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On the asymptotic behavior of solutions of anisotropic viscoelastic body

Yassine Letoufa[1,2], Hamid Benseridi[2], Salah Boulaaras[3,4*] and Mourad Dilmi[2]

This work is dedicated to Mr. Mahmoud ben Mouha Boulaaras and Ms. Fatma bent Tayeb Zeghdoud (1940–2021)—loving parents of the third author: Prof. SalAh Boulaaras

*Correspondence:
s.boularas@qu.edu.sa;
saleh_boulaares@yahoo.fr
[3]Department of Mathematics, College Of Sciences and Arts, ArRass, Qassim University, Buraydah, Saudi Arabia
[4]Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Oran, 31000, Algeria
Full list of author information is available at the end of the article

Abstract

The quasistatic problem of a viscoelastic body in a three-dimensional thin domain with Tresca's friction law is considered. The viscoelasticity coefficients and data for this system are assumed to vary with respect to the thickness ε. The asymptotic behavior of weak solution, when ε tends to zero, is proved, and the limit solution is identified in a new data system. We show that when the thin layer disappears, its traces form a new contact law between the rigid plane and the viscoelastic body. In which case, a generalized weak form equation is formulated, the uniqueness result for the limit problem is also proved.

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1 Introduction

Simulation and asymptotic behavior have been performed for many physical milieu that occupy thin layers of $\mathbb{R}^3$ in a lot of mechanical and mathematical papers in [1, 8, 9, 12], both for Newtonian and non-Newtonian fluids, including visco-plastic materials. Otherwise, the materials related to the linear theory of elasticity have been studied in research papers conducted by Benseridi et al. in [4–6, 16] and others in [2, 14, 16]. Here, the authors investigated several elasticity systems in thin layers in dimension 3 under various boundary conditions, through which they reached low-dimensional constitutive laws implicitly prevalent in many applications, particularly in lubrication problems, to describe the behavior of the phenomena that are already occurring in thin layers when the thickness is as small as 50 nm (e.g. [7]).

In this paper, we are interested in the asymptotic behavior of a quasistatic frictionless problem modeling the bilateral contact between a viscoelastic body and a rigid foundation in a thin layer represented by a domain Ω^ε in $\mathbb{R}^3$, where $(0 < \varepsilon < 1)$ is a small parameter that will tend to zero. Starting from the variational formulation giving the displacement and stress field, as formulated by Sofonea et al. in [10, 17], for such a material with a linear

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Kelvin–Voigt constitutive law of the form

$$\sigma^\varepsilon\left(u^\varepsilon\right) = \mathcal{A}^\varepsilon e\left(u^\varepsilon\right) + \mathcal{B}^\varepsilon e\left(\dot{u}^\varepsilon\right),$$

where $\mathcal{A}^\varepsilon$ is the viscosity tensor and $\mathcal{B}^\varepsilon$ is the elasticity tensor.

Our aim in this paper is to give an extension to our work in [13]. The novelty thing in this study is, firstly, that we take into account the heterogeneity and anisotropy of the milieu, a hypothesis that will cover a lot of materials in nature or industry, for example, wood, composite materials, and many biological materials, in which, although they appear to be homogeneous, their properties vary in all directions (see [6, 11]). Secondly, the body is assumed to have a viscous behavior, because the previously mentioned works are restricted only to the case of a homogeneous and isotropic elastic body by stress tensor $\sigma_{ij}^\varepsilon(u^\varepsilon) = 2\mu e_{ij}(u^\varepsilon) + \lambda \operatorname{Tr}(u^\varepsilon)\delta_{ij}$ in a thin domain. The boundary Γ^ε of Ω^ε is assumed to be composed of three portions, ω the bottom (in the $\mathbb{R}^2$ surface) of the body, and this is where our main interest lies, Γ_1^ε the upper surface, and Γ_L^ε the lateral surface. We consider Tresca-free boundary friction conditions on ω and Dirichlet boundary conditions on Γ_1^ε. However, we consider a traction boundary condition on Γ_L^ε.

The problem is converted into a one over a fixed domain explicitly written depending on the ε in the variational formulation. The model of the limit problem, when ε tends to zero, is then obtained. This study yields a new constitutive law, which takes into account the hypotheses of heterogeneity, anisotropy, and viscosity for the body. Furthermore, the effects of applied stress tensors on the ω boundary are determined, which will be subject to Tresca friction conditions. These models are very common in engineering literature, for example, see [1, 7, 8, 14] and the references cited therein.

The paper is organized as follows. In Sect. 2, the strong and weak formulation of the problem is given in terms of u^ε, and also the necessary assumptions which will be needed in the sequel are presented. In Sect. 3, we introduce a scaling, we find some estimates on the displacement and velocity which are independent of the parameter ε, the existence of a weak limit solution u^* is obtained. Finally, the new formula of our original problem is stated, the corresponding results for u^* with a specific weak form of the Reynolds equation are given in the last section.

2 Basic equation and weak formulation

Let ω be an bounded regular domain in the $x_1 O x_2$-plane, and let $h \in C^1(\omega)$ be a positive smooth function such that $0 < h_{\min} < h(x') < h_{\max}$ for all $x' = (x_1, x_2) \in \omega$. Let ε be a parameter taking values in a sequence of positive numbers converging to zero. Consider a viscoelastic body occupying the region Ω^ε,

$$\Omega^\varepsilon = \left\{\left(x', x_3\right) \in \mathbb{R}^3, \left(x', 0\right) \in \omega, 0 < x_3 < \varepsilon h\left(x'\right)\right\},$$

its boundary is $\Gamma^\varepsilon = \bar{\omega} \cup \bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_L^\varepsilon$, where Γ_1^ε is the upper surface defined by $x_3 = \varepsilon h(x')$, Γ_L^ε is the lateral boundary, and ω is the bottom of the domain.

Let $T > 0$. We denote ν to be the unit outward normal to Γ^ε. The body is assumed to be clamped on $\Gamma_1^\varepsilon \times (0, T)$ and surface tractions of density g^ε act on $\Gamma_L^\varepsilon \times (0, T)$.

On the bottom, the normal velocity is equal to zero but the tangential velocity is unknown and satisfies Tresca type fluid-solid boundary conditions with friction coefficient

k^ε, a given positive function. Moreover, a volume force of density $f^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon)$ acts on the body in $\Omega^\varepsilon \times (0, T)$. We denote by $\mathbb{R}_s^{n \times n}$ $(n = 2, 3)$ the space of symmetric tensors, and "·" is the scalar product in $\mathbb{R}^{n \times n}$ or $\mathbb{R}^n$, and $|\cdot|$ is the associated norm.

The classical formulation of the mechanical problem is as follows:

Find a displacement field $u^\varepsilon = (u_i^\varepsilon) : \Omega^\varepsilon \times [0, T] \longrightarrow \mathbb{R}^3$ and a stress field $\sigma^\varepsilon = (\sigma_{ij}^\varepsilon) : \Omega^\varepsilon \times [0, T] \longrightarrow \mathbb{R}_s^{3 \times 3}$, $i, j = 1, 2, 3$, such that

$$\sigma_{ij}^\varepsilon(u^\varepsilon) = A_{ijkl}^\varepsilon e_{kl}(u^\varepsilon) + B_{ijkl}^\varepsilon e_{kl}(\dot{u}^\varepsilon) \quad \text{in } \Omega^\varepsilon \times (0, T), \tag{2.1}$$

$$\frac{\partial \sigma_{ij}^\varepsilon}{\partial x_j} + f_i^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon \times (0, T), \tag{2.2}$$

$$u^\varepsilon = 0 \quad \text{on } \Gamma_1^\varepsilon \times (0, T), \tag{2.3}$$

$$\sigma \nu^\varepsilon = g^\varepsilon \quad \text{on } \Gamma_L^\varepsilon \times (0, T), \tag{2.4}$$

$$u_\nu^\varepsilon = 0 \quad \text{on } \omega \times (0, T), \tag{2.5}$$

$$\left. \begin{array}{l} |\sigma_\tau^\varepsilon| < k^\varepsilon \implies \dot{u}_\tau^\varepsilon = 0 \\ |\sigma_\tau^\varepsilon| = k^\varepsilon \implies \exists \lambda \geq 0 \text{ such that } \dot{u}_\tau^\varepsilon = -\lambda \sigma_\tau^\varepsilon \end{array} \right\} \quad \text{on } \omega \times (0, T), \tag{2.6}$$

$$u^\varepsilon(0) = u_0^\varepsilon \quad \text{in } \Omega^\varepsilon. \tag{2.7}$$

Here, equation (2.1) represents the linear constitutive law of Kelvin–Voigt [11, 17], where σ_{ij}^ε, A_{ijkl}^ε, and B_{ijkl}^ε denote the components of the stress tensor σ^ε, the elasticity tensor $\mathcal{A}^\varepsilon$, and the viscosity tensor $\mathcal{B}^\varepsilon$, respectively. The dot above represents the time derivative. Relations (2.2) are the quasi-static equations of motion, the indexes i, j, k, l run between 1 and 3, and the summation convention over repeated indexes is used. We denote by $e(\mathbf{v})$ the rate of deformation operator defined by

$$e(\mathbf{v}) = \left(e_{ij}(\mathbf{v}) \right), \quad e_{ij}(\mathbf{v}) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

We denote by $\sigma \nu^\varepsilon$ the Cauchy stress vector [11], and by

$$\sigma_\nu^\varepsilon = \sigma \nu^\varepsilon, \qquad \sigma_\tau^\varepsilon = \sigma \nu^\varepsilon - \left(\sigma_\nu^\varepsilon \right) \cdot \nu, \qquad u_\nu^\varepsilon = u^\varepsilon \cdot \nu, \qquad u_\tau^\varepsilon = u^\varepsilon - u_\nu^\varepsilon \nu,$$

respectively, the components of the normal, the tangential stress tensor on Γ^ε, the normal and the tangential of u^ε on Γ^ε, and finally, u_0^ε is the initial displacement.

To establish a weak formulation of the problem, we use the spaces $L^2(\Omega^\varepsilon)^3$, $H^1(\Omega^\varepsilon)^3$, and $L^2(\Omega^\varepsilon)_s^{3 \times 3}$. The inner products on the spaces $L^2(\Omega^\varepsilon)^3$ and $L^2(\Omega^\varepsilon)_s^{3 \times 3}$ are designed equally by $\langle \cdot, \cdot \rangle_{0, \Omega^\varepsilon}$, and let $\| \cdot \|_{0, \Omega^\varepsilon}$ be the associated norm, we denote by $\| \cdot \|_{1, \Omega^\varepsilon}$ the associated norm of the space $H^1(\Omega^\varepsilon)^3$. We define

$$V^\varepsilon = \left\{ \mathbf{v}^\varepsilon \in H^1\left(\Omega^\varepsilon \right)^3 : \mathbf{v}^\varepsilon = 0 \text{ on } \Gamma_1^\varepsilon, \mathbf{v}_\nu^\varepsilon = 0 \text{ on } \omega \right\}.$$

Following [10], V^ε is a real Hilbert space endowed with the inner product

$$\langle u, v \rangle_{V^\varepsilon} = \left\langle e(u), e(v) \right\rangle_{0, \Omega^\varepsilon}, \qquad \|v\|_{V^\varepsilon} = \langle v, v \rangle_{V^\varepsilon}^{1/2}.$$

From [17], see page 97, we use the real Banach space Q_∞^ε of fourth order tensor fields defined by

$$Q_\infty^\varepsilon = \left\{ C^\varepsilon = \left(C_{ijkl}^\varepsilon \right) / C_{ijkl}^\varepsilon = C_{jikl}^\varepsilon = C_{klij}^\varepsilon \in L^\infty\left(\Omega^\varepsilon \right), 1 \le i,j,k,l \le 3 \right\}$$

with the norm

$$\left\| C^\varepsilon \right\|_{Q_\infty^\varepsilon} = \max_{1 \le i,j,k,l \le 3} \left\| C_{ijkl}^\varepsilon \right\|_{L^\infty(\Omega^\varepsilon)}.$$

For every $C^\varepsilon \in Q_\infty^\varepsilon$ and $\xi \in L^2(\Omega^\varepsilon)_s^{3\times 3}$, the tensor $C^\varepsilon \xi$, of components $(C^\varepsilon \xi)_{ij} = \sum_{kl} C_{ijkl}^\varepsilon \xi_{kl}$ satisfies

$$C^\varepsilon \xi \in L^2\left(\Omega^\varepsilon \right)_s^{3\times 3}, \qquad \left\| C^\varepsilon \xi \right\|_{0,\Omega^\varepsilon} \le 3 \left\| C^\varepsilon \right\|_{Q_\infty^\varepsilon} \|\xi\|_{0,\Omega^\varepsilon}. \tag{2.8}$$

For every real Banach space H, we use the classical notation for the spaces $L^p(0,T;H)$ and $W^{1,p}(0,T;H)$, $1 \le p \le +\infty$.

Now, we list the assumptions imposed:

$$\begin{cases} A^\varepsilon \in Q_\infty^\varepsilon, \qquad B^\varepsilon \in Q_\infty^\varepsilon, \\ \exists m_a > 0, \forall \xi \in \mathbb{R}_s^{3\times 3}, \quad A^\varepsilon \xi \cdot \xi \ge m_a |\xi|^2 \quad \text{a.e. in } \Omega^\varepsilon, \\ \exists m_b > 0, \forall \xi \in \mathbb{R}_s^{3\times 3}, \quad B^\varepsilon \xi \cdot \xi \ge m_b |\xi|^2 \quad \text{a.e. in } \Omega^\varepsilon, \end{cases} \tag{2.9}$$

and

$$f^\varepsilon \in W^{1,2}\left(0,T;L^2\left(\Omega^\varepsilon \right)^3 \right), \qquad g^\varepsilon \in W^{1,2}\left(0,T;L^2\left(\Gamma_L^\varepsilon \right)^3 \right), \tag{2.10}$$

$$k^\varepsilon \in L^\infty(\omega), \quad k^\varepsilon(x) \ge 0 \text{ a.e. on } \omega, \tag{2.11}$$

moreover,

$$u_0^\varepsilon \in V^\varepsilon. \tag{2.12}$$

A formal application of Green's formula, using (2.1)–(2.8) leads to the weak formulation [10]:

Find $u^\varepsilon(t)$ in V^ε such that

$$\left\langle A^\varepsilon e\left(u^\varepsilon(t) \right), e\left(\mathbf{v}^\varepsilon - \dot{u}^\varepsilon(t) \right) \right\rangle_{0,\Omega^\varepsilon} + \left\langle B^\varepsilon e\left(\dot{u}^\varepsilon(t) \right), e\left(\mathbf{v}^\varepsilon - \dot{u}^\varepsilon(t) \right) \right\rangle_{0,\Omega^\varepsilon}$$

$$+ \int_\omega k^\varepsilon \left| \mathbf{v}^\varepsilon \right| dx' - \int_\omega k^\varepsilon \left| \dot{u}^\varepsilon(t) \right| dx'$$

$$\ge \int_{\Gamma_L^\varepsilon} g^\varepsilon(t).\left(\mathbf{v}^\varepsilon - \dot{u}^\varepsilon(t) \right) d\rho + \left\langle f^\varepsilon(t), \mathbf{v}^\varepsilon - \dot{u}^\varepsilon(t) \right\rangle_{0,\Omega^\varepsilon}$$

$$\forall \mathbf{v}^\varepsilon \in V^\varepsilon, \forall t \in [0,T], \tag{2.13}$$

$$u^\varepsilon(0) = u_0^\varepsilon, \tag{2.14}$$

where $d\rho$ represents the superficial measure on the lateral boundary Γ_L^ε.

Remark 2.1 If the assumptions of (2.9)–(2.12) hold, for any ε small fixed, then problem (2.13)–(2.14) admits a unique solution u^ε that satisfies (see [3, 10, 15, 19])

$$u^\varepsilon \in W^{1,2}(0, T; V^\varepsilon).$$

The aim of the next section is to describe the limit behavior of the displacement as ε tends to zero.

3 The rescaled problem and asymptotic behavior

To simplify the notation, everywhere in the sequel the indexes α, β, γ, and δ take values in the set $\{1, 2\}$. Moreover, $x = (x', z)$ denotes the generic point in $\mathbb{R}^3$. According to the change of variables $z = x_3/\varepsilon$, see e.g. [1, 8], we define the fixed domain Ω:

$$\Omega = \left\{(x', z) \in \mathbb{R}^3, (x', 0) \in \omega, 0 < z < h(x')\right\},$$

and we denote by $\Gamma = \bar{\omega} \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_L$ its boundary with

$$\Gamma_L = \left\{(x', z) \in \mathbb{R}^3, x' \in \partial\omega, 0 < z < h(x')\right\},$$
$$\Gamma_1 = \left\{(x', z) \in \mathbb{R}^3, (x', 0) \in \omega, z = h(x')\right\}.$$

We define the following functions in $\Omega \times [0, T]$:

$$\hat{u}_\alpha^\varepsilon(x', z, t) = u_\alpha^\varepsilon(x', x_3, t), \qquad \hat{u}_3^\varepsilon(x', z, t) = \varepsilon^{-1} u_3^\varepsilon(x', x_3, t).$$

Let us define the ε-independent tensors $\hat{A} = (\hat{A}_{ijkl})$, $\hat{B} = (\hat{B}_{ijkl})$:

$$\hat{A}(x', z) = A^\varepsilon(x', x_3), \qquad \hat{B}(x', z) = B^\varepsilon(x', x_3),$$

the vectors $\hat{f} = (\hat{f}_i)$, $\hat{g} = (\hat{g}_i)$, $\hat{u}_0 = (\hat{u}_{0i})$, and $\hat{k}$ assume the following dependence of the data:

$$\hat{f}(x', z, t) = \varepsilon^2 f^\varepsilon(x', x_3, t), \qquad \hat{g}(x', z, t) = \varepsilon^2 g^\varepsilon(x', x_3, t),$$
$$\hat{u}_{0\alpha}(x', z) = u_{0\alpha}^\varepsilon(x', x_3), \qquad \hat{u}_{03}(x', z) = \varepsilon^{-1} u_{03}^\varepsilon(x', x_3), \qquad \hat{k} = \varepsilon k^\varepsilon.$$

For this rescaling, we have the following function spaces:

$$Q_\infty = \left\{C = (C_{ijkl})/C_{ijkl} = C_{jikl} = C_{klij} \in L^\infty(\Omega), 1 \le i, j, k, l \le 3\right\},$$
$$V = \left\{\mathbf{v} \in H^1(\Omega)^3 : \mathbf{v} = 0 \text{ on } \Gamma_1, \mathbf{v}_\nu = 0 \text{ on } \omega\right\},$$
$$\Pi(V) = \left\{v = (v_1, v_2) \in H^1(\Omega)^2 : \mathbf{v} = (v_1, v_2, v_3) \in V\right\},$$

and

$$V_z = \left\{v = (v_1, v_2) \in L^2(\Omega)^2 : \frac{\partial v_\alpha}{\partial z} \in L^2(\Omega), v = 0 \text{ on } \Gamma_1\right\}.$$

V_z is a Banach space equipped with the norm

$$\|v\|_{V_z} = \left[\sum_{\alpha=1}^{2} \left(\|v_\alpha\|_{0,\Omega}^2 + \left\| \frac{\partial v_\alpha}{\partial z} \right\|_{0,\Omega}^2 \right) \right]^{1/2}.$$

Consider now the following functional **J** and the three-linear form $\Lambda \colon Q_\infty \times V \times V \longrightarrow \mathbb{R}$ given by

$$\mathbf{J}(\mathbf{v}) = \int_\omega \hat{k} |\mathbf{v}_\tau| \, dx', \quad \forall \mathbf{v} \in V;$$

$$\Lambda(\hat{C}, \hat{u}, \mathbf{v})$$

$$= \frac{1}{2} \int_\Omega \hat{C}_{\alpha\beta\gamma\theta} \varepsilon^2 \left(\frac{\partial \hat{u}_\gamma}{\partial x_\theta} + \frac{\partial \hat{u}_\theta}{\partial x_\gamma} \right) \frac{\partial v_\alpha}{\partial x_\beta} \, dx$$

$$+ \int_\Omega \hat{C}_{\alpha3\gamma\theta} \varepsilon \left(\frac{\partial \hat{u}_\gamma}{\partial x_\theta} + \frac{\partial \hat{u}_\theta}{\partial x_\gamma} \right) \frac{\partial v_\alpha}{\partial z} \, dx + \int_\Omega \hat{C}_{\alpha\beta\gamma3} \left(\varepsilon \frac{\partial \hat{u}_\gamma}{\partial z} + \varepsilon^3 \frac{\partial \hat{u}_3}{\partial x_\gamma} \right) \frac{\partial v_\alpha}{\partial x_\beta} \, dx$$

$$+ 2 \int_\Omega \hat{C}_{\alpha3\gamma3} \left(\frac{\partial \hat{u}_\gamma}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3}{\partial x_\gamma} \right) \frac{\partial v_\alpha}{\partial z} \, dx + \int_\Omega \hat{C}_{\alpha\beta33} \varepsilon^2 \frac{\partial \hat{u}_3}{\partial z} \frac{\partial v_\alpha}{\partial x_\beta} \, dx$$

$$+ \frac{1}{2} \int_\Omega \hat{C}_{33\alpha\beta} \varepsilon^2 \left(\frac{\partial \hat{u}_\alpha}{\partial x_\beta} + \frac{\partial \hat{u}_\beta}{\partial x_\alpha} \right) \frac{\partial v_3}{\partial z} \, dx + 2 \int_\Omega \hat{C}_{\alpha333} \varepsilon \frac{\partial \hat{u}_3}{\partial z} \frac{\partial v_\alpha}{\partial z} \, dx$$

$$+ \int_\Omega \hat{C}_{33\alpha3} \left(\varepsilon \frac{\partial \hat{u}_\alpha}{\partial z} + \varepsilon^3 \frac{\partial \hat{u}_3}{\partial x_\alpha} \right) \frac{\partial v_3}{\partial z} \, dx + \int_\Omega \hat{C}_{3333} \varepsilon^2 \frac{\partial \hat{u}_3}{\partial z} \frac{\partial v_3}{\partial z} \, dx,$$

$$\forall (\hat{C}, \hat{u}, \mathbf{v}) \in Q_\infty \times V \times V.$$

Problem (2.13)–(2.14) leads to the form in the following lemma.

Lemma 3.1 *The variational inequality* (2.13)–(2.14) *is equivalent to the following inequality*:

$$\Lambda\big(\hat{A}, \hat{u}^\varepsilon(t), \mathbf{v} - \partial_t \hat{u}^\varepsilon(t)\big) + \Lambda\big(\hat{B}, \partial_t \hat{u}^\varepsilon(t), \mathbf{v} - \partial_t \hat{u}^\varepsilon(t)\big) + \mathbf{J}(\mathbf{v}) - \mathbf{J}\big(\partial_t \hat{u}^\varepsilon(t)\big)$$

$$\geq \sum_{\alpha=1}^{2} \int_{\Gamma_L} \hat{g}_\alpha(t)\big(v_\alpha - \partial_t \hat{u}_\alpha^\varepsilon(t)\big) \, d\rho + \varepsilon \int_{\Gamma_L} \hat{g}_3(t)\big(v_3 - \partial_t \hat{u}_3^\varepsilon(t)\big) \, d\rho$$

$$+ \sum_{\alpha=1}^{2} \int_\Omega \hat{f}_\alpha(t)\big(v_\alpha - \partial_t \hat{u}_\alpha^\varepsilon(t)\big) \, dx + \varepsilon \int_\Omega \hat{f}_3(t)\big(v_3 - \partial_t \hat{u}_3^\varepsilon(t)\big) \, dx,$$

$$\forall \mathbf{v} \in V, \forall t \in [0, T], \tag{3.1}$$

$$\hat{u}^\varepsilon(0) = \hat{u}_0. \tag{3.2}$$

Proof Let u^ε be a solution of (2.13)–(2.14). For any $C^\varepsilon \in Q^\varepsilon$, using the symmetry of $(C^\varepsilon e(u^\varepsilon))_{ij}$, it follows that, for all $\mathbf{v}^\varepsilon$ in V^ε and $t \in [0, T]$, we have

$$\big\langle C^\varepsilon e\big(u^\varepsilon(t)\big), e\big(\mathbf{v}^\varepsilon\big)\big\rangle_{0,\Omega^\varepsilon} = \int_{\Omega^\varepsilon} C_{ijkl}^\varepsilon e_{kl}\big(u^\varepsilon(t)\big) e_{ij}\big(\mathbf{v}^\varepsilon\big) \, dx' \, dx_3$$

$$= \int_{\Omega^\varepsilon} C_{ijkl}^\varepsilon e_{kl}\big(u^\varepsilon(t)\big) \frac{\partial v_i^\varepsilon}{\partial x_j} \, dx' \, dx_3,$$

by passing to the fixed domain Ω, we find

$$\int_{\Omega^\varepsilon} \hat{C}_{ijkl} e_{kl}\big(u^\varepsilon(t)\big) \frac{\partial v_i}{\partial x_j} \, dx' \, dx_3 = \varepsilon^{-1} \Lambda\big(\hat{C}, \hat{u}^\varepsilon(t), \mathbf{v}\big)$$

with $\mathbf{v} \in V$, the rescaling of the function $\mathbf{v}^\varepsilon$.

Since the boundary Γ^ε and $\partial\omega$ are Lipschitz continuous, we can write the Γ_L^ε boundary into a union disjoint from $\Gamma_{L_i}^\varepsilon$, $i \in I$, such that each $\Gamma_{L_i}^\varepsilon$ is the graph of a function $\phi^{(i)}$, its points from the domain

$$D_i^\varepsilon = \big\{ (x_{\delta_i}, x_3) \in \mathbb{R}^2, a_i < x_{\delta_i} < b_i, 0 < x_3 < \varepsilon h\big(x'\big), x' \in \partial\omega \cap \bar{\Gamma}_{L_i}^\varepsilon \big\},$$

which are given by theorem of implicit function, with $\partial_3 \phi_i = 0$, a_i and b_i are real numbers, the indexes δ_i take the value 1 or 2. Putting

$$\hat{\phi}^{(i)}(x_{\delta_i}, z) = \phi^{(i)}(x_{\delta_i}, x_3),$$
$$D_i = \big\{ (x_{\delta_i}, z) \in \mathbb{R}^2, a_i < x_{\delta_i} < b_i, 0 < z < h\big(x'\big), x' \in \partial\omega \cap \bar{\Gamma}_{L_i} \big\}.$$

We do calculations on the lateral surface Γ_L^ε to get

$$\int_{\Gamma_L^\varepsilon} g^\varepsilon . \mathbf{v}^\varepsilon \, d\rho = \sum_{i \in I} \int_{D_i^\varepsilon} \big[\big(g^\varepsilon \circ \phi^{(i)}\big) . \big(\mathbf{v}^\varepsilon \circ \phi^{(i)}\big) \big](x_{\delta_i}, x_3) \sqrt{1 + |\partial_{\delta_i} \phi_i|^2} \, dx_{\delta_i} \, dx_3$$

$$= \varepsilon^{-1} \sum_{\alpha=1}^2 \sum_{i \in I} \int_{D_i} \big[\big(\hat{g}_\alpha \circ \hat{\phi}^{(i)}\big) . \big(v_\alpha \circ \hat{\phi}^{(i)}\big) \big](x_{\delta_i}, z) \sqrt{1 + \big|\partial_{\delta_i} \hat{\phi}^{(i)}\big|^2} \, dx_{\delta_i} \, dz$$

$$+ \sum_{i \in I} \int_{D_i} \big[\big(\hat{g}_3 \circ \hat{\phi}^{(i)}\big) . \big(v_3 \circ \hat{\phi}^{(i)}\big) \big](x_{\delta_i}, z) \sqrt{1 + \big|\partial_{\delta_i} \hat{\phi}^{(i)}\big|^2} \, dx_{\delta_i} \, dz$$

$$= \varepsilon^{-1} \sum_{\alpha=1}^2 \int_{\Gamma_L} \hat{g}_\alpha v_\alpha \, d\rho + \int_{\Gamma_L} \hat{g}_3 v_3.$$

Thus, by the previous relations, we easily obtain the equivalence between problem (3.1)–(3.2) and (2.13)–(2.14). □

Lemma 3.2 *Assume that u^ε is a solution of (2.13)–(2.14). Then*

$$\sum_{\alpha=1}^2 \left\| \frac{\partial \hat{u}_\alpha^\varepsilon}{\partial z} \right\|_{L^\infty(0,T;L^2(\Omega))}^2 + \varepsilon^2 \left\| \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right\|_{L^\infty(0,T;L^2(\Omega))}^2$$

$$+ \varepsilon^2 \sum_{\alpha,\beta=1}^2 \left\| \frac{\partial \hat{u}_\alpha^\varepsilon}{\partial x_\beta} \right\|_{L^\infty(0,T;L^2(\Omega))}^2 + \varepsilon^4 \sum_{\beta=1}^2 \left\| \frac{\partial \hat{u}_3^\varepsilon}{\partial x_\beta} \right\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq C; \tag{3.3}$$

$$\sum_{\alpha=1}^2 \left\| \frac{\partial^2 \hat{u}_\alpha^\varepsilon}{\partial z \partial t} \right\|_{L^2(0,T;L^2(\Omega))}^2 + \varepsilon^2 \left\| \frac{\partial^2 \hat{u}_3^\varepsilon}{\partial z \partial t} \right\|_{L^2(0,T;L^2(\Omega))}^2$$

$$+ \varepsilon^2 \sum_{\alpha,\beta=1}^2 \left\| \frac{\partial^2 \hat{u}_\alpha^\varepsilon}{\partial x_\beta \partial t} \right\|_{L^2(0,T;L^2(\Omega))}^2 + \varepsilon^4 \sum_{\alpha=1}^2 \left\| \frac{\partial^2 \hat{u}_3^\varepsilon}{\partial x_\beta \partial t} \right\|_{L^2(0,T;L^2(\Omega))}^2 \leq C, \tag{3.4}$$

where C denotes an independent constant of ε and t.

Proof First, we recall the Korn and Poincaré inequalities (see [1]), respectively,

$$\left\| e(v) \right\|_{0,\Gamma^\varepsilon} \ge C_K \| \nabla v \|_{0,\Omega^\varepsilon}, \quad \forall v \in V^\varepsilon, \tag{3.5}$$

$$\| v \|_{0,\Omega^\varepsilon} \le \varepsilon h_{\max} \| \nabla v \|_{0,\Omega^\varepsilon}, \quad \forall v \in V^\varepsilon, \tag{3.6}$$

where $C_K > 0$ is a constant independent of ε. By the Sobolev trace theorem and the above inequality, there exists $C_0 > 0$ depending only on Ω^ε and Γ^ε such that

$$\| v \|_{0,\Gamma_L^\varepsilon} \le C_0 \| \nabla v \|_{0,\Omega^\varepsilon}, \quad \forall v \in V^\varepsilon. \tag{3.7}$$

However, we can see that the constant C_0 does not depend on ε. To this end, for any function v, we denote its extension from Ω^ε to $\tilde{\Omega} = \omega \times]0, \tilde{h}[$ by

$$\bar{v} = \begin{cases} v & \text{on } \Omega^\varepsilon, \\ 0 & \text{on } \tilde{\Omega} - \Omega^\varepsilon, \end{cases}$$

where $\tilde{h} > h_{\max}$. We denote by $\tilde{\Gamma}_L$ the lateral boundary of $\tilde{\Omega}$, so we have $\| v \|_{0,\Gamma_L^\varepsilon} = \| \bar{v} \|_{0,\tilde{\Gamma}_L}$, $\| \nabla v \|_{0,\Omega^\varepsilon} = \| \nabla \bar{v} \|_{0,\tilde{\Omega}}$ such that (3.7) is valid for a constant independent of ε. Next, for any t in $[0, T]$, choosing in (2.13) $\mathbf{v}^\varepsilon = 0$ and integrating over $[0, t]$, it follows from (2.9) and (3.5) that

$$\frac{1}{2} m_a C_K \left\| \nabla u^\varepsilon(t) \right\|_{0,\Omega^\varepsilon}^2 + \frac{1}{2} m_b C_K \int_0^t \left\| \nabla \dot{u}^\varepsilon(s) \right\|_{0,\Omega^\varepsilon}^2 ds$$

$$\le \frac{1}{2} \langle A^\varepsilon e(u_0^\varepsilon), e(u_0^\varepsilon) \rangle_{0,\Omega^\varepsilon} + \int_0^t \langle f^\varepsilon(s), \dot{u}^\varepsilon(s) \rangle_{0,\Omega^\varepsilon} ds + \int_0^t \int_{\Gamma_L^\varepsilon} g^\varepsilon(s) . \dot{u}^\varepsilon(s) \, d\rho \, ds. \tag{3.8}$$

By the Cauchy–Schwarz inequality and (3.6)–(3.7), we have

$$\langle f^\varepsilon(s), \dot{u}^\varepsilon(s) \rangle_{0,\Omega^\varepsilon} \le \varepsilon h_{\max} \left\| f^\varepsilon(s) \right\|_{0,\Omega^\varepsilon} \left\| \nabla \dot{u}^\varepsilon(s) \right\|_{0,\Omega^\varepsilon},$$

$$\int_{\Gamma_L^\varepsilon} g^\varepsilon . \dot{u}^\varepsilon(s) \, d\rho \le C_0 \left\| g^\varepsilon(s) \right\|_{0,\Gamma_L^\varepsilon} \left\| \nabla \dot{u}^\varepsilon(s) \right\|_{0,\Omega^\varepsilon}.$$

Using now the Young inequality

$$ab \le \eta^{-2} \frac{a^2}{2} + \eta^2 \frac{b^2}{2}$$

for $\eta = \sqrt{m_b C_K / 2}$, $a = \varepsilon h_{\max} \| f^\varepsilon(s) \|_{0,\Omega^\varepsilon}$ and $b = \| \nabla \dot{u}^\varepsilon(s) \|_{0,\Omega^\varepsilon}$; then $\eta = \sqrt{m_b C_K / 2}$, $a = C_0 \| g^\varepsilon(s) \|_{0,\Gamma_L^\varepsilon}$ and $b = \| \nabla \dot{u}^\varepsilon(s) \|_{0,\Omega^\varepsilon}$, we deduce

$$\langle f^\varepsilon(s), \dot{u}^\varepsilon(s) \rangle_{0,\Omega^\varepsilon} \le \frac{(h_{\max})^2}{m_b C_K} \varepsilon^2 \left\| f^\varepsilon(s) \right\|_{0,\Omega^\varepsilon}^2 + \frac{m_b C_K}{4} \left\| \nabla \dot{u}^\varepsilon(s) \right\|_{0,\Omega^\varepsilon}^2,$$

$$\int_{\Gamma_L^\varepsilon} g^\varepsilon(s) . \dot{u}^\varepsilon(s) \, d\rho \le \frac{(C_0)^2}{m_b C_K} \left\| g^\varepsilon(s) \right\|_{0,\Gamma_L^\varepsilon}^2 + \frac{m_b C_K}{4} \left\| \nabla \dot{u}^\varepsilon(s) \right\|_{0,\Omega^\varepsilon}^2.$$

By two previous inequalities and (2.8), inequality (3.8) after multiplying by ε becomes

$$
\frac{1}{2} m_a C_K \varepsilon \left\| \nabla u^\varepsilon(t) \right\|_{0,\Omega^\varepsilon}^2 + \frac{1}{2} m_b C_K \int_0^t \varepsilon \left\| \nabla \dot{u}^\varepsilon(s) \right\|_{0,\Omega^\varepsilon}^2 \, ds
$$

$$
\leq \frac{3}{2} \varepsilon \left\| A^\varepsilon \right\|_{Q_\infty^\varepsilon} \left\| \nabla u_0^\varepsilon \right\|_{0,\Omega^\varepsilon}^2 + \frac{(h_{\max})^2}{m_b C_K} \int_0^t \varepsilon^3 \left\| f^\varepsilon(s) \right\|_{0,\Omega^\varepsilon}^2 \, ds
$$

$$
+ \frac{(C_0)^2}{m_b C_K} \int_0^t \varepsilon^3 \left\| g^\varepsilon(s) \right\|_{0,\Gamma_L^\varepsilon}^2 \, ds.
$$

Using the relations $\varepsilon^3 \| g^\varepsilon(s) \|_{0,\Gamma_L^\varepsilon}^2 = \| \hat{g}(s) \|_{0,\Gamma_L}^2$, $\varepsilon^3 \| f^\varepsilon(s) \|_{0,\Omega^\varepsilon}^2 = \| \hat{f}(s) \|_{0,\Omega}^2$, and $\varepsilon \| \nabla u_0^\varepsilon \|_{0,\Omega^\varepsilon}^2 \leq \| \nabla \hat{u}_0 \|_{0,\Omega}^2$, in the right-hand side of the last inequality and passing to the fixed domain Ω in the left-hand side, we get

$$
m_a \left(\sum_{\alpha=1}^2 \left\| \frac{\partial \hat{u}_\alpha^\varepsilon}{\partial z}(s) \right\|_{0,\Omega}^2 + \varepsilon^2 \left\| \frac{\partial \hat{u}_3^\varepsilon}{\partial z}(s) \right\|_{0,\Omega}^2 + \varepsilon^2 \sum_{\alpha,\beta=1}^2 \left\| \frac{\partial \hat{u}_\alpha^\varepsilon}{\partial x_\beta}(s) \right\|_{0,\Omega}^2 + \varepsilon^4 \sum_{\beta=1}^2 \left\| \frac{\partial \hat{u}_3^\varepsilon}{\partial x_\beta}(s) \right\|_{0,\Omega}^2 \right)
$$

$$
+ m_b \int_0^t \left(\sum_{\alpha=1}^2 \left\| \frac{\partial^2 \hat{u}_\alpha^\varepsilon}{\partial z \partial t}(s) \right\|_{0,\Omega}^2 + \varepsilon^2 \left\| \frac{\partial^2 \hat{u}_3^\varepsilon}{\partial z \partial t}(s) \right\|_{0,\Omega}^2 \right.
$$

$$
\left. + \varepsilon^2 \sum_{\alpha,\beta=1}^2 \left\| \frac{\partial^2 \hat{u}_\alpha^\varepsilon}{\partial x_\beta \partial t}(s) \right\|_{0,\Omega}^2 + \varepsilon^4 \sum_{\beta=1}^2 \left\| \frac{\partial^2 \hat{u}_3^\varepsilon}{\partial x_\beta \partial t}(s) \right\|_{0,\Omega}^2 \right) ds
$$

$$
\leq \frac{2}{C_K} \left\{ \frac{3}{2} \| \hat{A} \|_{Q_\infty} \| \nabla \hat{u}_0 \|_{0,\Omega}^2 + \frac{(h_{\max})^2}{m_b C_K} \int_0^t \| \hat{f}(s) \|_{0,\Omega}^2 \, ds + \frac{(C_0)^2}{m_b C_K} \int_0^t \| \hat{g}(s) \|_{0,\Gamma_L}^2 \, ds \right\}.
$$

So, we find estimates (3.3) and (3.4) with $m = \min(m_a, m_b)$ and

$$
C = \frac{2}{m C_K} \left\{ \frac{3}{2} \| \hat{A} \|_{Q_\infty} \| \nabla \hat{u}_0 \|_{0,\Omega}^2 + \frac{(h_{\max})^2}{m_b C_K} \| \hat{f} \|_{W^{1,2}(0,T;L^2(\Omega)^3)}^2 \right.
$$

$$
\left. + \frac{(C_0)^2}{m_b C_K} \| \hat{g} \|_{W^{1,2}(0,T;L^2(\Gamma_L)^3)}^2 \right\}. \qquad \square
$$

As a consequence of estimates (3.3)–(3.4), we obtain the convergence of the solution $\hat{u}^\varepsilon$ of problem (3.1)–(3.2).

Corollary 3.1 *There exists $u^* = (u_1^*, u_2^*)$ in $W^{1,2}(0,T;V_z)$ such that*

$$
\left(\hat{u}_1^\varepsilon, \hat{u}_2^\varepsilon \right) \rightharpoonup \left(u_1^\star, u_2^\star \right) \quad \text{weakly in } W^{1,2}(0,T;V_z), \tag{3.9}
$$

$$
\varepsilon \hat{u}_3^\varepsilon \rightharpoonup 0, \qquad \varepsilon \frac{\partial \hat{u}_\alpha^\varepsilon}{\partial x_\beta} \rightharpoonup 0, \qquad \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_\beta} \rightharpoonup 0, \qquad \varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0
$$

weakly in $W^{1,2}\left(0,T;L^2(\Omega)\right)$. $\tag{3.10}$

Proof We note that (3.3) and (3.4) imply that, for $\alpha = 1, 2$,

$$
\left\| \frac{\partial \hat{u}_\alpha^\varepsilon}{\partial z} \right\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq C, \qquad \left\| \frac{\partial^2 \hat{u}_\alpha^\varepsilon}{\partial z \partial t} \right\|_{L^2(0,T;L^2(\Omega))}^2 \leq C. \tag{3.11}
$$

Applying Poincaré's inequality in the domain $\Omega \times (0, T)$, with a simple comparison of the two relations in (3.11), we find

$$\left\| \hat{u}_\alpha^\varepsilon \right\|_{L^\infty(0,T;L^2(\Omega))} \leq h_{\max} \left\| \frac{\partial \hat{u}_\alpha^\varepsilon}{\partial z} \right\|_{L^\infty(0,T;L^2(\Omega))} \leq h_{\max} C,$$

$$\left\| \frac{\partial \hat{u}_\alpha^\varepsilon}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))} \leq h_{\max} \left\| \frac{\partial^2 \hat{u}_\alpha}{\partial z \partial t} \right\|_{L^2(0,T;L^2(\Omega))} \leq h_{\max} C.$$

Clearly, $(\hat{u}_\alpha^\varepsilon)_{\alpha=1,2}$ is bounded in $W^{1,2}(0,T;V_z) \cap L^\infty(0,T;V_z)$; furthermore, convergence (3.9) can be easily deduced by the injection $W^{1,2}(0,T;V_z) \hookrightarrow C(0,T;V_z)$ as in [17, Lemma 2.2]. Also (3.10) follows from (3.3)–(3.4) and (3.9). □

4 Main results and limit problem

In this section, we give the satisfied equations of u^* in $\Omega \times [0, T]$, and we can show the corresponding boundary conditions obtained for system (2.1)–(2.7). For the rest of this article, we denote by $\langle \cdot, \cdot \rangle$ the inner product on the space $L^2(\Omega)^2$.

Theorem 4.1 *u^* satisfies the following variational inequality*:

$$\left\langle A^* \frac{\partial u^*}{\partial z}(t), \frac{\partial}{\partial z}(v - \dot{u}^*(t)) \right\rangle + \left\langle B^* \frac{\partial \dot{u}^*}{\partial z}(t), \frac{\partial}{\partial z}(v - \dot{u}^*(t)) \right\rangle + \int_\omega \hat{k}\left(|v| - |\dot{u}^*(t)|\right) dx'$$

$$\geq \sum_{\alpha=1}^2 \int_{\Gamma_L} \hat{g}_\alpha(t)\left(v_\alpha - \dot{u}_\alpha^*(t)\right) d\rho + \sum_{\alpha=1}^2 \int_\Omega \hat{f}_\alpha(t)\left(v_\alpha - \dot{u}_\alpha^*(t)\right) dx$$

$$\forall v \in \Pi(V), \forall t \in [0, T], \tag{4.1}$$

$$u^*(0) = \hat{\underline{u}}_0, \tag{4.2}$$

where the matrices A^*, B^* are given by

$$A^* = 2 \begin{pmatrix} \hat{A}_{1313} & \hat{A}_{1323} \\ \hat{A}_{2313} & \hat{A}_{2323} \end{pmatrix}, \qquad B^* = 2 \begin{pmatrix} \hat{B}_{1313} & \hat{B}_{1323} \\ \hat{B}_{2313} & \hat{B}_{2323} \end{pmatrix}$$

and $\hat{\underline{u}}_0 = (\hat{u}_{01}, \hat{u}_{02})$.

   Moreover, we have

$$-\frac{\partial}{\partial z}\left\{ A^* \frac{\partial u^*}{\partial z} + B^* \frac{\partial \dot{u}^*}{\partial z} \right\} = (\hat{f}_1, \hat{f}_2) \quad \text{in } L^2\left(0, T; L^2(\Omega)^2\right), \tag{4.3}$$

$$u^*(0) = \hat{\underline{u}}_0.$$

Proof Let $t \in [0, T]$. By the integral of (3.1) relative to t, we find for every $\mathbf{v} \in V$

$$\int_0^t \left[\Lambda\left(\hat{A}, \hat{u}^\varepsilon(s), \mathbf{v}\right) + \Lambda\left(\hat{B}, \partial_t \hat{u}^\varepsilon(s), \mathbf{v}\right) + \mathbf{J}(\mathbf{v}) \right] ds + \frac{1}{2}\Lambda(\hat{A}, \hat{u}_0, \hat{u}_0)$$

$$\geq \frac{1}{2}\Lambda\left(\hat{A}, \hat{u}^\varepsilon(t), \hat{u}^\varepsilon(t)\right) + \int_0^t \Lambda\left(\hat{B}, \partial_t \hat{u}^\varepsilon(s), \partial_t \hat{u}^\varepsilon(s)\right) ds + \int_0^t \mathbf{J}\left(\partial_t \hat{u}^\varepsilon(s)\right) ds$$

$$+ \sum_{\alpha=1}^2 \int_0^t \int_{\Gamma_L} \hat{g}_\alpha(s)\left(v_\alpha - \partial_t \hat{u}_\alpha^\varepsilon(s)\right) d\rho\, ds + \varepsilon \int_0^t \int_{\Gamma_L} \hat{g}_3(s)\left(v_3 - \partial_t \hat{u}_3^\varepsilon(s)\right) d\rho\, ds$$

$$+ \sum_{\alpha=1}^{2} \int_0^t \int_\Omega \hat{f}_\alpha(s)\big(v_\alpha - \partial_t \hat{u}_\alpha^\varepsilon(s)\big)\, dx\, ds + \varepsilon \int_0^t \int_\Omega \hat{f}_3(s)\big(v_3 - \partial_t \hat{u}_3^\varepsilon(s)\big)\, dx\, ds. \qquad (4.4)$$

Since the form $\Lambda(\hat{C}, \cdot, \cdot)$ is a symmetry and V-elliptic, by using the convergence (3.10), we have

$$\liminf_{\varepsilon \to 0} \Lambda\big(\hat{A}, \hat{u}^\varepsilon(t), \hat{u}^\varepsilon(t)\big) \geq 2 \sum_{\alpha, \gamma = 1}^{2} \int_\Omega \hat{A}_{\alpha 3 \gamma 3} \frac{\partial u_\gamma^*}{\partial z} \frac{\partial u_\alpha^*}{\partial z}\, dx.$$

This formula can be rewritten using the matrix form A^* as follows:

$$\liminf_{\varepsilon \to 0} \Lambda\big(\hat{A}, \hat{u}^\varepsilon(t), \hat{u}^\varepsilon(t)\big) \geq \left\langle A^* \frac{\partial u^*}{\partial z}(t), \frac{\partial u^*}{\partial z}(t) \right\rangle. \qquad (4.5)$$

An argument similar to that used for (4.5) shows that the functional

$$t \longrightarrow \int_0^t \Lambda\big(\hat{B}, \partial_t \hat{u}^\varepsilon(s), \partial_t \hat{u}^\varepsilon(s)\big)\, ds$$

is lower semi-continuous for the weak topology of $W^{1,2}(0, T; V_z)$. Then

$$\liminf_{\varepsilon \to 0} \int_0^t \Lambda\big(\hat{B}, \partial_t \hat{u}^\varepsilon(s), \partial_t \hat{u}^\varepsilon(s)\big)\, ds \geq \int_0^t \left\langle B^* \frac{\partial \dot{u}^*}{\partial z}(s), \frac{\partial \dot{u}^*}{\partial z}(s) \right\rangle ds. \qquad (4.6)$$

From (4.5)–(4.6) and (3.9)–(3.10), and by the semi-continuity of $t \longrightarrow \int_0^t \mathbf{J}(\dot{u}^*(s))\, ds$, we let ε tend to 0 in (4.4) to obtain

$$\left\langle A^* \frac{\partial u^*}{\partial z}(t), \frac{\partial v}{\partial z} \right\rangle + \int_0^t \left\langle B^* \frac{\partial \dot{u}^*}{\partial z}(s), \frac{\partial v}{\partial z} \right\rangle ds + \int_0^t \mathbf{J}(v)\, ds + \frac{1}{2} \left\langle A^* \frac{\partial \hat{\underline{u}}_0}{\partial z}, \frac{\partial \hat{\underline{u}}_0}{\partial z} \right\rangle$$

$$\geq \frac{1}{2} \left\langle A^* \frac{\partial u^*}{\partial z}(t), \frac{\partial u^*}{\partial z}(t) \right\rangle + \int_0^t \left\langle B^* \frac{\partial \dot{u}^*}{\partial z}(s), \frac{\partial \dot{u}^*}{\partial z}(s) \right\rangle ds + \int_0^t \mathbf{J}(\dot{u}^*(s))\, ds$$

$$+ \sum_{\alpha=1}^{2} \int_0^t \int_{\Gamma_L} \hat{g}_\alpha(s)\big(v_\alpha - \dot{u}_\alpha^*(s)\big)\, d\rho\, ds + \sum_{\alpha=1}^{2} \int_0^t \int_\Omega \hat{f}_\alpha(s)\big(v_\alpha - \dot{u}_\alpha^*(s)\big)\, dx\, ds$$

with

$$u^*(0) = \hat{\underline{u}}_0.$$

The condition $u^*(0) = \hat{\underline{u}}_0$ is an immediate consequence of (3.9) and (3.2). Thus, by the following equality

$$\int_0^t \left\langle A^* \frac{\partial u^*}{\partial z}(s), \frac{\partial \dot{u}^*}{\partial z}(s) \right\rangle ds = \frac{1}{2} \left\langle A^* \frac{\partial u^*}{\partial z}(t), \frac{\partial u^*}{\partial z}(t) \right\rangle - \frac{1}{2} \left\langle A^* \frac{\partial \hat{\underline{u}}_0}{\partial z}, \frac{\partial \hat{\underline{u}}_0}{\partial z} \right\rangle, \qquad (4.7)$$

we conclude that

$$\int_0^t \left[\left\langle A^* \frac{\partial}{\partial z} u^*(s), \frac{\partial}{\partial z}\big(v - \dot{u}^*(s)\big) \right\rangle + \left\langle B^* \frac{\partial}{\partial z} \dot{u}^*(s), \frac{\partial}{\partial z}\big(v - \dot{u}^*(s)\big) \right\rangle + \mathbf{J}(v) - \mathbf{J}(\dot{u}^*(s)) \right] ds$$

$$\geq \sum_{\alpha=1}^{2} \int_0^t \int_{\Gamma_L} \hat{g}_\alpha(s)\big(v_\alpha - \dot{u}_\alpha^*(s)\big)\, d\rho\, ds + \sum_{\alpha=1}^{2} \int_0^t \int_\Omega \hat{f}_\alpha(s)\big(v_\alpha - \dot{u}_\alpha^*(s)\big)\, dx\, ds.$$

Now, it remains only to use the same method as that used in [11] to derive relation (4.1). To prove (4.3), we use the Green formula in (4.1), and this completes the proof of Theorem 4.1. □

Theorem 4.2 *The limit problem* (4.1)–(4.2) *has a unique solution* u^* *in* $W^{1,2}(0,T;V_z)$.

Proof Let $u^{*,1}$, $u^{*,2}$ be two solutions of (4.1)–(4.2), and let $t \in [0,T]$. Taking $v = u^{*,2}$ in (4.1)–(4.2) and $v = u^{*,1}$ in the inequality relating to $u^{*,2}$, it follows by posing $\mathbf{w}^* = u^{*,2} - u^{*,1}$ that

$$\left\langle A^* \frac{\partial \mathbf{w}^*}{\partial z}(t), \frac{\partial \dot{\mathbf{w}}^*}{\partial z}(t) \right\rangle + \left\langle B^* \frac{\partial \dot{\mathbf{w}}^*}{\partial z}(t), \frac{\partial \dot{\mathbf{w}}^*}{\partial z}(t) \right\rangle \leq 0.$$

Since $u^{*,1}(0) = u^{*,2}(0) = \hat{\underline{u}}_0$, using (4.7), we deduce that

$$\frac{1}{2}\left\langle A^* \frac{\partial \mathbf{w}^*}{\partial z}(t), \frac{\partial \mathbf{w}^*}{\partial z}(t) \right\rangle + \int_0^t \left\langle B^* \frac{\partial \dot{\mathbf{w}}^*}{\partial z}(s), \frac{\partial \dot{\mathbf{w}}^*}{\partial z}(s) \right\rangle ds \leq 0. \qquad (4.8)$$

We must now check that the matrices A^*, B^* are elliptic. Let $\eta = (\eta_\alpha)_{\alpha=1,2} \in \mathbb{R}^2$, we return now to hypotheses (2.8)–(2.9). By choosing symmetric tensors ξ given by $\xi_{\alpha\beta} = 0$ for $\alpha, \beta = 1, 2$, $\xi_{33} = 0$ and $\xi_{\alpha3} = \xi_{3\alpha} = \eta_\alpha$ for $\alpha = 1, 2$, we get

$$\hat{A}_{ijkl}\xi_{kl}\xi_{ij} = 2\hat{A}_{\alpha3\beta3}\xi_{\beta3}\xi_{\alpha3} + 2\hat{A}_{\alpha333}\xi_{33}\xi_{\alpha3} + 2\hat{A}_{33\alpha3}\xi_{\alpha3}\xi_{33} + \hat{A}_{3333}\xi_{33}\xi_{33}$$

$$= A^*_{\alpha\beta}\eta_\beta\eta_\alpha.$$

In the same way, derive the last relation for the matrix B^*. Consequently, $|\xi|^2 = 2|\eta|^2$ leads to

$$A^*\eta.\eta \geq 2m_a|\eta|^2, \qquad B^*\eta.\eta \geq 2m_b|\eta|^2$$

for all $\eta \in \mathbb{R}^2$.

Hence, inequality (4.8) becomes

$$2m_a \left\| \frac{\partial \mathbf{w}^*}{\partial z}(t) \right\|_{0,\Omega}^2 + 2m_b \int_0^t \left\| \frac{\partial \dot{\mathbf{w}}^*}{\partial z}(s) \right\|_{0,\Omega}^2 ds \leq 0.$$

Since $m_a, m_b > 0$, and from Poincaré's inequality, we obtain

$$\left\| \mathbf{w}^*(t) \right\|_{0,\Omega}^2 \leq h_{\max}^2 \left\| \frac{\partial \mathbf{w}^*}{\partial z}(t) \right\|_{0,\Omega}^2 = 0,$$

$$\int_0^t \left\| \dot{\mathbf{w}}^*(s) \right\|_{0,\Omega}^2 ds \leq h_{\max}^2 \int_0^t \left\| \frac{\partial \dot{\mathbf{w}}^*}{\partial z}(s) \right\|_{0,\Omega}^2 ds = 0,$$

we deduce that $\mathbf{w}^* = 0$ in $L^2(0,T;V_z)$ and $\dot{\mathbf{w}}^* = 0$ in $L^2(0,T;V_z)$, which concludes the uniqueness of problem (4.1)–(4.2). □

Theorem 4.3 *Under the assumptions of Theorem 4.2, the traces* $\{\mathbf{s}^*, \boldsymbol{\pi}^*\}$ *with* $\mathbf{s}^* = (s_1^*, s_2^*)$ *and* $\boldsymbol{\pi}^* = (\pi_1^*, \pi_2^*)$ *defined by*

$$\mathbf{s}^*(x', t) = u^*(x', 0, t);$$

$$\boldsymbol{\pi}^*(x', t) = A^*(x', 0)\frac{\partial u^*}{\partial z}(x', 0, t) + B^*(x', 0)\frac{\partial \dot{u}^*}{\partial z}(x', 0, t) \tag{4.9}$$

satisfy the following limit form of the Tresca boundary conditions:

$$\int_\omega \hat{k}\left|\psi + \frac{\partial \mathbf{s}^*}{\partial t}\right| - \left|\frac{\partial \mathbf{s}^*}{\partial t}\right| dx' - \int_\omega \boldsymbol{\pi}^*.\psi\,dx' \geq 0 \quad \forall \psi \in L^2(\omega)^2, \forall t \in [0, T], \tag{4.10}$$

$$\left.\begin{array}{l} |\boldsymbol{\pi}^*| < \hat{k} \Rightarrow \frac{\partial \mathbf{s}^*}{\partial t} = 0 \\ |\boldsymbol{\pi}^*| = \hat{k} \Rightarrow \exists \lambda > 0 \text{ such that } \frac{\partial \mathbf{s}^*}{\partial t} = \lambda \boldsymbol{\pi}^* \end{array}\right\} \quad a.e.\ in\ \omega \times [0, T]. \tag{4.11}$$

Proof For every $t \in [0, T]$, we chose in the variational inequality (4.1) $v = \dot{u}^*(t) + \psi$, where ψ in $H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)^2$, and using the Green formula, we find

$$-\int_\Omega \frac{\partial}{\partial z}\left\{A^*(x', z)\frac{\partial u^*}{\partial z}(x', z, t) + B^*(x', z)\frac{\partial \dot{u}^*}{\partial z}(x', z, t)\right\} \cdot \psi\,dx'\,dz$$

$$-\int_\omega \left(A^*(x', 0)\frac{\partial u^*}{\partial z}(x', 0, t) + B^*(x', 0)\frac{\partial \dot{u}^*}{\partial z}(x', 0, t)\right) \cdot \psi(x', 0)\,dx'$$

$$+\int_\omega \hat{k}\left(|\psi + \dot{u}^*(t)| - |\dot{u}^*(t)|\right)dx' \geq \sum_{\alpha=1}^2 \int_\Omega \hat{f}_\alpha(t) \cdot \psi_\alpha\,dx'\,dz.$$

Adding the last formula and (4.1), (4.9) leads to

$$\int_\omega \hat{k}\left(|\psi + \dot{u}^*(t)| - |\dot{u}^*(t)|\right)dx' - \int_\omega \boldsymbol{\pi}^*(x', t) \cdot \psi(x', 0)\,dx' = 0$$

for all ψ in $H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)^2$.

As in [1], this inequality remains valid for any ψ in $\mathcal{D}(\omega)^2$, and by density of $\mathcal{D}(\omega)$ in $L^2(\omega)$, we could therefore deduce (4.10). To prove (4.11), we use an argument similar to that used in the proof of Theorem 4.2 in [1]. □

Theorem 4.4 *Suppose that the components* $\hat{A}_{\alpha 3\beta 3}$, $\hat{B}_{\alpha 3\beta 3}$ *for* $1 \leq \alpha, \beta \leq 2$ *depend only on the variable* x', *we have the following weak form*:

$$\int_\omega \left(\int_0^h \left[A^* u^*(x', z, t) + B^* \dot{u}^*(x', z, t)\right]dz + \frac{h^2}{2}\boldsymbol{\pi}^*(x', t) - \tilde{F}(x', t)\right).\nabla \psi(x')\,dx' = 0,$$

$$\forall \psi \in H^1(\omega), \forall t \in [0, T], \tag{4.12}$$

where the vector $\tilde{F} = (\tilde{F}_\alpha)_{\alpha=1,2}$ *is given by*

$$\tilde{F}_\alpha(x'; t) = \int_0^h F_\alpha(x', z, t)\,dz - hF_\alpha(x', h, t) \quad and$$

$$F_\alpha(x', z, t) = \int_0^z \int_0^w \hat{f}_\alpha(x', y, t)\,dy\,dw. \tag{4.13}$$

Proof Integrating equality (4.3) over $[0, z]$ two times and taking into account $\hat{A}_{\alpha 3 \beta 3}$ and $\hat{B}_{\alpha 3 \beta 3}$ depending only on x', we infer

$$
\begin{aligned}
& -A^*(x') u^*(x', z, t) - B^*(x') \dot{u}^*(x', z, t) + A^*(x') \mathbf{s}^*(x', t) \\
& \quad + B^*(x') \dot{\mathbf{s}}^*(x', t) + z \boldsymbol{\pi}^*(x', t) = F(x', z, t)
\end{aligned}
\tag{4.14}
$$

for all $t \in [0, T]$. As $u_\alpha^*(x', h(x'), t) = 0$, $\alpha = 1, 2$, we have

$$
A^*(x') \mathbf{s}^*(x', t) + B^*(x') \dot{\mathbf{s}}^*(x', t) + h \boldsymbol{\pi}^*(x', t) = F(x', h, t).
\tag{4.15}
$$

We integrate (4.14) from 0 to $h(x')$ to obtain

$$
\begin{aligned}
& -\int_0^{h(x')} \left(A^*(x') u^*(x', z, t) + B^*(x') \dot{u}^*(x', z, t) \right) dz \\
& \quad + h A^*(x') \mathbf{s}^*(x') + h B^*(x') \dot{\mathbf{s}}^*(x', t) + \frac{h^2}{2} \boldsymbol{\pi}^*(x', t) \\
& = \int_0^{h(x')} F(x', z, t) \, dz.
\end{aligned}
$$

From this equality and (4.15), we derive the relation

$$
\int_0^{h(x')} \left[A^*(x') u^*(x', z, t) + B^*(x') \dot{u}^*(x', z, t) \right] dz + \frac{h^2}{2} \boldsymbol{\pi}^*(x', t) - \tilde{F}(x', t) = 0
\tag{4.16}
$$

such that $\tilde{F}$ is already defined in (4.13). Let us finally get the weak form (4.12) if we multiply (4.16) by $\nabla \psi(x')$ and integrate it in ω. □

5 Conclusion

The key to this work lies in the relation between the tensors A^ε, B^ε and the matrices A^*, B^*, which played a major role in the passage from u^ε to u^*. This permits us to deduce that at the limit the phenomenon can be described by the following two-dimensional constitutive law:

$$
\sigma^*(u^*) = A^* \frac{\partial u^*}{\partial z} + B^* \frac{\partial \dot{u}^*}{\partial z}.
$$

Such a constitutive law maintains the classical physical and algebraic properties, so we can deduce a corresponding inverse law, see for example [11, 17].

Moreover, this law and its traces meet the basic equation of motion in (4.2)–(4.3) and the boundary conditions of the Tresca friction in (4.10)–(4.11). The phenomenon is described mathematically by the weak formula (4.12), known as the Reynolds equation, and has been proved in many papers for the particular cases, see [1, 2, 4, 7, 12].

Specifically, if $A^\varepsilon = 0$, the model matches the case of the anisotropic linearized elasticity system and has been studied in [13, 14]. Other cases in [1, 5, 6, 16, 18] related to a homogeneous and isotropic system can also be recovered in a similar manner to [13].

Through the results of this work, we are sure that the behavior of the posed problem for "small" parameters ε is characterized by a clearly defined physical-mathematical model.

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this manuscript. The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally in this article. They have all read and approved the final manuscript.

Author details
[1]Department of Mathematics, University of El Oued, Po Box 789, El Oued, 39000, Algeria. [2]Applied Mathematics Laboratory, Department of Mathematics, Faculty of Sciences, University of Ferhat ABBAS-Sétif 1, Sétif, 19000, Algeria. [3]Department of Mathematics, College Of Sciences and Arts, ArRass, Qassim University, Buraydah, Saudi Arabia. [4]Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Oran, 31000, Algeria.

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