|U_h^n\|_1^2$ under the initial condition $u(x,y,0) = \sin\pi x \sin\pi y$ is plotted in Fig. 2, here $t \in [0,0.1]$, $h = \frac{1}{20}$, $\Delta t = 1.0\text{e}{-05}$.

Then a larger initial condition ($u(x,y,0) = 100\sin\pi x\sin\pi y$) is considered, and E^n is displayed in Fig. 3, here t, h, and Δt are the same as above.

From Figs. 2–3 we can see that the energy is stable, which is consistent with the conclusion of Theorem 2.1.

Example 2 We consider the following inhomogeneous BBMB equation:

$$\begin{cases} u_t - \Delta u_t - \Delta u - \nabla \cdot \vec{f}(u) = g(x,y,t), & (x,y,t) \in [0,1] \times [0,1] \times (0,T], \\ u(x,y,0) = 2xy(x-1)(y-1), & (x,y) \in [0,1] \times [0,1], \end{cases}$$

here $g(x,y,t)$ could be computed by the exact solution $u(x,y,t) = (1+e^{-t})xy(x-1)(y-1)$.

First of all, we take $\|I_h u^n - U_h^n\|_1$ as an example to validate the unconditional stability. Here fix $h = \frac{1}{100}$ and choose $\Delta t = \frac{h}{10}, \frac{h}{20}, \frac{h}{40}, \frac{h}{80}$, respectively, we provide the results of $\|I_h u^n - U_h^n\|_1$ at $t = 0.1$, 0.5, and 1.0 in Table 1.

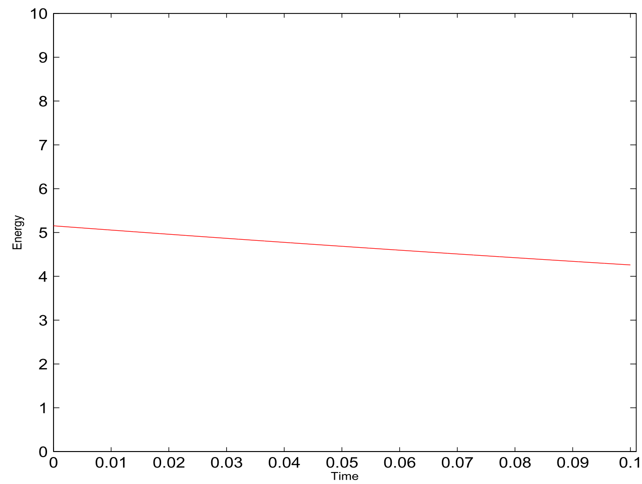


Figure 2 E^n under the initial condition $u(x,y,0) = \sin \pi x \sin \pi y$

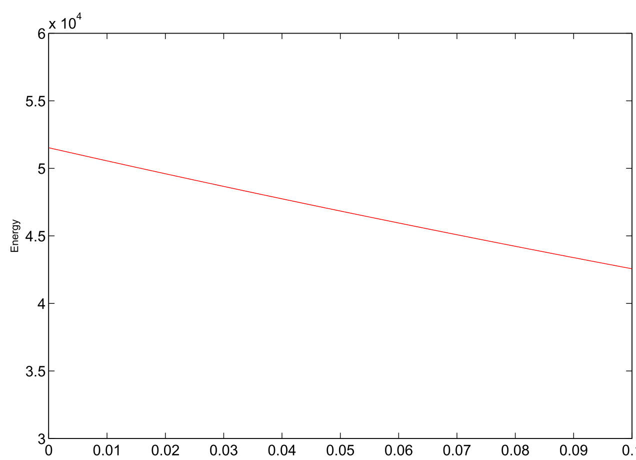


Figure 3 E^n under the initial condition $u(x,y,0) = 100 \sin \pi x \sin \pi y$

Table 1 Error results of $\|I_h u^n - U_h^n\|_1$ with $h = \frac{1}{100}$

t	$\Delta t = \frac{h}{10}$	$\Delta t = \frac{h}{20}$	$\Delta t = \frac{h}{40}$	$\Delta t = \frac{h}{80}$
0.1	1.0893e–06	1.0958e–06	1.0991e–06	1.1007e–06
0.5	4.6014e–06	4.6203e–06	4.6295e–06	4.6340e–06
1.0	7.5644e–06	7.5844e–06	7.5939e–06	7.5986e–06

From Table 1, we can observe that $\|I_h u^n - U_h^n\|_1$ is stable at a certain time and free from the ratio between Δt and h.

Moreover, the convergence, superclose, and superconvergence results at $t = 0.1, 0.5$, and 1.0 are listed in Tables 2–4, respectively. At the same time, we describe the error reduction results in Figs. 4–6, respectively. To confirm the convergence order, we choose $\Delta t = h$.

From Tables 2–4 and Figs. 4–6, we can see that $\|u^n - U_h^n\|_1$ is convergent at rate of $O(h)$, $\|I_h u^n - U_h^n\|_1$ and $\|u^n - I_{2h} U_h^n\|_1$ are convergent at rate of $O(h^2)$, which coincides with the conclusions of Theorems 3.1–3.2.

Table 2 Numerical results of u at $t = 0.1$

$m \times n$	$\|u^n - U_h^n\|_1$	order	$\|I_h u^n - U_h^n\|_1$	order	$\|u^n - I_{2h} U_h^n\|_1$	order
4×4	7.2863e–02	/	6.4179e–04	/	7.1337e–04	/
8×8	3.5745e–02	1.0274	1.5938e–04	2.0095	1.6411e–04	2.1199
16×16	1.7779e–02	1.0075	4.0251e–05	1.9854	4.0554e–05	2.0168
32×32	8.8776e–03	1.0019	1.0329e–05	1.9622	1.0349e–05	1.9703
64×64	4.4373e–03	1.0005	2.4987e–06	2.0468	2.4999e–06	2.0489

Table 3 Numerical results of u at $t = 0.5$

$m \times n$	$\|u^n - U_h^n\|_1$	order	$\|I_h u^n - U_h^n\|_1$	order	$\|u^n - I_{2h} U_h^n\|_1$	order
4×4	6.1063e–02	/	2.7140e–03	/	3.0059e–03	/
8×8	3.0096e–02	1.0207	6.7986e–04	1.9971	6.9905e–04	2.1043
16×16	1.4988e–02	1.0057	1.7159e–04	1.9862	1.7281e–04	2.0161
32×32	7.4865e–03	1.0014	4.3815e–05	1.9694	4.3893e–05	1.9771
64×64	3.7423e–03	1.0004	1.0696e–05	2.0338	1.0701e–05	2.0357

Table 4 Numerical results of u at $t = 1.0$

$m \times n$	$\|u^n - U_h^n\|_1$	order	$\|I_h u^n - U_h^n\|_1$	order	$\|u^n - I_{2h} U_h^n\|_1$	order
4×4	5.1679e–02	/	4.4691e–03	/	4.9353e–03	/
8×8	2.5582e–02	1.0144	1.1293e–03	1.9845	1.1598e–03	2.0891
16×16	1.2756e–02	1.0039	2.8497e–04	1.9865	2.8691e–04	2.0152
32×32	6.3737e–03	1.0010	7.2411e–05	1.9765	7.2534e–05	1.9838
64×64	3.1863e–03	1.0003	1.7824e–05	2.0201	1.7832e–05	2.0219

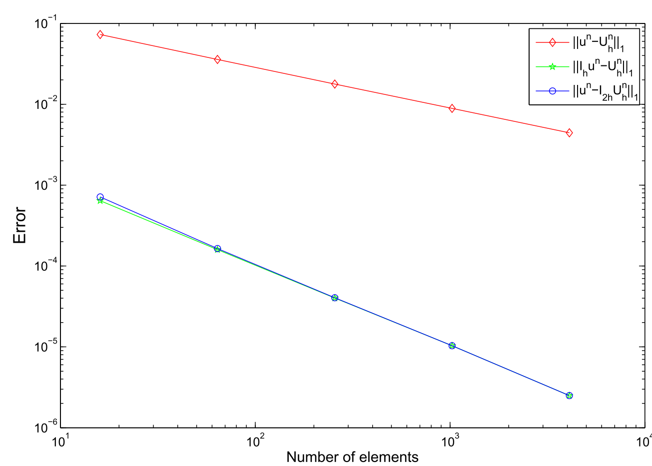


Figure 4 Numerical results for u at $t = 0.1$

Example 3 We consider the inhomogeneous BBMB equation in larger domains.

Firstly, we introduce the following equation:

$$\begin{cases} u_t - \Delta u_t - \Delta u - \nabla \cdot \vec{f}(u) = g(x,y,t), & (x,y,t) \in [0,5] \times [0,5] \times (0,T], \\ u(x,y,0) = 2xy(x-5)(y-5), & (x,y) \in [0,5] \times [0,5], \end{cases}$$

where the exact solution $u(x,y,t) = (1 + e^{-t})xy(x-5)(y-5)$.

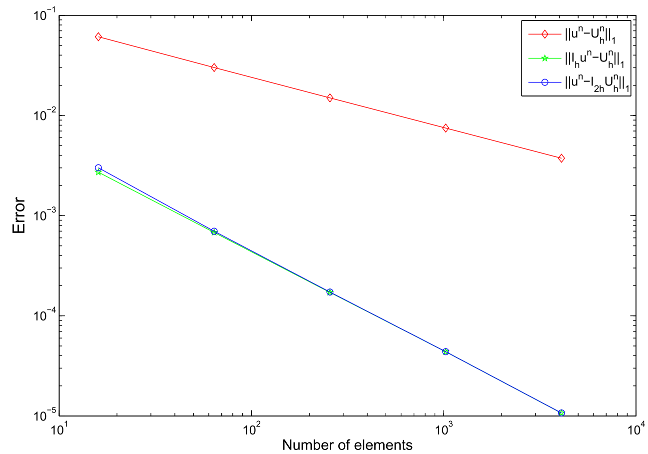


Figure 5 Numerical results for u at $t = 0.5$

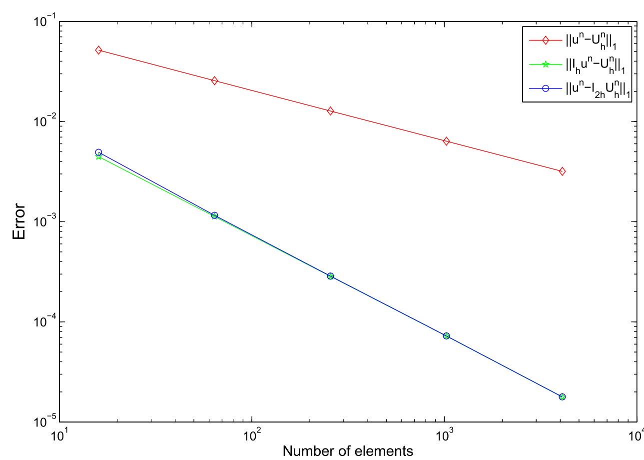


Figure 6 Numerical results for u at $t = 1.0$

Table 5 Error results of $\|I_hu^n - U_h^n\|_1$ with $h = \frac{1}{20}$

t	$\Delta t = \frac{h}{10}$	$\Delta t = \frac{h}{20}$	$\Delta t = \frac{h}{40}$	$\Delta t = \frac{h}{80}$
1	6.7407e–02	6.7791e–02	6.7913e–02	6.7714e–02
2	3.7426e–02	3.7565e–02	3.7626e–02	3.7493e–02
3	2.7182e–02	2.7260e–02	2.7196e–02	2.7198e–02

Table 6 Numerical results of u at $t = 1$

$m \times n$	$\|u^n - U_h^n\|_1$	order	$\|I_hu^n - U_h^n\|_1$	order	$\|u^n - I_{2h}U_h^n\|_1$	order
16 × 16	8.0941	/	1.5482	/	1.6887	/
32 × 32	4.0095	1.0134	4.9625e–01	1.6414	5.2125e–01	1.6959
64 × 64	1.9943	1.0075	1.1871e–01	2.0636	1.2037e–01	2.1144
128 × 128	9.9582e–01	1.0019	1.6704e–02	2.8291	1.6761e–02	2.8442

Here we take $t = 1, 2$, and 3, respectively, the unconditional stability is validated in Table 5, the convergence, superclose, and superconvergence results are listed in Tables 6–8, respectively.

Table 7 Numerical results of u at $t = 2$

$m \times n$	$\|u^n - U_h^n\|_1$	order	$\|I_h u^n - U_h^n\|_1$	order	$\|u^n - I_{2h} U_h^n\|_1$	order
16 × 16	6.6821	/	1.1840	/	1.2678	/
32 × 32	3.3145	1.0114	3.2218e–01	1.8778	3.3841e–01	1.9055
64 × 64	1.6538	1.0029	7.4794e–01	2.1068	7.5961e–02	2.1554
128 × 128	8.2649e–01	1.0007	1.5762e–02	2.2464	1.5813e–02	2.2641

Table 8 Numerical results of u at $t = 3$

$m \times n$	$\|u^n - U_h^n\|_1$	order	$\|I_h u^n - U_h^n\|_1$	order	$\|u^n - I_{2h} U_h^n\|_1$	order
16 × 16	6.1503	/	9.9968e–01	/	1.0411	/
32 × 32	3.0608	1.0067	2.4141e–01	2.0499	2.4919e–01	2.0627
64 × 64	1.5287	1.0015	6.2188e–02	1.9567	6.2752e–02	1.9895
128 × 128	7.6418e–01	1.0003	1.4664e–02	2.0843	1.4697e–02	2.0941

Table 9 Numerical results of u at $t = 1$

$m \times n$	$\|u^n - U_h^n\|_1$	order	$\|I_h u^n - U_h^n\|_1$	order	$\|u^n - I_{2h} U_h^n\|_1$	order
32 × 32	29.0467	/	12.6079	/	13.3593	/
64 × 64	13.7423	1.0797	4.1904	1.5891	4.3447	1.6205
128 × 128	6.6382	1.0497	1.2041	1.7990	1.2251	1.8262
256 × 256	3.2728	1.0202	2.5485e–01	2.2402	2.5658e–01	2.2555

Table 10 Numerical results of u at $t = 2$

$m \times n$	$\|u^n - U_h^n\|_1$	order	$\|I_h u^n - U_h^n\|_1$	order	$\|u^n - I_{2h} U_h^n\|_1$	order
32 × 32	25.9705	/	14.2278	/	15.2578	/
64 × 64	11.6481	1.1567	4.2769	1.7340	4.4823	1.7672
128 × 128	5.5048	1.0813	9.8468e–01	2.1188	1.0043	2.1579
256 × 256	2.7120	1.0213	1.4919e–01	2.7224	1.5014e–01	2.7419

Table 11 Numerical results of u at $t = 3$

$m \times n$	$\|u^n - U_h^n\|_1$	order	$\|I_h u^n - U_h^n\|_1$	order	$\|u^n - I_{2h} U_h^n\|_1$	order
32 × 32	24.9412	/	14.8079	/	15.8557	/
64 × 64	10.8178	1.2051	4.1108	1.8488	4.3097	1.8793
128 × 128	5.0430	1.1010	5.9951e–01	2.7775	6.0899e–01	2.8230
256 × 256	2.5055	1.0091	9.4431e–02	2.6664	9.4848e–02	2.6827

Secondly, we consider the following equation:

$$\begin{cases} u_t - \Delta u_t - \Delta u - \nabla \cdot \vec{f}(u) = g(x,y,t), & (x,y,t) \in [0,8] \times [0,8] \times (0,T], \\ u(x,y,0) = 2xy(x-8)(y-8), & (x,y) \in [0,8] \times [0,8], \end{cases}$$

here $u(x,y,t) = (1 + e^{-t})xy(x-8)(y-8)$.

The convergence, superclose, and superconvergence results at $t = 1, 2$, and 3 are provided in Tables 9–11, respectively.

From Tables 5–11, we can see that under the large initial condition, the numerical results are also in good agreement with our theoretical analysis.

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Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contribution
LW carried out theoretical calculation, participated in the design of the study, and drafted the manuscript. XL and HY participated in its design and helped to draft the manuscript. Both authors read and approved the final manuscript.

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