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Boundary value problems for a second-order difference equation involving the mean curvature operator

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Abstract

In this paper, we consider the existence of multiple solutions for discrete boundary value problems involving the mean curvature operator by means of Clark's Theorem, where the nonlinear terms do not need any asymptotic and superlinear conditions at 0 or at infinity. Further, the existence of a positive solution has been considered by the strong comparison principle. As an application, some examples are given to illustrate the obtained results.

Keywords: Discrete boundary value problems; Mean curvature operator; Palais–Smale condition; Critical-point theory

1 Introduction

Denote the sets of integers and real numbers by $\mathbb{Z}$, $\mathbb{R}$, respectively. For $a, b \in \mathbb{Z}$, $\mathbb{Z}(a, b)$ denotes the discrete interval $\{a, a+1 \dots, b\}$ if $a \leq b$. Due to geometric and physical motivations, many authors [1–3] have studied the existence results for the prescribed mean curvature equations with Dirichlet boundary conditions

$$\begin{cases} -(\phi_c(u'))' = f(u), & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $f : [0,\infty) \to [0,\infty)$ is a continuous function, ϕ_c is the mean curvature operator defined by $\phi_c(\xi) = \frac{\xi}{\sqrt{1+\kappa\xi^2}}$ with $\kappa > 0$. For general background on the mean curvature operator, we refer to [4, 5].

Because of the wide applications of difference equations in various research fields such as computer science, economics, biology, and other fields [6–11], many authors have obtained excellent results for difference equations, for example, positive solutions [12–15], homoclinic solutions [16–21], and ground-state solutions [22, 23]. In particular, Guo and Yu [24] first used the critical-point theory to study the existence of a periodic solution for the following discrete problem

$$-\Delta^2 u(t-1) = f\big(t, u(t)\big), \quad t \in \mathbb{Z}, \tag{2}$$

where Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, $\Delta^2 u(t) = \Delta(\Delta u(t))$, $f(t,\cdot) \in C(R,R)$ for each $t \in \mathbb{Z}$. Also, the critical-point theory is an important tool to deal with the existence of solutions for the discrete boundary value problems [25–27]. However, few works have been done concerning the discrete problems (1). In [15], Zhou and Ling proved the existence results for the boundary value problem

$$\begin{cases} -\Delta(\phi_c(\Delta u(t-1))) = f(t,u(t)), & t \in \mathbb{Z}(1,T), \\ u(0) = u(T+1) = 0, \end{cases} \tag{3}$$

where T is a given positive integer, $f(t,\cdot) \in C(R,R)$ for each $t \in \mathbb{Z}(1,T)$. Under some suitable oscillating assumption on the nonlinearity f at infinity, they investigated the existence of infinitely many positive solutions.

The aim of this paper is to study the existence of multiple solutions for the following nonlinear difference equations with mean curvature operator

$$\begin{cases} -\Delta(\phi_c(\Delta u(t-1))) + q(t)u(t) = \lambda f(t,u(t)), & t \in \mathbb{Z}(1,T), \\ u(0) = u(T+1) = 0, \end{cases} \tag{4}$$

where $q(t) \in \mathbb{R}^+$ for each $t \in \mathbb{Z}(1,T)$ and $\lambda > 0$ is a positive parameter. Based on a version of Clark's Theorem [28, 29], we investigate the existence of multiple solutions of (4).

Let f satisfy the following hypotheses:

(a_1) $f(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(t,0) = 0$ for each $t \in \mathbb{Z}(1,T)$;

(a_2) $\liminf_{\xi \to +\infty} f(t,\xi) < 0$ and there exists a positive constant α such that

$$0 < f(t,\xi) \quad \text{for all } (t,\xi) \in \mathbb{Z}(1,T) \times (0,\alpha);$$

(a_3) $f(t,\xi)$ is odd in ξ for any $t \in \mathbb{Z}(1,T)$.

Example 1.1 It is easy to find some suitable functions satisfying assumptions (a_1)–(a_3). Let

$$f(t,\xi) = \sin\xi \quad \text{for all } (t,\xi) \in \mathbb{Z}(1,T) \times \mathbb{R}.$$

We note that $\sin\xi$ is a continuous odd function on $\mathbb{R}$. Thus, (a_1) and (a_3) hold. Lastly, if we set $\alpha = \frac{\pi}{4}$, then

$$\liminf_{\xi \to +\infty} \sin\xi = -1 < 0 \quad \text{and} \quad 0 < \sin\xi \quad \text{for all } \xi \in \left(0, \frac{\pi}{4}\right).$$

Therefore, (a_2) holds.

Obviously, if u is a solution of (4), then $-u$ is also a solution of (4) by (a_3). We say that $\pm u$ is a pair of solutions.

2 Preliminaries

Consider the T-dimensional real space

$$E = \left\{u : [0, T+1] \to \mathbb{R} \text{ such that } u(0) = u(T+1) = 0\right\},$$

endowed with the norm

$$\|u\| = \left(\sum_{t=0}^{T} \left| \Delta u(t) \right|^2 \right)^{\frac{1}{2}}.$$

From functional analysis theory, we know that E is a real Banach space. Moreover, we define the following two equivalent norms on E,

$$\|u\|_2 = \left(\sum_{t=1}^{T} \left| u(t) \right|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \|u\|_\infty = \max_{t \in \mathbb{Z}(1,T)} \left\{ \left| u(t) \right| \right\}.$$

Let J be a C^1 functional on E. A sequence $\{u_n\} \subset E$ is called a Palais–Smale sequence (P.S. sequence for short) for J if $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$. We say J satisfies the Palais–Smale condition (P.S. condition for short) if any P.S. sequence for J possesses a convergent subsequence in E.

Let θ be the zero element of Banach space E. Let Σ denote the family of sets $A \subset E \backslash \{\theta\}$ such that A is closed in E and symmetric with respect to θ, i.e., $u \in A$ implies $-u \in A$.

The following Clark's Theorem will be used to prove our main result.

Lemma 2.1 ([28, 29]) *Let E be a real Banach space, $J \in C^1(E, \mathbb{R})$ with J even, bounded from below, and satisfying the P.S. condition. Suppose $J(\theta) = 0$, there is a set $K \subset \Sigma$ such that K is homeomorphic to S^{j-1} by an odd map, and $\sup_K J < 0$. Then, J possesses at least j distinct pairs of critical points.*

First, the following comparison principle is necessary for the positive solutions.

Lemma 2.2 *Let $u, v \in E$. If*

$$-\Delta\left(\phi_c\left(\Delta u(t-1)\right)\right) + q(t)u(t) \geq -\Delta\left(\phi_c\left(\Delta v(t-1)\right)\right) + q(t)v(t) \quad \textit{for all } t \in \mathbb{Z}(1,T), \tag{5}$$

then $u \geq v$ in $\mathbb{Z}(1, T)$.

Proof Arguing indirectly, if not, there exist some $j_0 \in \mathbb{Z}(1, T)$ such that $u(j_0) < v(j_0)$. Set that

$$j := \max\left\{ j_0 | j_0 \in \mathbb{Z}(1,T) \text{ such that } u(j_0) < v(j_0) \right\}.$$

If $u(j-1) > v(j-1)$, by $\phi_c(s)$ being increasing in s, we have

$$-\Delta\left(\phi_c\left(\Delta u(j-1)\right)\right) + q(j)u(j) < -\Delta\left(\phi_c\left(\Delta v(j-1)\right)\right) + q(j)v(j), \tag{6}$$

which contradicts (5).

If $u(j-1) \leq v(j-1)$, we first consider the case for $u(j) - u(j-1) < v(j) - v(j-1)$, which implies (6). This contradicts (5), since it remains to consider that $u(j-1) \leq v(j-1)$ and $u(j) - u(j-1) > v(j) - v(j-1)$. First, we assume that $u(j-2) > v(j-2)$ holds, then

$$-\Delta\left(\phi_c\left(\Delta u(j-2)\right)\right) + q(j-1)u(j-1) < -\Delta\left(\phi_c\left(\Delta v(j-2)\right)\right) + q(j-1)v(j-1). \tag{7}$$

If $u(j-2) \le v(j-2)$ and $u(j-1)-u(j-2) < v(j-1)-v(j-2)$, we obtain (7), again contradicting (5). Then, $u(j) < v(j)$ can happen only if $u(j-2) \le v(j-2)$ and $u(j-1) - u(j-2) > v(j-1) - v(j-2)$.

By repeating the above process, $u(j) < v(j)$ can happen only if $u(2) \le v(2)$ and $u(3)-u(2) > v(3) - v(2)$. In this case, if $u(1) > v(1)$, or $u(1) \le v(1)$ and $u(2) - u(1) < v(2) - v(1)$, we have

$$-\Delta\big(\phi_c\big(\Delta u(1)\big)\big) + q(2)u(2) < -\Delta\big(\phi_c\big(\Delta v(1)\big)\big) + q(2)v(2), \tag{8}$$

which contradicts (5). If $u(1) \le v(1)$ and $u(2) - u(1) > v(2) - v(1)$, we have

$$-\Delta\big(\phi_c\big(\Delta u(0)\big)\big) + q(1)u(1) < -\Delta\big(\phi_c\big(\Delta v(0)\big)\big) + q(1)v(1), \tag{9}$$

which contradicts (5). Hence, $u(j) \ge v(j)$ for all $j \in \mathbb{Z}(1, T)$. The proof is completed. □

Lemma 2.3 *Let $u \in E$, if*

$$-\Delta\big(\phi_c\big(\Delta u(t-1)\big)\big) + q(t)u(t) \ge 0,$$
$$u(t) = 0, \qquad u(t \pm 1) \ge 0,$$

then $u(t \pm 1) = 0$.

Proof By the above assumptions, we have

$$0 \le \phi_c\big(\Delta u(t)\big) \le \phi_c\big(\Delta u(t-1)\big) \le 0,$$
$$\Delta u(t) \ge 0, \qquad \Delta u(t-1) \le 0.$$

Combining with the monotonicity of ϕ_c, we have $u(t \pm 1) = 0$. The proof is completed. □

In particular, let $v = 0$ in Lemma 2.2, the strong comparison principle is given by the two lemmas above.

Lemma 2.4 *Let $u \in E$, if $u \ne \theta$ and*

$$-\Delta\big(\phi_c\big(\Delta u(t-1)\big)\big) + q(t)u(t) \ge 0 \quad \textit{for all } t \in \mathbb{Z}(1, T),$$

then $u > 0$ in $\mathbb{Z}(1, T)$.

Lemma 2.5 *Fix $u \in E$, if $u(j) \le 0$ for some $j \in \mathbb{Z}(1, T)$ and*

$$-\Delta\big(\phi_c\big(\Delta u(j-1)\big)\big) + q(j)u(j) \ge 0, \tag{10}$$

then $u > 0$ in $\mathbb{Z}(1, T)$ or $u = 0$.

Proof Let u be a nontrivial function satisfying (10). We assume first that $j = 1$ and $u(1) \le 0$, by (10), then we obtain

$$0 \ge \frac{u(1)}{\sqrt{1 + \kappa u(1)^2}} + q(1)u(1) \ge \frac{u(2) - u(1)}{\sqrt{1 + \kappa(\Delta u(1))^2}}.$$

Hence, $u(2) \le u(1) \le 0$. Again, we apply (10) with $j = 2$ to conclude that $u(3) \le u(2) \le 0$. Hence, we have $u(T) \le u(T-1) \le \cdots \le u(3) \le u(2) \le u(1) \le 0$. If $u(T) = 0$, then u is a trivial function, contradicting the definition of u. If $u(T) < 0$, by (10) with $j = T$, we obtain

$$0 \ge \frac{u(T) - u(T-1)}{\sqrt{1 + \kappa(\Delta u(T-1))^2}} + q(T)u(T) \ge \frac{-u(T)}{\sqrt{1 + \kappa u(T)^2}} > 0. \tag{11}$$

This is absurd, hence $u(1) > 0$. By a similar argument, if $u(2) < 0$, then $u(T) < 0$, but we obtain $u(T) = 0$ from (11). Thus, we have $u(2) \ge 0$. Repeating the above computation, we have $u(3) \ge 0, u(4) \ge 0, \dots, u(T-1) \ge 0$. Now, we show $u(T) \ge 0$. If $u(T) < 0$, since $u(T) < 0 \le u(T-1)$, again we obtain (11) from (10). This implies $u(T) = 0$, showing $u(T) \ge 0$. Hence, $u(1) > 0$ and $u(j) \ge 0$ for all $j \in \mathbb{Z}(2, T)$. If $u(2) = 0$, we have

$$0 > \frac{-u(1)}{\sqrt{1 + \kappa u(1)^2}} \ge \frac{u(3)}{\sqrt{1 + \kappa u(3)^2}} \ge 0,$$

contradicting $u(1) > 0$. By a similar argument, we obtain $u(3) > 0, u(4) > 0, \dots, u(T-1) > 0$. If $u(T) = 0$, we have $0 > \frac{-u(T-1)}{\sqrt{1+\kappa u(T-1)^2}} \ge 0$. Then, $u(T-1) = 0$, which is absurd, hence $u > 0$. □

3 Main results

Now, we state our main results.

Theorem 3.1 *Assume that* (a_1)–(a_3) *hold, then there exists a positive constant* $\bar{\lambda}$, *when* $\lambda > \bar{\lambda}$, *and problem* (4) *admits at least* T *distinct pairs of nontrivial solutions. Furthermore, there exists a positive constant* M *such that each solution* u *satisfies* $\|u\|_\infty \le M$.

Proof Using the condition (a_2), there exists a positive real sequence $\{d_n\}$ with $\lim_{n\to\infty} d_n = +\infty$ such that

$$\lim_{n\to+\infty} f(t, d_n) < 0 \quad \text{for all } t \in \mathbb{Z}(1, T).$$

We can find a positive integer n_0 such that $M = d_{n_0} > \alpha$ and $f(t, M) < 0$, where α comes from (a_2). First, we consider the following boundary value problem

$$\begin{cases} -\Delta(\phi_c(\Delta u(t-1))) + q(t)u(t) = \lambda \hat{f}(t, u(t)), & t \in \mathbb{Z}(1, T), \\ u(0) = u(T+1) = 0, \end{cases} \tag{12}$$

where $\hat{f}(t, \xi)$ is a truncation function defined by

$$\hat{f}(t, \xi) = \begin{cases} f(t, M) & \text{if } \xi > M, \\ f(t, \xi) & \text{if } |\xi| \le M, \\ f(t, -M) & \text{if } \xi < -M. \end{cases}$$

We show that if u satisfies problem (12), then $\|u\|_\infty \le M$ and u is a solution of problem (4). Arguing indirectly, there exists a $k_0 \in \mathbb{Z}(1, T)$ such that $|u(k_0)| > M$ and $|u(t)| \le M$ for $t \in \mathbb{Z}(1, k_0 - 1)$. If $u(k_0) > M$, then $\hat{f}(t, u(k_0)) = f(t, M) < 0$. We have

$$-\Delta\bigl(\phi_c\bigl(\Delta u(k_0 - 1)\bigr)\bigr) + q(k_0)u(k_0) < 0,$$

or

$$\frac{u(k_0)-u(k_0+1)}{\sqrt{1+\kappa(\Delta u(k_0))^2}} < -\frac{u(k_0)-u(k_0-1)}{\sqrt{1+\kappa(\Delta u(k_0-1))^2}} - q(k_0)u(k_0) < 0,$$

which implies that $u(k_0+1) > u(k_0) > M$. By repeating the above process, we obtain

$$u(t) > u(t-1) > M \quad \text{for all } t \in \mathbb{Z}(k_0+1, T).$$

Further,

$$0 = u(T+1) > u(T) > M,$$

which is a contradiction. If $u(k_0) < -M$, we can similarly obtain a contradiction. Thus, $\|u\|_\infty \leq M$ holds.

Define the functional $\hat{J}$ on E as follows:

$$\hat{J}(u) = \sum_{t=0}^{T}\left(\left(\frac{\sqrt{1+\kappa(\Delta u(t))^2}-1}{\kappa}\right) + \frac{q(t)u^2(t)}{2}\right) - \lambda\sum_{t=1}^{T}\hat{F}\big(t,u(t)\big), \tag{13}$$

where $\hat{F}(t,\xi) = \int_0^\xi \hat{f}(t,s)\,ds$, $(t,\xi) \in \mathbb{Z}(1,T)\times\mathbb{R}$. It is easy to verify that $\hat{J} \in C^1(E,\mathbb{R})$ and is even. By using $u(0) = u(T+1) = 0$, we can compute the Frećhet derivative,

$$\langle \hat{J}'(u), v\rangle = \sum_{t=1}^{T}\Big(-\Delta\big(\phi_c\big(\Delta u(t-1)\big)\big) + q(t)u(t) - \lambda\hat{f}\big(t,u(t)\big)\Big)v(t),$$

for all $u, v \in E$. It is clear that the critical points of $\hat{J}$ are the solutions of problem (12). In what follows, we will prove that $\hat{J}$ has at least T distinct pairs of nonzero critical points by Lemma 2.1.

For any sequence $\{u_n\} \subset E$, if $\{\hat{J}(u_n)\}$ is bounded and $\hat{J}'(u_n) \to 0$ as $n \to +\infty$, we claim that $\{u_n\}$ is bounded. In fact, there exists a positive constant $C \in \mathbb{R}$ such that $|\hat{J}(u_n)| \leq C$. Since E is a finite-dimensional real Banach space, there is $\|u\|_2 \leq \|u\| \leq 2\|u\|_2$ for all $u \in E$ (see [30]). Assume that $\|u_n\| \to +\infty$ as $n \to +\infty$, then

$$\begin{aligned}
C &\geq \hat{J}(u_n) \\
&= \sum_{t=0}^{T}\left(\left(\frac{\sqrt{1+\kappa(\Delta u(t))^2}-1}{\kappa}\right) + \frac{q(t)u_n^2(t)}{2}\right) - \lambda\sum_{t=1}^{T}\hat{F}\big(t,u_n(t)\big) \\
&\geq \sum_{t=1}^{T}\frac{q(t)u_n(t)^2}{2} - \lambda\sum_{|u_n(t)|\leq M}\big|\hat{F}\big(t,u_n(t)\big)\big| - \lambda\sum_{|u_n(t)|>M}\big|\hat{F}\big(t,u_n(t)\big)\big| \\
&\geq \frac{q_*}{2}\|u_n\|_2^2 - \lambda D\sum_{|u_n(t)|\leq M}\big|u_n(t)\big| - \lambda\sum_{t=0}^{T}\left|\int_0^M \hat{f}(t,s)\,ds\right| - \lambda\sum_{|u_n(t)|>M}\left|\int_M^{u_n(t)}\hat{f}(t,s)\,ds\right| \\
&\geq \frac{q_*}{2}\|u_n\|_2^2 - \lambda D\sum_{|u_n(t)|\leq M}\big|u_n(t)\big| - \lambda D\sum_{|u_n(t)|>M}\big|u_n(t)\big| - 2\lambda TDM
\end{aligned}$$

$$= \frac{q_*}{2}\|u_n\|_2^2 - \lambda D\sum_{t=1}^{T}\bigl|u_n(t)\bigr| - 2\lambda TDM$$
$$\geq \frac{q_*}{8}\|u_n\|^2 - \lambda DT^{\frac{1}{2}}\|u_n\| - 2\lambda TDM \to +\infty \quad \text{as } n\to +\infty,$$

where $q_* = \min_{t\in\mathbb{Z}(1,T)} q(t)$ and $D = \max|f(t,u)|$ for $(t,u)\in\mathbb{Z}(1,T)\times[-M,M]$. This is impossible, since C is a fixed constant. Thus, $\{u_n\}$ is bounded in E. This implies that $\{u_n\}$ has a convergent subsequence. Then, the functional $\hat{J}$ satisfies the P.S. condition.

Moreover, the coerciveness of $\hat{J}$,

$$\hat{J}(u) \geq \frac{q_*}{8}\|u\|^2 - \lambda DT^{\frac{1}{2}}\|u\| - 2\lambda TDM \to +\infty \quad \text{as } \|u\|\to +\infty$$

implies that $\hat{J}$ is bounded from below.

Let $\{e_i\}_{i=1}^T$ be a base of E and $\|e_i\| = 1$ for each $i\in\mathbb{Z}(1,T)$. We define

$$A(\rho) = \left\{\sum_{i=1}^{T}\beta_i e_i \Big| \sum_{i=1}^{T}|\beta_i|^2 = \rho^2\right\}, \quad \rho > 0.$$

Obviously, $\theta\notin A(\rho)$, $A(\rho)$ is closed in E and symmetric with respect to θ. We note that $A(\rho)$ is homeomorphic to S^{T-1} for any $\rho>0$. For $u\in A(\rho)$, we see that

$$\|u\|^2 = \sum_{t=0}^{T}\left|\sum_{i=1}^{T}\beta_i\Delta e_i(t)\right|^2 \leq \sum_{t=0}^{T}\left(\sum_{i=1}^{T}|\beta_i|^2\sum_{i=1}^{T}\bigl|\Delta e_i(t)\bigr|^2\right)$$
$$= \rho^2\sum_{i=1}^{T}\|e_i\|^2 \leq \rho^2(T+1), \quad \rho>0.$$

Take $\rho = \frac{\alpha}{T+1}$, thus

$$\|u\|_\infty \leq \sum_{t=0}^{T}\bigl|\Delta u(t)\bigr| \leq (T+1)^{\frac{1}{2}}\|u\| \leq (T+1)\rho < \alpha < M.$$

For $u\in A(\frac{\alpha}{T+1})$, we note that $u\neq\theta$ and $\hat{f}(t,u(t)) = f(t,u(t))$. By (a_2) and (a_3), then

$$\sum_{t=1}^{T}\hat{F}\bigl(t,u(t)\bigr) = \sum_{\{t\in\mathbb{Z}(1,T)|u(t)>0\}}\hat{F}\bigl(t,u(t)\bigr) + \sum_{\{t\in\mathbb{Z}(1,T)|u(t)<0\}}\hat{F}\bigl(t,u(t)\bigr)$$
$$= \sum_{\{t\in\mathbb{Z}(1,T)|u(t)>0\}}\int_0^{u(t)} f(t,s)\,ds + \sum_{\{t\in\mathbb{Z}(1,T)|u(t)<0\}}\int_0^{-u(t)} f(t,-s)d(-s)$$
$$= \sum_{\{t\in\mathbb{Z}(1,T)|u(t)>0\}}\int_0^{u(t)} f(t,s)\,ds + \sum_{\{t\in\mathbb{Z}(1,T)|u(t)<0\}}\int_0^{-u(t)} f(t,s)\,ds$$
$$> 0.$$

Let $\tau = \inf_{u\in A(\frac{\alpha}{T+1})}\sum_{t=1}^{T}\hat{F}(t,u(t))$ and $\bar{\lambda} = \frac{(2+q^*)\alpha^2}{2T\tau}$. By (a_2), we know $\tau>0$. If $\lambda>\bar{\lambda}$, then

$$\hat{J}(u) = \sum_{t=0}^{T}\left(\left(\frac{\sqrt{1+\kappa(\Delta u(t))^2}-1}{\kappa}\right) + \frac{q(t)u^2(t)}{2}\right) - \lambda\sum_{t=1}^{T}\hat{F}\bigl(t,u(t)\bigr)$$

$$\begin{aligned}
&\le \sum_{t=0}^{T}\left|\Delta u(t)\right|^2 + \frac{q^*}{2}\|u\|_2^2 - \lambda \sum_{t=1}^{T}\hat{F}\big(t,u(t)\big) \\
&\le \frac{2+q^*}{2}\|u\|^2 - \lambda \sum_{t=1}^{T}\hat{F}\big(t,u(t)\big) \\
&\le \frac{(2+q^*)\alpha^2}{2T} - \lambda\tau \\
&< 0,
\end{aligned}$$

where $q^* = \max_{t\in\mathbb{Z}(1,T)} q(t)$. Since all conditions of Lemma 2.1 hold, problem (4) admits at least T distinct pairs of nontrivial solutions. The proof is completed. □

At the end of this section, we try to prove a pair of constant-sign solutions of the original problem. We first introduce the following assumptions:

(a_1') $f(t,\cdot)$ is a continuous functional on $\mathbb{R}\backslash\{0\}$ for each $t\in\mathbb{Z}(1,T)$;

(a_2') $0<q^*\le f(t,\xi), \forall(t,\xi)\in\mathbb{Z}(1,T)\times(0,\alpha)$ for some $\alpha>0$, where $q^* = \max_{t\in\mathbb{Z}(1,T)} q(t)$;

(a_3') $f(t,\xi) = -f(t,-\xi)$ in $\xi\neq 0$ for any $t\in\mathbb{Z}(1,T)$.

We note that the function $f(t,\cdot)$ is locally bounded from below for each $t\in\mathbb{Z}(1,T)$ in the right-hand side of 0 from (a_2') and problem (4) has no trivial solution. When θ is not the solution of the problem, many problems become more complicated [31, 32]. For example, we put $f(t,\xi) = \frac{1}{\sqrt[3]{\xi}}$, $\alpha = 1$ and $q^* = \frac{1}{2}$, then $0<\frac{1}{2}<\frac{1}{\sqrt[3]{\xi}}, \forall(t,\xi)\in\mathbb{Z}(1,T)\times(0,1)$, which satisfies the conditions (a_1')–(a_3').

Let

$$\mu_1 = \inf_{u\in E\backslash\{\theta\}} \frac{\sum_{t=0}^{T}\frac{(\Delta u(t))^2}{\sqrt{1+\kappa(\Delta u(t))^2}}}{\|u\|_2^2}.$$

We observe that $\frac{\sum_{t=0}^{T}\frac{(\Delta u(t))^2}{\sqrt{1+\kappa(\Delta u(t))^2}}}{\|u\|_2^2}$ is positive in $E\backslash\{\theta\}$. Thus, $\mu_1\ge 0$.

Theorem 3.2 *Assume that* (a_1')–(a_3') *hold and*

$$\limsup_{|\xi|\to\infty}\frac{f(t,\xi)}{\xi} < \mu_1 + q_*, \quad t\in\mathbb{Z}(1,T). \tag{14}$$

Then, problem (4) *has a positive solution and a negative solution for each* $\lambda\in(0,\frac{\mu_1+q_*}{\mu})$, *where* $\mu\in(0,\mu_1+q_*)$.

Proof We consider the following problem

$$\begin{cases} -\Delta(\phi_c(\Delta u(t-1))) + q(t)u(t) = \lambda q^*, & t\in\mathbb{Z}(1,T), \\ u(0) = u(T+1) = 0. \end{cases} \tag{15}$$

Define the variational framework of problem (15)

$$J_{q^*}(u) = \sum_{t=0}^{T}\left(\left(\frac{\sqrt{1+\kappa(\Delta u(t))^2}-1}{\kappa}\right) + \frac{q(t)u^2(t)}{2}\right) - \lambda q^*\sum_{t=1}^{T}u(t),$$

then we have

$$\begin{aligned} J_{q^*}(u) &= \sum_{t=0}^{T}\left(\left(\frac{\sqrt{1+\kappa(\Delta u(t))^2}-1}{\kappa}\right)+\frac{q(t)u^2(t)}{2}\right)-\lambda q^*\sum_{t=1}^{T}u(t) \\ &= \sum_{t=0}^{T}\left(\frac{(\Delta u(t))^2}{\sqrt{1+\kappa(\Delta u(t))^2}+1}+\frac{q(t)u^2(t)}{2}\right)-\lambda q^*\sum_{t=1}^{T}u(t) \\ &\geq \sum_{t=0}^{T}\frac{(\Delta u(t))^2}{2\sqrt{1+\kappa(\Delta u(t))^2}}+\frac{q_*}{2}\|u\|_2^2-\lambda q^*\sqrt{T}\|u\|_2 \\ &\geq \frac{\mu_1+q_*}{2}\|u\|_2^2-\lambda q^*\sqrt{T}\|u\|_2 \to +\infty \quad \text{as } \|u\|_2 \to +\infty. \end{aligned}$$

Hence, J_{q^*} is coercive on E and has a global minimum point u_0 that is its critical point. Combining with Lemma 2.4, u_0 is a positive solution of problem (15). We take $\varepsilon > 0$ so small that $u_1(t) = \varepsilon u_0(t) < \alpha$.

Define a continuous function as follows:

$$f_{u_1}(t,\xi) = \begin{cases} f(t,\xi) & \text{if } \xi \geq u_1(t), \\ f(t,u_1(t)) & \text{if } \xi < u_1(t). \end{cases}$$

By (14), there exist a $\mu \in [0, \mu_1 + q_*)$ and $M_1 > u_1(t)$ such that

$$f(t,\xi) \leq \mu\xi, \quad (t,\xi) \in \mathbb{Z}(1,T) \times (M_1,\infty). \tag{16}$$

Thus,

$$f_{u_1}(t,\xi) \begin{cases} \leq f(t,u_1(t)) + \max_{(t,\xi)\in\mathbb{Z}(1,T)\times[u_1(t),M_1]} f(t,\xi) + \mu\xi & \text{if } \xi \geq 0, \\ \geq q^* & \text{if } \xi < 0, \end{cases} \tag{17}$$

and

$$\limsup_{|\xi|\to\infty} \frac{f_{u_1}(t,\xi)}{\xi} \leq \mu, \quad t \in \mathbb{Z}(1,T). \tag{18}$$

Next, we claim that the following problem

$$\begin{cases} -\Delta(\phi_c(\Delta u(t-1))) + q(t)u(t) = \lambda f_{u_1}(t,u(t)), & t \in \mathbb{Z}(1,T), \\ u(0) = u(T+1) = 0 \end{cases} \tag{19}$$

admits a positive solution u and $u > u_1 > 0$.

We define the following variational framework corresponding to problem (19)

$$\hat{J}(u) = \sum_{t=0}^{T}\left(\left(\frac{\sqrt{1+\kappa(\Delta u(t))^2}-1}{\kappa}\right)+\frac{q(t)u^2(t)}{2}\right)-\lambda\sum_{t=1}^{T}F_{u_1}\big(t,u(t)\big),$$

where $F_{u_1}(t,\xi) = \int_0^\xi f_{u_1}(t,s)\,ds$, $(t,\xi) \in \mathbb{Z}(1,T) \times \mathbb{R}$.

Using (18), there is one positive constant $\overline{M}$ such that

$$F_{u_1}(t,\xi) \le \frac{\mu}{2}|\xi|^2 + \overline{M}. \tag{20}$$

Let $\eta = \frac{\mu_1+q_*}{\mu}$. For $\eta > \lambda > 0$, we have

$$\begin{aligned}
\hat{J}(u) &= \sum_{t=0}^{T}\left(\left(\frac{\sqrt{1+\kappa(\Delta u(t))^2}-1}{\kappa}\right) + \frac{q(t)u^2(t)}{2}\right) - \lambda\sum_{t=1}^{T}F_{u_1}\big(t,u(t)\big) \\
&\ge \sum_{t=0}^{T}\frac{(\Delta u(t))^2}{\sqrt{1+\kappa(\Delta u(t))^2}+1} + \frac{q_*}{2}\|u\|_2^2 - \frac{\lambda\mu}{2}\|u\|_2^2 - \lambda T\overline{M} \\
&\ge \sum_{t=0}^{T}\frac{(\Delta u(t))^2}{2\sqrt{1+\kappa(\Delta u(t))^2}} + \frac{q_*}{2}\|u\|_2^2 - \frac{\lambda\mu}{2}\|u\|_2^2 - \lambda T\overline{M} \\
&\ge \frac{\mu}{2}(\eta-\lambda)\|u\|_2^2 - \lambda T\overline{M} \to +\infty \quad \text{as } \|u\|_2 \to +\infty.
\end{aligned}$$

This shows that $\hat{J}$ is also coercive. As the functional is coercive and continuous, it has a global minimum point $u \in E$, which is a critical point. By (17) and Lemma 2.5, u is a positive solution. Moreover, if we can show $u > u_1$, then u must be one positive solution of problem (4). First, we assume that $u \le u_1$ for every $t \in \mathbb{Z}(1,T)$. Since

$$-\Delta\big(\phi_c\big(\Delta u(t-1)\big)\big) + q(t)u(t) = \lambda f\big(t,u_1(t)\big) \ge \lambda q^* = -\Delta\big(\phi_c\big(\Delta u_0(t-1)\big)\big) + q(t)u_0(t),$$

by Lemma 2.2, we obtain $u \ge u_0 > u_1$, contradicting the assumption above. Secondly, we consider that u and u_1 are not ordered vectors. There exist some $j_0 \in \mathbb{Z}(1,T)$ such that $u(j_0) < u_1(j_0)$. Let

$$j := \max\big\{j_0 | j_0 \in \mathbb{Z}(1,T) \text{ such that } u(j_0) < u_1(j_0)\big\}.$$

From the proof of Lemma 2.2, if $u(j) < u_1(j)$ holds, we have the following inequality

$$\lambda f\big(i,u_1(i)\big) = -\Delta\big(\phi_c\big(\Delta u(i-1)\big)\big) + q(i)u(i) < -\Delta\big(\phi_c\big(\Delta u_1(i-1)\big)\big) + q(i)u_1(i), \tag{21}$$

where $i = 1,2,j$. Since $q^* \le f(i,u_1(i))$ from the proof of Lemma 2.2 and (a_2'), this implies

$$\lambda q^* = -\Delta\big(\phi_c\big(\Delta u_0(i-1)\big)\big) + q(i)u_0(i) < -\Delta\big(\phi_c\big(\Delta u_1(i-1)\big)\big) + q(i)u_1(i). \tag{22}$$

We see from (22) that

$$\begin{aligned}
0 &\le q(i)\big(u_0(i) - u_1(i)\big) \\
&< \Delta\big(\phi_c\big(\Delta u_0(i-1)\big)\big) - \Delta\big(\phi_c\big(\Delta u_1(i-1)\big)\big) \\
&= \left(\frac{\varepsilon\Delta u_0(i-1)}{\sqrt{1+\kappa(\varepsilon\Delta u_0(i-1))^2}} - \frac{\Delta u_0(i-1)}{\sqrt{1+\kappa(\Delta u_0(i-1))^2}}\right) \\
&\quad + \left(\frac{\Delta u_0(i)}{\sqrt{1+\kappa(\Delta u_0(i))^2}} - \frac{\varepsilon\Delta u_0(i)}{\sqrt{1+\kappa(\varepsilon\Delta u_0(i))^2}}\right).
\end{aligned}$$

By the strict monotonicity of ϕ_c, we find that if $\Delta u_0(i-1) > 0$, then $\Delta u_0(i) > 0$. That is, if $\Delta u_1(i-1) > 0$, then $\Delta u_1(i) > 0$.

Further, we estimate the inequality (22) from the following three cases. First, if $\Delta u_1(i-1) \le \Delta u_1(i)$, we have

$$\lambda q^* < -\Delta\big(\phi_c\big(\Delta u_1(i-1)\big)\big) + q(i)u_1(i) < \varepsilon q(i)u_0(i). \tag{23}$$

For the second case, if $\Delta u_1(i-1) > \Delta u_1(i) > 0$, then

$$\begin{aligned}
\lambda q^* &< -\Delta\big(\phi_c\big(\Delta u_1(i-1)\big)\big) + q(i)u_1(i) \\
&= \frac{\Delta u_1(i-1)}{\sqrt{1+\kappa(\Delta u_1(i-1))^2}} - \frac{\Delta u_1(i)}{\sqrt{1+\kappa(\Delta u_1(i))^2}} + q(i)u_1(i) \\
&\le \Delta u_1(i-1) + q(i)u_1(i) \\
&\le \varepsilon\big(\Delta u_0(i-1) + q(i)u_0(i)\big).
\end{aligned} \tag{24}$$

For the last case, if $0 > \Delta u_1(i-1) > \Delta u_1(i)$, then

$$\begin{aligned}
\lambda q^* &< -\Delta\big(\phi_c\big(\Delta u_1(i-1)\big)\big) + q(i)u_1(i) \\
&= \frac{\Delta u_1(i-1)}{\sqrt{1+\kappa(\Delta u_1(i-1))^2}} - \frac{\Delta u_1(i)}{\sqrt{1+\kappa(\Delta u_1(i))^2}} + q(i)u_1(i) \\
&\le -\Delta u_1(i) + q(i)u_1(i) \\
&\le \varepsilon\big(-\Delta u_0(i) + q(i)u_0(i)\big).
\end{aligned} \tag{25}$$

We note that $\Delta u_1(i-1) = 0$ or $\Delta u_1(i) = 0$ still satisfies the above cases. Obviously, when ε is taken sufficiently small, (23), (24), and (25) cannot hold. These are contradictions. Thus, $u \ge u_1$. u is one positive solution of problem (4). Moreover, we see that $-u$ is a negative solution of problem (4) because of (a_3'). This completes the proof. □

Example 3.1 Let $\lambda = 1$, we consider the problem

$$\begin{cases} -\Delta(\phi_c(\Delta u(t-1))) + q(t)u(t) = \sin(u(t)), & t \in \mathbb{Z}(1,T), \\ u(0) = u(T+1) = 0, \end{cases} \tag{26}$$

we note that the conditions (a_1)–(a_3) hold from Example 1.1, and $\bar{\lambda}$ can be less than 1 by the definition, thus the problem (26) admits at least T distinct pairs of nontrivial solutions by Theorem 3.1.

In fact, in [30], such a problem can be found when $\kappa = 0$ and $q(t) = 0$ for each $t \in \mathbb{Z}(1,T)$ in Corollary 5.1. We see that the conditions of Theorem 3.1 are different from the conditions of Corollary 5.1 of [30], and we find more solutions of problem (26).

Example 3.2 Let $\kappa = 0$ and $T = 3$. We consider the problem

$$\begin{cases} -\Delta^2 u(t-1) + q(t)u(t) = \lambda \frac{1}{\sqrt[3]{u(t)}}, & t \in \mathbb{Z}(1,3), \\ u(0) = u(4) = 0. \end{cases} \tag{27}$$

Put $\alpha = 1$, $q_* = \frac{1}{4}$ and $q^* = \frac{1}{2}$. The conditions (a_1')–(a_3') hold from the previous example. We have $\mu_1 = 2 - \sqrt{2}$ from [30]. Clearly,

$$\limsup_{|\xi|\to\infty} \frac{f(t,\xi)}{\xi} = \limsup_{|\xi|\to\infty} \frac{1}{\xi^{4/3}} = 0 < \frac{9}{4} - \sqrt{2}, \quad t \in \mathbb{Z}(1,T).$$

All conditions of Theorem 3.2 are verified. If we take $\mu > 0$ sufficiently small, then the problem (27) has a positive solution and a negative solution for each $\lambda \in (0,+\infty)$.

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Author contribution
ZW proposed the idea of this paper and performed all the steps of the proofs. QX wrote the whole paper. All authors read and approved the final manuscript.

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