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Multiple solutions for a system involving an anisotropic variable exponent operator

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Abstract

In this paper, the existence of a solution for an anisotropic variable exponent system is obtained and proved under general hypotheses. By considering additional conditions, it is proved a multiplicity result. The proofs are based on an application of appropriated L^∞ estimates, a sub-supersolution argument, and the Mountain Pass Theorem.

Keywords: Anisotropic problem; Electrorheological fluids; Maximum principle; Variable exponents; Weak solution; Sub-supersolutions

1 Introduction

In this paper, we are interested in nonnegative solutions for the anisotropic system

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) = a(x)u^{\alpha(x)-1} + F_u(x,u,v) & \text{in } \Omega, \\ -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{q_i(x)-2}\frac{\partial v}{\partial x_i}\right) = b(x)v^{\beta(x)-1} + F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{S}$$

where, unless otherwise stated, Ω is a bounded domain in $\mathbb{R}^N (N \geq 3)$ with smooth boundary, $p_i, q_i \in C(\overline{\Omega})$, $2 \leq p_i(x) \leq p_+(x) < \overline{p}^\star(x)$, $2 \leq q_i(x) \leq q_+(x) < \overline{q}^\star(x)$, $i = 1, \ldots, N$, $p_+(x) := \max\{p_1(x), \ldots, p_N(x)\}$, $q_+(x) := \max\{q_1(x), \ldots, q_N(x)\}$ for any $x \in \overline{\Omega}$ with $\overline{p}(x) := N/\sum_{i=1}^N (1/p_i(x))$ and $\overline{p}^*(x) = N\overline{p}(x)/(N - \overline{p}(x))$ if $\overline{p}(x) < N$ and $\overline{p}(x) = +\infty$ if $N \geq p(x)$, $\overline{q}(x) := N/\sum_{i=1}^N (1/q_i(x))$ and $\overline{q}^*(x) = N\overline{q}(x)/(N - \overline{q}(x))$ if $\overline{q}(x) < N$ and $\overline{q}(x) = +\infty$ if $N \geq q(x)$, $\alpha, \beta \in C(\overline{\Omega})$ are nonnegative functions with $1 \leq \alpha(x), \beta(x)$ for all $x \in \overline{\Omega}$, $F : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function and

(H) $a, b \in L^\infty(\Omega)$ and $a(x), b(x) > 0$ a.e. in Ω;

(F_1) There is $\delta > 0$ with

$$F_s(x,s,t) \geq \left(1 - s^{\alpha(x)-1}\right)a(x), \quad \text{for all } 0 \leq s \leq \delta \text{ a.e. in } \Omega$$

and

$$F_t(x,s,t) \geq \left(1 - t^{\beta(x)-1}\right)b(x), \quad \text{for all } 0 \leq t \leq \delta \text{ a.e. in } \Omega.$$

(F_2) There is $r \in C(\overline{\Omega})$ with $1 < r(x)$, for all $x \in \overline{\Omega}$ and

$$F_s(x,s,t) \le a(x)\big(s^{r(x)-1} + t^{r(x)-1} + 1\big), \quad \text{for all } 0 \le s, \text{ a.e. in } \Omega$$

and

$$F_t(x,s,t) \le b(x)\big(s^{r(x)-1} + t^{r(x)-1} + 1\big), \quad \text{for all } 0 \le t, \text{ a.e. in } \Omega.$$

It will be considered that a weak solution for the system (S) is a pair $(u,v) \in Z$, where $Z := W_0^{1,\overrightarrow{p(x)}}(\Omega) \times W_0^{1,\overrightarrow{q(x)}}(\Omega)$ with $u(x), v(x) \ge 0$ a.e in Ω satisfying

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega a(x)v^{\alpha(x)-1}\varphi + F_u(x,u,v)\varphi$$

and

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial v}{\partial x_i}\right|^{q_i(x)-2} \frac{\partial v}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega b(x)u^{\beta(x)-1}\psi + F_v(x,u,v)\psi,$$

for all $(\varphi,\psi) \in Z$.

Denote by $\|\cdot\|_\infty$ the norm in the space $L^\infty(\Omega)$. Through minimization and sub-supersolutions arguments, it is obtained the existence result below.

Theorem 1.1 *Suppose that the hypotheses* (H), (F_1)-(F_2) *hold. Then, there exists* $\nu > 0$ *such that the system* (S) *has a solution for* $\max\{\|a\|_\infty, \|b\|_\infty\} < \nu$.

Consider the functions $p_\infty(x) := \max\{\overline{p}^\star(x), p_+(x)\}$, $p_-(x) := \min\{p_1(x),\dots,p_n(x)\}$, $q_\infty(x) := \max\{\overline{q}^\star(x), q_+(x)\}$, $q_-(x) := \min\{q_1(x),\dots,q_n(x)\}$, $x \in \overline{\Omega}$ and denote $l^- := \inf_\Omega l$ and $l^+ := \sup_\Omega l$ for a function $l \in C(\overline{\Omega})$. Under the Ambrosetti-Rabinowitz type condition,

(F_3) it holds that $\alpha^- > 1$, $\alpha^+, r^+ < p_\infty^-$ with $\alpha^+ < p_-^-$ or $p_+^+ < \alpha^-$, and there are constants $t_0 > 0$ and $\theta > p_+^+$ such that

$$0 < \theta F(x,t) \le f(x,t)t, \quad \text{a.e. in } \Omega, \text{for all } t \ge t_0,$$

we have the multiplicity result below.

(F_3) It hold the inequalities $r^+ < \min\{p_\infty^-, q_\infty^-\}$, $\alpha^+ < p_\infty^-$, $\beta^+ < q_\infty^-$, and there are $0 < \theta < \frac{1}{p_+^+}$, $0 < \xi < \frac{1}{q_+^+}$ and $k_0 > 0$ such that

$$F(x,s,t) \le \theta s F_s(x,s,t) + \xi t F_t(x,s,t)$$

a.e. in Ω for any $|(s,t)| \ge k_0$ with $s,t \ge 0$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$ and $F(x,s,t) := \int_0^s F_\tau(x,\tau,t)\,d\tau + \int_0^t F_\tau(x,t,\tau)\,d\tau$.

Theorem 1.2 *Suppose that the hypotheses* (H), (f_1)-(f_3) *hold. Consider that one of the conditions below holds.*

(i) *It holds that* $p_+^+ < \alpha^-$ *and* $q_+^+ < \beta^-$ *or* $\beta^+ < q_-^-$.

(ii) *It holds that* $\alpha^+ < p_-^-$ *and* $\beta^+ < q_-^-$ *or* $q_+^+ < \beta^-$.

Then, there exists $\eta > 0$ *such that system* (*S*) *has at least two solutions for* $\max\{\|a\|_\infty, \|b\|_\infty\} < \eta$.

In the last decades, Partial Differential Equations with variable exponents have been attracting the attention of several scientists due to their applicability in several relevant models. The main application of this kind of equation is in the study of electrorheological fluids. As mentioned in [1], the study of such fluids arose when fluids that stop spontaneously were discovered, also known as Bingham fluids. In the classical reference [2], due to W. Winslow, it was presented one of the main properties of electrorheological fluids. Parallel and string-like formations arise in this kind of fluid when it is considered the presence of an electrical field. This pattern is known as the*Winslow effect*. Moreover, the electrical field can raise the viscosity of the fluid by five orders of magnitude, see reference [1]. As pointed out in the interesting work [3], several studies with electrorheological fluids have been considered in NASA laboratories.

On the other hand, Anisotropic Partial Differential Equations can also be applied in several models. For example, in the classical reference [4], a model was presented that was applied for both image enhancement and denoising in terms of anisotropic problems as well as allowed the preservation of significant image features. We also quote the applicability in the study of the spread of epidemic disease in heterogeneous environments. In Physics, such an equation can be applied to consider the dynamics of fluids with different conductivities in different directions. We point out the references [4–7] for more details.

An important fact is that there is increasing interest in anisotropic problems with variable exponents. In the paper [8], the regularity of solutions of a stationary system is obtained, which is motivated by the theory of electrorheological fluids. In [9], a strong maximum principle is gained in the variable exponent setting, generalizing the classical principal of the Laplacian operator. The paper [10] presents the mathematical theory, which allows considering problems involving anisotropic operators with variable exponents. Moreover, several applications were considered. We also point out the interesting references [11–20] and the paper [21], which provides an overview concerning elliptic variational problems with nonstandard growth conditions and refers to different kinds of nonuniformly elliptic operators. See also [1, 22] for a complete presentation of the theory of the Sobolev spaces with variable exponents and its applications.

The study of the system (*S*) is motivated by the problem considered in the reference [23], where it was proved, in an anisotropic setting, versions of Theorems 1.1 and 1.2 with $\alpha, \beta \equiv 2$, and [24], where it was considered a scalar version of the system.

Regarding the remainder of the paper, we mention that in Sect. 2, it is considered some preliminary facts regarding the theory of the anisotropic variable spaces. The proofs of Theorems 1.1 and 1.2 are provided in Sects. 3 and 4, respectively.

2 Preliminaries

Consider $p \in C_+(\overline{\Omega}) := \{p \in C(\overline{\Omega}); \inf_\Omega p > 1\}$, where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain. The Lebesgue space with a variable exponent is defined by

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable}; \int_\Omega |u(x)|^{p(x)} < \infty\right\},$$

with the Luxemburg's norm

$$\|u\|_{p(x)} := \inf\left\{\lambda > 0;\ \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \leq 1\right\}.$$

It holds that $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a Banach space.

In what follows, we point out some results that can be found, for example, in [25].

Proposition 2.1 *Consider a function $p \in C_+(\overline{\Omega})$ and define $\rho(u) := \int_\Omega |u|^{p(x)}\,dx$. For $u, u_n \in L^{p(x)}(\Omega)$, $n \in \mathbb{N}$, the assertions below hold.*

(i) *If $u \neq 0$ in $L^{p(x)}(\Omega)$, then $\|u\|_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$;*
(ii) *If $\|u\|_{p(x)} < 1$ $(>1;\ =1)$, then $\rho(u) < 1$ $(>1;\ =1)$;*
(iii) *If $\|u\|_{p(x)} > 1$, then $\|u\|_{p(x)}^{p^-} \leq \rho(u) \leq \|u\|_{p(x)}^{p^+}$;*
(iv) *If $\|u\|_{p(x)} < 1$, then $\|u\|_{p(x)}^{p^+} \leq \rho(u) \leq \|u\|_{p(x)}^{p^-}$.*

Theorem 2.2 *Consider functions $p, q \in C_+(\overline{\Omega})$. The statements below hold.*

(i) *If $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ in Ω, then $|\int_\Omega uv\,dx| \leq (\frac{1}{p^-} + \frac{1}{q^-})\|u\|_{p(x)}\|v\|_{q(x)}$;*
(ii) *If $q(x) \leq p(x)$ in Ω and $|\Omega| < \infty$, then $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

Some results on anisotropic variable exponents [10] will be presented below. Consider functions $p_i \in C_+(\overline{\Omega})$, $i = 1, \dots, N$. Define

$$\overrightarrow{p(x)} := \big(p_1(x), \dots, p_N(x)\big) \in \big(C_+(\overline{\Omega})\big)^N$$

and consider the functions

$$p_+(x) := \max\big\{p_1(x), \dots, p_N(x)\big\} \quad \text{and} \quad p_-(x) := \min\big\{p_1(x), \dots, p_N(x)\big\}, \quad x \in \overline{\Omega}. \tag{2.1}$$

The anisotropic variable exponent Sobolev space is defined by

$$W^{1,\overrightarrow{p(x)}}(\Omega) := \left\{u \in L^{p_+(x)}(\Omega);\ \frac{\partial u}{\partial x_i} \in L^{p_i(x)}(\Omega), i = 1, \dots, N\right\},$$

which is a Banach space with the norm

$$\|u\|_* := \|u\|_{p_+(x)} + \sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(x)}. \tag{2.2}$$

If $p_i^- > 1$, $i = 1, \dots, N$, then it holds that $W^{1,\overrightarrow{p(x)}}(\Omega)$ is reflexive, see, for instance, [10, Theorem 2.2].

Denote by $W_0^{1,\overrightarrow{p(x)}}(\Omega)$ the Banach space defined by the closure of $C_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}}(\Omega)$ with the norm (2.2).

Define the functions $\overline{p}(x) := N/\sum_{i=1}^N (1/p_i(x))$ and $\overline{p}^*(x) = N\overline{p}(x)(N - \overline{p}(x))$ if $\overline{p}(x) < N$ and $\overline{p}(x) = +\infty$ if $N \geq p(x)$. Under the condition $p(x) < \overline{p}^*(x)$ for all $x \in \overline{\Omega}$, it holds the Poincaré type inequality below

$$\|u\|_{p^+(x)} \leq C \sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(x)} \quad \text{for all } u \in W_0^{1,\overrightarrow{p(x)}}(\Omega), \tag{2.3}$$

where C is a positive constant that does not depend on $u \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$. Thus, it holds that the norm defined by

$$\|u\|_{1,\overrightarrow{p}(x)} := \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i(x)}, \quad u \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$$

is equivalent to the one given in (2.2).

An important fact is that it holds the compact embedding

$$W_0^{1,\overrightarrow{p(x)}}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \tag{2.4}$$

for a function $q \in C_+(\overline{\Omega})$ with $q(x) < p_\infty(x)$, for all $x \in \overline{\Omega}$, where $p_\infty(x) := \max\{\overline{p}^\star(x), p_+(x)\}$.

The results below, which will play an important role in our arguments, can be found in [24].

Lemma 2.3 *Consider a function $a \in L^\infty(\Omega)$. The problem*

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{p_i(x)-2}\frac{\partial u}{\partial x_i}) = a & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases}$$

has an unique solution in $W_0^{1,\overrightarrow{p(x)}}(\Omega)$.

Lemma 2.4 *Consider functions $u, v \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$ such that*

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{p_i(x)-2}\frac{\partial u}{\partial x_i}) \le -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial v}{\partial x_i}|^{p_i(x)-2}\frac{\partial v}{\partial x_i}) & in\ \Omega, \\ u \le v & on\ \partial\Omega, \end{cases}$$

where $u \le v$ on $\partial\Omega$ means that $(u-v)^+ := \max\{0, u-v\} \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$. Then it holds that $u(x) \le v(x)$ a.e. in Ω.

Lemma 2.5 *Let $u_\lambda \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$ be the unique solution to the problem*

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{p_i(x)-2}\frac{\partial u}{\partial x_i}) = \lambda & in\ \Omega, \\ u = 0 & on\ \Omega, \end{cases}$$

where $\lambda > 0$ is a constant. Define $\sigma := \frac{p_-^-}{2|\Omega|^{\frac{1}{N}}} K_0$, where K_0 is the best constant of the continuous embedding $W_0^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$, which depends only on Ω and N. If $\lambda < \sigma$, then $u \in L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \le K_\star \lambda^{\frac{1}{p_+^+-1}}$ and $\|u\|_{L^\infty(\Omega)} \le K^\star \lambda^{\frac{1}{p_-^- -1}}$ when $\lambda \ge \sigma$, where $K^\star$ and $K_\star$ are positive constants depending only on Ω, N and p_i, $i = 1, \dots, N$.

3 Proof of Theorem 1.1

The proof of Theorem 1.1 will be split into some steps. The first one consists of obtaining appropriated sub-supersolutions for the system (*S*). After this, the existence of solutions for an auxiliary system will be proved, which solves (*S*).

In what follows, it will be considered the definition of sub-supersolution for the system (S) and an auxiliary lemma.

It will be considered that $(\underline{u},\underline{v}),(\overline{u},\overline{v}) \in (W_0^{1,\overrightarrow{p(x)}}(\Omega)\cap L^\infty(\Omega)) \times (W_0^{1,\overrightarrow{q(x)}}(\Omega)\cap L^\infty(\Omega))$ is a sub-supersolution pair for the system (S) if $\underline{u}(x) \leq \overline{u}(x)$, $\underline{v}(x) \leq \overline{v}(x)$ a.e. in Ω and

$$\begin{cases} \int_\Omega \sum_{i=1}^N |\frac{\partial \underline{u}}{\partial x_i}|^{p_i(x)-2} \frac{\partial \underline{u}}{\partial x_i}\frac{\partial \varphi}{\partial x_i} \leq \int_\Omega a(x)\underline{u}^{\alpha(x)-1}\varphi + F_u(x,\underline{u},w)\varphi & \text{for all } w \in [\underline{v},\overline{v}], \\ \int_\Omega \sum_{i=1}^N |\frac{\partial \underline{v}}{\partial x_i}|^{q_i(x)-2} \frac{\partial \underline{v}}{\partial x_i}\frac{\partial \psi}{\partial x_i} \leq \int_\Omega b(x)\underline{v}^{\beta(x)-1}\psi + F_v(x,w,\underline{v})\psi & \text{for all } w \in [\underline{u},\overline{u}] \end{cases} \tag{3.1}$$

and

$$\begin{cases} \int_\Omega \sum_{i=1}^N |\frac{\partial \overline{u}}{\partial x_i}|^{p_i(x)-2} \frac{\partial \overline{u}}{\partial x_i}\frac{\partial \varphi}{\partial x_i} \geq \int_\Omega a(x)\overline{u}^{\alpha(x)-1}\varphi + F_u(x,\overline{u},w)\varphi & \text{for all } w \in [\underline{v},\overline{v}], \\ \int_\Omega \sum_{i=1}^N |\frac{\partial \overline{v}}{\partial x_i}|^{q_i(x)-2} \frac{\partial \overline{v}}{\partial x_i}\frac{\partial \psi}{\partial x_i} \geq \int_\Omega b(x)\overline{v}^{\beta(x)-1}\psi + F_v(x,w,\overline{v})\psi & \text{for all } w \in [\underline{u},\overline{u}], \end{cases}$$

is verified for all nonnegative functions $\varphi \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$, $\psi \in W_0^{1,\overrightarrow{q(x)}}(\Omega)$, where $[u,v] := \{w : \Omega \to \mathbb{R}$ measurable; $u(x) \leq w(x) \leq v(x)$ a.e in $\Omega\}$ for $u,v \in \mathcal{S}(\Omega)$ with $u(x) \leq v(x)$ a.e in Ω.

In the next result, it is obtained appropriated sub-supersolutions for (S).

Lemma 3.1 *Suppose that de hypotheses* (H) *and* $(F_1)-(F_2)$ *are satisfied. Then, there exists* $\rho > 0$ *such that the problem* (S) *admits a sub-supersolution pairs* $(\underline{u},\underline{v}),(\overline{u},\overline{v}) \in (W_0^{1,\overrightarrow{p(x)}}(\Omega)\cap L^\infty(\Omega)) \times (W_0^{1,\overrightarrow{q(x)}}(\Omega)\cap L^\infty(\Omega))$, *satisfying* $\max\{\|\underline{u}\|_\infty,\|\underline{v}\|_\infty\} \leq \delta$ *with* δ *as described in* (F_1), *whenever* $\max\{\|a\|_\infty,\|b\|_\infty\} < \rho$.

Proof The lemmas 2.3 and 2.5 imply that there are unique nonnegative solutions $\underline{u},\overline{u} \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$ and $\underline{v},\overline{v} \in W_0^{1,\overrightarrow{q(x)}}(\Omega)$ such that

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial \underline{u}}{\partial x_i}|^{p_i(x)-2}\frac{\partial \underline{u}}{\partial x_i}) = a(x) & \text{in } \Omega, \\ \underline{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial \underline{v}}{\partial x_i}|^{q_i(x)-2}\frac{\partial \underline{v}}{\partial x_i}) = b(x) & \text{in } \Omega, \\ \underline{v} = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial \overline{u}}{\partial x_i}|^{p_i(x)-2}\frac{\partial \overline{u}}{\partial x_i}) = 1 + a(x) & \text{in } \Omega, \\ \overline{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial \overline{v}}{\partial x_i}|^{q_i(x)-2}\frac{\partial \overline{v}}{\partial x_i}) = 1 + b(x) & \text{in } \Omega, \\ \overline{v} = 0 & \text{on } \partial\Omega, \end{cases}$$

such that $\max\{\|\underline{u}\|_\infty,\|\underline{v}\|_\infty\} \leq K \max\{\|a\|_\infty^{\frac{1}{p_-^- -1}}, \|a\|_\infty^{\frac{1}{p_+^+ -1}}, \|b\|_\infty^{\frac{1}{q_-^- -1}}, \|b\|_\infty^{\frac{1}{q_+^+ -1}}\}$, with $K > 0$ being a constant that does not depend on a and b. Consider there is $\rho > 0$, depending only on K, such that $\max\{\|\underline{u}\|_\infty,\|\underline{v}\|_\infty\} \leq \delta/2$, when $\max\{\|a\|_\infty,\|b\|_\infty\} < \rho$.

From Lemma 2.4, we have $0 < \underline{u}(x) \leq \overline{u}(x)$, $0 < \underline{v}(x) \leq \overline{v}(x)$ a.e. in Ω.

Consider nonnegative functions $\varphi \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$ and $\psi \in W_0^{1,\overrightarrow{q(x)}}(\Omega)$. From the definition of $\underline{u}$ and $\overline{v}$ and the hypothesis (F_1), it follows that

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial \underline{u}}{\partial x_i}\frac{\partial \varphi}{\partial x_i} - \int_\Omega a(x)\underline{u}^{\alpha(x)-1}\varphi - \int_\Omega F_s(x,\underline{u},w)\varphi \\ &\quad \le \int_\Omega a(x)\varphi - \int_\Omega a(x)\underline{u}^{\alpha(x)-1}\varphi - \int_\Omega \left(1-\underline{u}^{\alpha(x)-1}\right)a(x)\varphi \\ &\quad = 0,\end{aligned}$$

for all $w \in [\underline{v},\overline{v}]$ and

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N \left|\frac{\partial \underline{v}}{\partial x_i}\right|^{q_i(x)-2} \frac{\partial \underline{v}}{\partial x_i}\frac{\partial \psi}{\partial x_i} - \int_\Omega b(x)\underline{v}^{\beta(x)-1}\psi - \int_\Omega F_t(x,w,\underline{v})\psi \\ &\quad \le \int_\Omega b(x)\psi - \int_\Omega b(x)\underline{v}^{\beta(x)-1}\psi - \int_\Omega \left(1-\underline{v}^{\beta(x)-1}\right)b(x)\varphi \\ &\quad = 0\end{aligned}$$

for all $w \in [\underline{u},\overline{u}]$. Using (F_2), we obtain that

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N \left|\frac{\partial \overline{u}}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial \overline{u}}{\partial x_i}\frac{\partial \varphi}{\partial x_i} - \int_\Omega a(x)\overline{u}^{\alpha(x)-1}\varphi - \int_\Omega F_s(x,\overline{u},w)\varphi \\ &\quad \ge \int_\Omega \left(1-C_1\|a\|_\infty\right)\varphi, \quad w \in [\underline{v},\overline{v}]\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N \left|\frac{\partial \overline{v}}{\partial x_i}\right|^{q_i(x)-2} \frac{\partial \overline{v}}{\partial x_i}\frac{\partial \psi}{\partial x_i} - \int_\Omega b(x)\overline{v}^{\beta(x)-1}\psi - \int_\Omega F_t(x,w,\overline{v})\psi \\ &\quad \ge \int_\Omega \left(1-C_2\|b\|_\infty\right)\psi, \quad w \in [\underline{u},\overline{u}],\end{aligned} \tag{3.3}$$

where

$$C_1 := \max\left\{\|\overline{u}\|_\infty^{\alpha^+-1}, \|\overline{u}\|_\infty^{\alpha^--1}\right\} + \max\left\{\|\overline{u}\|_\infty^{r^+-1}, \|\overline{u}\|_\infty^{r^--1}\right\} + \max\left\{\|\overline{v}\|_\infty^{r^+-1}, \|\overline{v}\|_\infty^{r^--1}\right\}$$

and

$$C_2 := \max\left\{\|\overline{v}\|_\infty^{\beta^+-1}, \|\overline{v}\|_\infty^{\beta^--1}\right\} + \max\left\{\|\overline{v}\|_\infty^{r^+-1}, \|\overline{v}\|_\infty^{r^--1}\right\} + \max\left\{\|\overline{u}\|_\infty^{r^+-1}, \|\overline{u}\|_\infty^{r^--1}\right\}.$$

Considering, if necessary, $\rho > 0$ smaller such that $\max\{C_1\|a\|_\infty, C_2\|b\|_\infty\} \le 1$, if $\max\{\|a\|_\infty, \|b\|_\infty\} < \rho$, it will follow that the right-hand sides in (3.2) and (3.3) are nonnegative, providing the result. □

Proof of Theorem 1.1 Consider the sub-supersolution pair

$$(\underline{u},\underline{v}), (\overline{u},\overline{v}) \in \left(W_0^{1,\overrightarrow{p(x)}}(\Omega)\cap L^\infty(\Omega)\right) \times \left(W_0^{1,\overrightarrow{q(x)}}(\Omega)\cap L^\infty(\Omega)\right),$$

provided in the proof of Lemma 3.1. Define the operators $T: W_0^{1,\overrightarrow{p(x)}}(\Omega) \to W_0^{1,\overrightarrow{p(x)}}(\Omega)$ and $S: W_0^{1,\overrightarrow{q(x)}}(\Omega) \to W_0^{1,\overrightarrow{q(x)}}(\Omega)$

$$Tu(x) := \begin{cases} \overline{u}(x), & \text{if } u(x) > \overline{u}(x), \\ u(x), & \text{if } \underline{u}(x) \le u(x) \le \overline{u}(x) , \\ \underline{u}(x), & \text{if } u(x) < \underline{u}(x), \end{cases} \qquad Sv(x) := \begin{cases} \overline{v}(x), & \text{if } v(x) > \overline{v}(x), \\ v(x), & \text{if } \underline{v}(x) \le v(x) \le \overline{v}(x) \\ \underline{v}(x), & \text{if } v(x) < \underline{v}(x), \end{cases}$$

and the auxiliary system

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{p_i(x)-2}\frac{\partial u}{\partial x_i}) = G_u(x,u,v) & \text{in } \Omega, \\ -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial v}{\partial x_i}|^{q_i(x)-2}\frac{\partial v}{\partial x_i}) = G_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{S'}$$

where

$$\begin{aligned} G_u(x,u,v) &:= a(x)\big(Tu(x)\big)^{\alpha(x)-1} + F_u\big(x, Tu(x), Sv(x)\big), \\ G_v(x,u,v) &:= b(x)\big(Tv(x)\big)^{\beta(x)-1} + F_v\big(x, Tu(x), Sv(x)\big). \end{aligned} \tag{3.4}$$

Consider $W := W_0^{1,\overrightarrow{p(x)}}(\Omega) \times W_0^{1,\overrightarrow{q(x)}}(\Omega)$ with the norm $\|(u,v)\| := \|u\|_{1,\overrightarrow{p(x)}} + \|v\|_{1,\overrightarrow{q(x)}}$, which is a Banach space. The solutions of (S') coincide with the critical points of the C^1 functional defined by

$$\begin{aligned} J(u,v) := {} & \int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + \int_\Omega \sum_{i=1}^N \frac{1}{q_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{q_i(x)} \\ & - \int_\Omega G(x,u,v), \quad (u,v) \in W, \end{aligned} \tag{3.5}$$

where $G(x,s,t) := \int_0^s G_\tau(x,\tau,t)\,d\tau + \int_0^t G_\tau(x,s,\tau)\,d\tau$. We have that J is a coercive and sequentially weakly lower semicontinuous. Consider the set

$$A := \big\{(u,v) \in W; \underline{u}(x) \le u(x) \le \overline{u}(x), \underline{v}(x) \le v(x) \le \overline{v}(x) \text{ a.e in } \Omega\big\},$$

which is closed and convex and hence weakly closed in W. Thus, it follows that $J|_A$ attains its infimum at some function $u_0 \in A$. Similar reasoning with respect to the proof of [26, Theorem 2.4] provides that $J'(u_0) = 0$, which proves the result. □

4 Proof of Theorem 1.2

Let $\underline{u} \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$ and $\underline{v} \in W_0^{1,\overrightarrow{q(x)}}(\Omega)$ be the function given in Lemma 3.1. Consider $\widetilde{T}: W_0^{1,p(x)}(\Omega) \to W_0^{1,p(x)}(\Omega)$ and $\widetilde{S}: W_0^{1,q(x)}(\Omega) \to W_0^{1,q(x)}(\Omega)$ defined by

$$\widetilde{T}u(x) := \begin{cases} u(x), & \text{if } \underline{u}(x) \le u(x), \\ \underline{u}(x), & \text{if } u(x) < \underline{u}(x), \end{cases} \qquad \widetilde{S}v(x) := \begin{cases} v(x), & \text{if } \underline{v}(x) \le v(x), \\ \underline{v}(x), & \text{if } v(x) < \underline{v}(x), \end{cases}$$

the functions $\widetilde{G}_u(x,u,v) := a(x)(\widetilde{T}u)^{\alpha(x)-1} + F_u(x,\widetilde{T}u,\widetilde{S}v)$, $\widetilde{G}_v(x,u,v) := b(x)(\widetilde{S}v)^{\beta(x)-1} + F_v(x,\widetilde{T}u,\widetilde{S}v)$, $u \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$, $v \in W_0^{1,\overrightarrow{q(x)}}(\Omega)$ and the problem

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{p_i(x)-2}\frac{\partial u}{\partial x_i}) = \widetilde{G}_u(x,u,v) & \text{in } \Omega, \\ -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial v}{\partial x_i}|^{q_i(x)-2}\frac{\partial v}{\partial x_i}) = \widetilde{G}_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\widetilde{S})$$

whose solutions are given by the critical points of the C^1 functional

$$L(u,v) := \int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + \int_\Omega \sum_{i=1}^N \frac{1}{q_i(x)}\left|\frac{\partial v}{\partial x_i}\right|^{q_i(x)} - \int_\Omega \widetilde{G}(x,u,v), \quad (u,v) \in W,$$

where W was defined in the proof of Theorem 1.1 and

$$\widetilde{G}(x,s,t) := \int_0^s \widetilde{G}_\tau(x,\tau,t)\,d\tau + \int_0^t \widetilde{G}_\tau(x,s,\tau)\,d\tau.$$

Lemma 4.1 *The Palais-Smale condition is satisfied by the functional L.*

Proof Consider $(u_n,v_n) \subset W$ a sequence such that $L'(u_n,v_n) \to 0$ and $L(u_n,v_n) \to c$ for some $c \in \mathbb{R}$. With respect to the first part of (i), note that (F_3) holds with $\overline{\theta},\overline{\xi} > 0$ such that $\max\{\frac{1}{\alpha^-},\theta\} < \overline{\theta} < \frac{1}{p_+^+}$ and $\max\{\frac{1}{\beta^-},\xi\} < \overline{\xi} < \frac{1}{q_+^+}$. Applying (H), (F_1)-(F_3), Propositions 2.1, embedding (2.4), the boundedness of the functions $\underline{u}$ and $\underline{v}$ and arguing as in the proof of [1, Theorem 36] (see also inequality 3.2 of [24]), we obtain that there are constants $C_i > 0$, $i = 1,\dots,4$ such that

$$\begin{aligned} C_1 + o_n(1)\|(u_n,v_n)\| &\geq L'(u_n,v_n)(u_n,v_n) - \overline{\theta}L'(u_n,v_n)(u_n,0) - \overline{\xi}L'(u_n,v_n)(0,v_n) \\ &\geq C_2\left(\|u_n\|_{1,\overrightarrow{p(x)}}^{p_-^-} + \|v_n\|_{1,\overrightarrow{q(x)}}^{q_-^-}\right) - C_3\|(u_n,v_n)\| \\ &\quad + \int_{\{u_n \geq \underline{u}\}} \left(\overline{\theta} - \frac{1}{\alpha(x)}\right) a(x) {u_n}^{\alpha(x)} \\ &\quad + \int_{\{v_n \geq \underline{v}\}} \left(\overline{\xi} - \frac{1}{\beta(x)}\right) b(x) {v_n}^{\beta(x)} \\ &\geq C_2\left(\|u_n\|_{1,\overrightarrow{p(x)}}^{p_-^-} + \|v_n\|_{1,\overrightarrow{q(x)}}^{q_-^-}\right) - C_4\|(u_n,v_n)\|, \end{aligned}$$

which provide that (u_n,v_n) is bounded in W.

With respect to the second case of (i), that is $\beta^+ < q_-^-$, we have constants $C_i > 0$, $i = 1,\dots,5$ with

$$\begin{aligned} C_1 + o_n(1)\|(u_n,v_n)\| &\geq L'(u_n,v_n)(u_n,v_n) - \overline{\theta}L'(u_n,v_n)(u_n,0) - \xi L'(u_n,v_n)(0,v_n) \\ &\geq C_2\left(\|u_n\|_{1,\overrightarrow{p(x)}}^{p_-^-} + \|v_n\|_{1,\overrightarrow{q(x)}}^{q_-^-}\right) - C_4\|(u_n,v_n)\| \\ &\quad - C_5 \max\left\{\|v_n\|_{\beta(x)}^{\beta^+}, \|v_n\|_{\beta(x)}^{\beta^-}\right\}, \end{aligned}$$

where $\overline{\theta} > 0$ was provided in the first part of the proof (i). Thus, the continuous embedding $W_0^{1,\overrightarrow{q(x)}}(\Omega) \hookrightarrow L^{\beta(x)}(\Omega)$, which is given by (2.4), implies that

$$C_1 + o_n(1)\big\|(u_n, v_n)\big\| + C_2\big\|(u_n, v_n)\big\| \geq C_3\big(\|u_n\|^{p_-^-}_{1,\overrightarrow{p(x)}} + \|v_n\|^{q_-^-}_{1,\overrightarrow{q(x)}}\big) - C_4 \max\big\{\|v_n\|^{\beta^+}_{1,\overrightarrow{q(x)}}, \|v_n\|^{\beta^-}_{1,\overrightarrow{q(x)}}\big\},$$

for constants $C_i > 0$, $i = 1, \ldots, 5$. Since $\beta^+ < q^-$, we obtain that the sequence (u_n, v_n) is bounded in W.

Thus, for a subsequence still denoted by (u_n, v_n), we obtain that

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,\overrightarrow{p(x)}}(\Omega), \\ u_n(x) \to u(x) & \text{a.e. in } \Omega, \\ u_n \to u & \text{in } L^{h(x)}(\Omega) \end{cases} \quad \text{and} \quad \begin{cases} v_n \rightharpoonup v & \text{in } W_0^{1,\overrightarrow{q(x)}}(\Omega), \\ v_n(x) \to v(x) & \text{a.e. in } \Omega, \\ v_n \to v & \text{in } L^{k(x)}(\Omega), \end{cases} \tag{4.1}$$

for all $h, k \in C(\overline{\Omega})$ with $1 < h^- \leq h^+ < (p^\star)^-$, $1 < k^- \leq k^+ < (q^\star)^-$ and some pair $(u, v) \in W$. From Lebesgue's Dominated Convergence Theorem and (4.1), it follows that

$$\int_\Omega \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2}\frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) \to 0,$$
$$\int_\Omega \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{q_i-2}\frac{\partial v_n}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{q_i-2}\frac{\partial v}{\partial x_i}\right)\left(\frac{\partial v_n}{\partial x_i} - \frac{\partial v}{\partial x_i}\right) \to 0.$$

Since $p_-^-, q_-^- \geq 2$, we have the result by the inequality (see, for instance, [27, page 97])

$$\big\langle |x|^{m-2}x - |y|^{m-2}y, x - y\big\rangle \geq \frac{1}{2^{m-2}}|x - y|^m \tag{4.2}$$

for all $x, y \in \mathbb{R}^N$ and $m \geq 2$, where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product in $\mathbb{R}^N$. □

The next result provides the Mountain Pass Geometry for the functional L.

Lemma 4.2 *If the hypotheses* (H), (F_1)*-*(F_3) *hold, then for* $\max\{\|a\|_\infty, \|b\|_\infty\}$ *small enough, the claims below are true.*

(i) *There are constants* $R, \sigma > 0$ *with* $R > \|(\underline{u}, \underline{v})\|$ *such that*

$$L(\underline{u}, \underline{v}) < 0 < \sigma \leq \inf_{(u,v) \in \partial B_R(0)} L(u, v).$$

(ii) *There is* $e \in W \setminus \overline{B_{2R}(0)}$ *such that* $L(e) < \sigma$.

Proof The inequalities $p_-^-, q_-^- > 1$ and (3.1) provide that $L(\underline{u}, \underline{v}) < 0$. Consider $(u, v) \in W$ with $\|(u, v)\| \geq 1$. From the embeddings $W_0^{1,\overrightarrow{p(x)}}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$, $W_0^{1,\overrightarrow{q(x)}}(\Omega) \hookrightarrow L^{\beta(x)}(\Omega)$ and Proposition 2.1, it follows that

$$L(u, v) \geq K_1\big\|(u, v)\big\|^\iota - K_2 - K_3\big\|(u, v)\big\| - \|a\|_\infty K_4\big(\max\big\{\|u\|^{\alpha^+}_{1,\overrightarrow{p(x)}}, \|u\|^{\alpha^-}_{1,\overrightarrow{p(x)}}\big\}$$

$$
\begin{aligned}
&+\max\left\{\|u\|^{r^+}_{1,\overrightarrow{p(x)}},\|u\|^{r^-}_{1,\overrightarrow{p(x)}}\right\}\Big)\\
&-\|b\|_\infty K_5\Big(\max\left\{\|v\|^{\beta^+}_{1,\overrightarrow{q(x)}},\|v\|^{\beta^-}_{1,\overrightarrow{q(x)}}\right\}\\
&+\max\left\{\|v\|^{r^+}_{1,\overrightarrow{q(x)}},\|v\|^{r^-}_{1,\overrightarrow{q(x)}}\right\}\Big),
\end{aligned}
$$

for positive constants $K_i > 0$, $i = 1,\dots,5$, where $\iota := \min\{p^-, q^-\}$. If necessary, decrease $\max\{\|a\|_\infty, \|b\|_\infty\}$ in a such way that $\|(\underline{u},\underline{v})\| < 1$, which is possible applying the functions $\varphi = \underline{u}$ and $\psi = \underline{v}$ in the inequality (3.1) and using Lemma 2.5. Fix $\sigma > 0$ and let $R > 1$ be a constant such that $K_1R^\iota - K_3R \geq 2\sigma$. Considering $\max\{\|a\|_\infty, \|a\|_\infty\}$ small enough such that $K_4\|a\|_\infty(R^{\alpha^+} + R^{r^+}) + K_5\|b\|_\infty(R^{\beta^+} + R^{r^+}) \leq \sigma$, it follows that $L(u,v) \geq \sigma$ for $(u,v) \in W$ with $\|(u,v)\| = R$, which provides (i).

With respect to (ii), note that the hypothesis (F_3) and the inequality $\frac{1}{\theta} > p_+^+$, provide constants $K_i > 0$, $i = 1,\dots,4$ and $t > 0$ large enough such that $L(t\underline{u},0) \leq C_1t^{p_+^+} - C_2t^{\alpha^-} - C_3t^{\frac{1}{\theta}} + C_4 < 0$ and $\|(t\underline{u},0)\| > 2R$. □

Proof of Theorem 1.2 Let $(\underline{u},\underline{v}), (\overline{u},\overline{v}) \in W$ be the pairs given in Lemma 3.1. Consider $(u_1,v_1) \in W$, the solution to the system (S) provided in Theorem 1.1, which minimizes the functional $J|_A$, where J was given in (3.5) and

$$
A = \left\{(u,v) \in W; \underline{u}(x) \leq u(x) \leq \overline{u}(x), \underline{v}(x) \leq v(x) \leq \overline{v}(x) \text{ a.e. in } \Omega\right\}.
$$

The Lemmas 4.1 and 4.2 provide that the hypotheses of the Mountain Pass Theorem [28, Theorem 2.1] are verified by the functional L. Therefore,

$$
c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} L\big(\gamma(t)\big), \quad \text{where } \Gamma := \left\{\gamma \in C\big([0,1],W\big); \gamma(0) = (\underline{u},\underline{v}), \gamma(1) = e\right\}
$$

is a critical value of L, i.e., $L'(u_2,v_2) = 0$ and $L(u_2,v_2) = c$, for some $(u_2,v_2) \in W$. From the definition of G_u and G_v provided in (3.4), we obtain that $J(u,v) = L(u,v)$ for $(u,v) \in \{(w,z) \in W; 0 \leq w(x) \leq \overline{u}(x), 0 \leq z(x) \leq \overline{v}(x) \text{ a.e in } \Omega\}$. Thus, it follows that $J(\underline{u},\underline{v}) = L(\underline{u},\underline{v})$ and $L(u_1,v_1) = J(u_1,v_1) = \inf_{(u,v)\in A} J(u,v)$. Recall that $L(\underline{u},\underline{v}) < 0$. Thus, if $u_2(x) \geq \underline{u}(x)$, $v_2(x) \geq \underline{v}(x)$ a.e. in Ω, then it follows that (S) has two weak solutions $(u_1,v_1), (u_2,v_2) \in W$ with $L(u_1,v_1) \leq L(\underline{u},\underline{v}) < 0 < \sigma \leq c = L(u_2,v_2)$, where $\sigma > 0$ given in Lemma 4.2.

We affirm that $u_2(x) \geq \underline{u}(x)$, $v_2(x) \geq \underline{v}(x)$ a.e. in Ω. In order to prove such inequality, consider the test functions $(\underline{u} - u_2)^+ \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$, $(\underline{v} - v_2)^+ \in W_0^{1,\overrightarrow{q(x)}}(\Omega)$ and $w \in [\underline{v},\overline{v}]$, $z \in [\underline{u},\overline{u}]$. It follows from $(\widetilde{S})$ and (3.1) that

$$
\begin{aligned}
&\int_\Omega \sum_{i=1}^N \left|\frac{\partial u_2}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u_2}{\partial x_i}\frac{\partial(\underline{u}-u_2)^+}{\partial x_i} + \int_\Omega \sum_{i=1}^N \left|\frac{\partial v_2}{\partial x_i}\right|^{q_i(x)-2} \frac{\partial v_2}{\partial x_i}\frac{\partial(\underline{v}-v_2)^+}{\partial x_i}\\
&\quad= \int_{\{u_2<\underline{u}\}} a(x)\underline{u}^{\alpha(x)-1} + F_u(x,u_2,w) + \int_{\{v_2<\underline{v}\}} b(x)\underline{v}^{\alpha(x)-1} + F_v(x,z,v_2),\\
&\int_\Omega \sum_{i=1}^N \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial \underline{u}}{\partial x_i}\frac{\partial(\underline{u}-u_2)^+}{\partial x_i} + \int_\Omega \sum_{i=1}^N \left|\frac{\partial \underline{v}}{\partial x_i}\right|^{q_i(x)-2} \frac{\partial \underline{v}}{\partial x_i}\frac{\partial(\underline{v}-v_2)^+}{\partial x_i}.
\end{aligned}
$$

Therefore,

$$\begin{aligned} &\int_{\{\underline{u}>u_2\}} \sum_{i=1}^{N} \left(\left| \frac{\partial \underline{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i} - \left| \frac{\partial u_2}{\partial x_i} \right|^{p_i-2} \frac{\partial u_2}{\partial x_i} \right) \left(\frac{\partial \underline{u}}{\partial x_i} - \frac{\partial u_2}{\partial x_i} \right) \leq 0, \\ &\int_{\{\underline{v}>v_2\}} \sum_{i=1}^{N} \left(\left| \frac{\partial \underline{v}}{\partial x_i} \right|^{p_i-2} \frac{\partial \underline{v}}{\partial x_i} - \left| \frac{\partial v_2}{\partial x_i} \right|^{p_i-2} \frac{\partial v_2}{\partial x_i} \right) \left(\frac{\partial \underline{v}}{\partial x_i} - \frac{\partial v_2}{\partial x_i} \right) \leq 0. \end{aligned} \tag{4.3}$$

From inequality (4.2) and (4.3), it follows that

$$\begin{aligned} &\int_{\Omega} \left| \frac{\partial}{\partial x_i} (\underline{u} - u_2)^+ \right|^{p_i(x)} = 0, \\ &\int_{\Omega} \left| \frac{\partial}{\partial x_i} (\underline{v} - v_2)^+ \right|^{q_i(x)} = 0, \end{aligned}$$

for $i = 1, \dots, N$, which provides that $\frac{\partial}{\partial x_i}(\underline{u} - u_2)^+(x) = \frac{\partial}{\partial x_i}(\underline{v} - v_2)^+(x) = 0$ a.e. in Ω. Thus, it follows from (2.4) that $(\underline{u} - u_2)^+(x) = (\underline{v} - v_2)^+(x) = 0$ a.e. in Ω, which proves the claim. □

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