Boundary Value Problems
a SpringerOpen Journal

RESEARCH
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High perturbations of a new Kirchhoff problem involving the p-Laplace operator

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Abstract

In the present work we are concerned with the existence and multiplicity of solutions for the following new Kirchhoff problem involving the p-Laplace operator:

$$\begin{cases} -(a - b\int_\Omega |\nabla u|^p\,dx)\Delta_p u = \lambda |u|^{q-2}u + g(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $a, b > 0$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator, $1 < p < N$, $p < q < p^* := (Np)/(N-p)$, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain. Under suitable conditions on g, we show the existence and multiplicity of solutions in the case of high perturbations (λ large enough). The novelty of our work is the appearance of new nonlocal terms which present interesting difficulties.

MSC: 35J60; 35J20

Keywords: p-Laplace operator; New Kirchhoff problem; Variational method

1 Introduction and main result

In this paper, we are concerned with the existence and multiplicity of solutions for the following problem:

$$\begin{cases} -(a - b\int_\Omega |\nabla u|^p\,dx)\Delta_p u = \lambda |u|^{q-2}u + g(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $a, b > 0$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator, $1 < p < N$, $p < q < p^* := (Np)/(N-p)$, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain, λ is a real parameter.

In what follows, we suppose that the continuous function g satisfies the following conditions:

(g_1) $g \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$ and

$$\bigl|g(x,s)\bigr| \leq C\bigl(1 + |s|^{r-1}\bigr) \quad \forall (x,s) \in \Omega \times \mathbb{R},$$

where $C > 0$ and $p < r < p^*$;

(g_2) $g(x,s) = o(|s|^{p-1})$ as $s \to 0$;
(g_3) (AR-condition) there exist $\theta \in (p, p^*)$ and $T > 0$ such that

$$0 < \theta G(x,s) \leq g(x,s)s \quad \text{for all } |s| \geq T, x \in \Omega,$$

where $G(x,s) = \int_0^s g(x,t)\,dt$;
(g_3^*) $\lim_{|x|\to\infty} \frac{G(x,t)}{|t|^p} = \infty$ uniformly in $x \in \Omega$;
(g_3^{**}) $G(x,0) \in L^1(\Omega)$ and there exists $\sigma \in L^1(\Omega)$ such that

$$\mathcal{G}(x,t) \leq \mathcal{G}(x,s) + \sigma(x)$$

for all $0 \leq t \leq s$ or $s \leq t \leq 0$, where $\mathcal{G}(x,t) := g(x,t)t - pG(x,t)$;
(g_4) $g(x,-s) = -g(x,s)$ for all $(x,s) \in \Omega \times \mathbb{R}$.

The novelty of our work is the fact that we combine several different phenomena in one problem. The features of this paper are the following:

(1) The continuous function g may satisfy the Ambrosetti–Rabinowitz condition or not.
(2) The presence of the new nonlocal term $(a - b\int_\Omega |\nabla u|^2\,dx)$.

To the best of our knowledge, there few papers proving the existence and multiplicity of solutions with the combined effects generated by the above features.

Recently, nonlocal problems and operators have been widely studied in the literature and have attracted the attention of a lot of mathematicians coming from different research areas. A typical model proposed by Kirchhoff [4] serves as a generalization of the classic D'Alembert wave equation by taking into account the effects of the changes in the length of the strings during the vibrations. Thanks to the pioneering work of Lions [13], a lot of attention has been drawn to these nonlocal problems during the last decade. After that, some studies on this kind of problems have been performed by using different approaches, see [2, 6–12, 14, 15, 18, 19, 23] and the references therein.

In this paper, we mainly consider a new Kirchhoff problem involving the p-Laplace operator, that is, the form with a nonlocal coefficient $(a - b\int_\Omega |\nabla u|^p\,dx)$. Its background is derived from negative Young's modulus, when the atoms are pulled apart rather than compressed together and the strain is negative. Recently, the authors in [22] first studied this kind of problem

$$\begin{cases} -(a - b\int_\Omega |\nabla u|^2\,dx)\Delta u = |u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $2 < p < 2^* := (2N)/(N-2)$, and they obtained the existence of solutions by using the mountain pass lemma. Furthermore, some interesting results have been obtained for this kind of Kirchhoff-type problem. We refer the readers to [1, 3, 5, 16, 20–22] and the references therein.

From the above-mentioned papers, it is a natural question to see what results can be recovered when this new Kirchhoff problem involves the p-Laplace operator. Compared to the above papers, some difficulties arise in our paper when dealing with problem (1.1), because of the appearance of the nonlocal coefficient $(a - b\int_\Omega |\nabla u|^p\,dx)$ which provokes some mathematical difficulties, and these make the study of problem (1.1) particularly interesting. In addition, we need more delicate estimates which are not trivial, the method of this paper is obviously different from the literature works mentioned above.

If the nonlinear term g satisfies the AR-condition, we have the following existence results of problem (1.1).

Theorem 1.1 *Suppose that assumptions* (g_1)–(g_3) *are fulfilled. Then there exists* $\lambda_1 > 0$ *such that, for any* $\lambda \geq \lambda_1$, *problem* (1.1) *admits a nontrivial solution.*

Theorem 1.2 *Suppose that assumptions* (g_1)–(g_3) *and* (g_4) *are fulfilled. Then there exists* $\lambda_2 > 0$ *such that, for any* $\lambda \geq \lambda_2$, *problem* (1.1) *admits a sequence of solutions with unbounded energy.*

If the nonlinear term g satisfies not the AR-condition, our next main result in this paper is the following.

Theorem 1.3 *Suppose that assumptions* (g_1)–(g_2), (g_3^*), (g_3^{**}), *and* (g_5) *are fulfilled. Then there exists* $\lambda_3 > 0$ *such that, for any* $\lambda \geq \lambda_3$, *problem* (1.1) *admits a sequence of solutions with unbounded energy.*

Remark 1.1 The main feature of problem (1.1) contains a nonlocal coefficient $(a - b\int_\Omega |\nabla u|^2\,dx)$, there is no doubt that we encounter serious difficulties because of the lack of compactness. To overcome the challenge, we must estimate precisely the value of c and give a threshold value. So the variational technique for problem (1.1) becomes more delicate. To the best of our knowledge, the present paper results have not been covered yet in the literature.

This paper is organized as follows. In Sect. 2, we give some necessary preliminary knowledge on the functional setting and prove the Palais–Smale compactness condition. In Sect. 3, we prove Theorem 1.1 by using the mountain pass theorem. In Sect. 4, we prove Theorems 1.2 and 1.3 via the symmetric mountain pass theorem where the nonlinear term g satisfies the AR-condition or not, respectively.

2 Preliminaries and compactness results

We seek weak solutions to problem (1.1) in $W_0^{1,p}(\Omega)$ which is the usual Sobolev space with respect to the norm $\|u\| = \int_\Omega |\nabla u|^p\,dx$. We then have that $W_0^{1,p}(\Omega)$ is continuously and compactly embedded into the Lebesgue space $L^\tau(\Omega)$ endowed the norm $|u|_\tau = (\int_\Omega |u|^\tau\,dx)^{\frac{1}{\tau}}$, $p < \tau < p^*$. Denote by S_τ the best constant for this embedding, that is,

$$S_\tau |u|_\tau \leq \|u\|, \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.1}$$

In particular, if S is the best constant for the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, then it is defined by

$$S = \inf\left\{\int_\Omega |\nabla u|^p\,dx : u \in W_0^{1,p}(\Omega), \int_\Omega |u|^{p^*}\,dx = 1\right\}. \tag{2.2}$$

We consider the energy functional $J_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) = \frac{a}{p}\int_\Omega |\nabla u|^p\,dx - \frac{b}{2p}\left(\int_\Omega |\nabla u|^p\,dx\right)^2 - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \int_\Omega G(x,u)\,dx. \tag{2.3}$$

It is well known that a critical point of J_λ is a weak solution of problem (1.1) and the functional J_λ is of class C^1 in $W_0^{1,p}(\Omega)$ (see [17]). Denote by J_λ' the derivative operator of J_λ in the weak sense. Then

$$\begin{aligned}\langle J_\lambda'(u), v\rangle &= a\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx - b\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx \\ &\quad - \lambda\int_\Omega |u|^{q-2}uv\,dx - \int_\Omega g(x,u)v\,dx, \quad \forall u,v \in W_0^{1,p}(\Omega).\end{aligned}$$

We recall that a C^1 functional J_λ on the Banach space $W_0^{1,p}(\Omega)$ is said to satisfy the Palais–Smale condition at level c ($(PS)_c$ in short) if every sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ satisfying $J_\lambda(u_n) \to c$ and $J_\lambda'(u_n) \to 0$ ($n \to \infty$) has a convergent subsequence.

Lemma 2.1 *Assume that* (g_1)–(g_3) *hold. Then the functional* J_λ *satisfies the* $(PS)_c$ *condition, where* $c \in (-\infty, \frac{a^2}{2pb})$.

Proof First, let $\{u_n\}_n \subset W_0^{1,p}(\Omega)$ be a $(PS)_c$ sequence associated with the functional J_λ, that is,

$$J_\lambda(u_n) \to c \quad \text{and} \quad J_\lambda'(u_n) \to 0 \quad \text{as } n \to \infty.$$

Then, $\{u_n\}_n$ is bounded in $W_0^{1,p}(\Omega)$. In fact, arguing by contradiction, we assume that, passing eventually to a subsequence, still denoting by $\{u_n\}_n$, we have $\|u_n\| \to +\infty$ as $n \to +\infty$. By (g_3) we have

$$\begin{aligned} c + o(1)\|u_n\| &= J_\lambda(u_n) - \frac{1}{q}\langle J_\lambda'(u_n), u_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)a\|u_n\|^p + \left(\frac{1}{q} - \frac{1}{2p}\right)b\|u_n\|^{2p} \\ &\quad + \int_\Omega \left[\frac{1}{q}g(x,u_n)u_n - G(x,u_n)\right]dx \\ &\geq \left(\frac{1}{p} - \frac{1}{q}\right)\|u_n\|^p + \left(\frac{1}{q} - \frac{1}{2p}\right)b\|u_n\|^{2p} - C|\Omega|. \end{aligned} \tag{2.4}$$

Thus, (2.4) leads to contradiction since $1 < p$. Therefore, there exists $u \in W_0^{1,p}(\Omega)$ such that up to a subsequence $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $u_n \to u$ strongly in $L^s(\Omega)$ with $p \leq s < p^*$.

In the following, we will prove $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$.

In fact, by the Hölder inequality, one has

$$\begin{aligned}\int_\Omega |u_n|^{q-2}u_n(u_n-u)\,dx &\le \int_\Omega |u_n|^{q-1}(u_n-u)\,dx \\ &\le \left(\int_\Omega |u_n|^q\,dx\right)^{\frac{q-1}{q}}\left(\int_\Omega |u_n-u|^q\,dx\right)^{\frac{1}{q}} \\ &\to 0 \quad \text{as } n\to\infty,\end{aligned}$$

and thus

$$\lim_{n\to\infty}\int_\Omega |u_n|^{q-2}u_n(u_n-u)\,dx = 0. \tag{2.5}$$

On the other hand, by conditions (g_1) and (g_2), we have that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left|g(x,u_n)\right| \le \varepsilon |u_n|^{p-1} + C_\varepsilon |u_n|^{r-1}.$$

This fact implies that

$$\begin{aligned}\left|\int_\Omega g(x,u_n)(u_n-u)\,dx\right| &\le \varepsilon\int_\Omega |u_n|^{p-1}|u_n-u|\,dx + C_\varepsilon\int_\Omega |u_n|^{r-1}|u_n-u|\,dx \\ &\le \varepsilon\left(\int_\Omega |u_n|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |u_n-u|^p\,dx\right)^{\frac{1}{p}} \\ &\quad + C_\varepsilon\left(\int_\Omega |u_n|^r\,dx\right)^{\frac{r-1}{r}}\left(\int_\Omega |u_n-u|^r\,dx\right)^{\frac{1}{r}} \\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Thus

$$\lim_{n\to\infty}\int_\Omega g(x,u_n)(u_n-u)\,dx = 0. \tag{2.6}$$

From

$$\langle J_\lambda'(u_n), u_n-u\rangle \to 0$$

that

$$\begin{aligned}&\left(a-b\int_\Omega |\nabla u_n|^p\,dx\right)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\,dx \\ &\quad -\lambda\int_\Omega |u_n|^{q-2}u_n(u_n-u)\,dx - \int_\Omega g(x,u_n)(u_n-u)\,dx \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Therefore, we can deduce from (2.5) and (2.6) that

$$\left(a-b\int_\Omega |\nabla u_n|^p\,dx\right)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\,dx \to 0 \quad \text{as } n\to\infty. \tag{2.7}$$

Since $\{u_n\}_n$ is bounded in $W_0^{1,p}(\Omega)$, passing to a subsequence, if necessary, we may assume that

$$\int_\Omega |\nabla u_n|^p\,dx \to d \geq 0 \quad \text{as } n \to \infty.$$

Next, we distinguish the following two steps.

Step I: $d \neq \frac{a}{b}$.

In this step, we have

$$\left(a - b\int_\Omega |\nabla u_n|^p\,dx\right) \nrightarrow 0 \quad \text{as } n \to \infty.$$

This means that $\{a - b\int_\Omega |\nabla u_n|^p\,dx\}$ is bounded.

Step II: $d = \frac{a}{b}$.

In this step, we have

$$a - b\int_\Omega |\nabla u_n|^p\,dx \to 0. \tag{2.8}$$

Define

$$\psi(u) = \frac{\lambda}{q}\int_\Omega |u|^q\,dx + \int_\Omega G(x,u)\,dx.$$

Then

$$\langle \psi'(u), v\rangle = \lambda\int_\Omega |u|^{q-2}uv\,dx + \int_\Omega g(x,u)v\,dx.$$

It follows that

$$\langle \psi'(u_n) - \psi'(u), v\rangle = \lambda\int_\Omega \left(|u_n|^{q-2}u_n - |u|^{q-2}u\right)v\,dx - \int_\Omega \left(g(x,u_n) - g(x,u)u\right)v\,dx$$

for all $v \in W_0^{1,p}(\Omega)$. From the Hölder inequality and the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ ($p \leq s < p^*$) is compact, one has

$$\lambda\int_\Omega \left(|u_n|^{q-2}u_n - |u|^{q-2}u\right)v\,dx \to 0 \quad \text{as } n \to \infty$$

and

$$\int_\Omega \left(g(x,u_n) - g(x,u)u\right)v\,dx \to 0 \quad \text{as } n \to \infty.$$

Thus

$$\langle \psi'(u_n) - \psi'(u), v\rangle \to 0 \quad \text{as } n \to \infty. \tag{2.9}$$

Since

$$\langle J'_\lambda(u_n), v\rangle = \left(a - b\int_\Omega |\nabla u_n|^p\,dx\right)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v\,dx + \langle \psi'(u_n), v\rangle,$$

$$\langle J_\lambda'(u_n), v\rangle \to 0 \quad \text{and} \quad \left(a - b\int_\Omega |\nabla u_n|^p\,dx\right) \to 0.$$

Hence $\psi'(u_n) \to 0$ as $n \to \infty$, i.e.,

$$\langle \psi'(u), v\rangle = \lambda \int_\Omega |u|^{q-2}uv\,dx + \int_\Omega g(x,u)v\,dx = 0 \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

Then we have

$$\lambda|u|^{q-2}u + g(x,u) = 0, \quad \text{a.e. } x \in \Omega.$$

By the fundamental lemma of the variational method (see [17]), it follows that $u = 0$. So

$$\psi(u_n) = \frac{\lambda}{q}\int_\Omega |u_n|^q\,dx + \int_\Omega G(x,u_n)\,dx \to \frac{\lambda}{q}\int_\Omega |u|^q\,dx + \int_\Omega G(x,u)\,dx = 0.$$

Hence, we see that

$$J_\lambda(u_n) = \frac{a}{p}\|u_n\|^p - \frac{b}{2p}\|u_n\|^{2p} - \frac{\lambda}{q}\int_\Omega |u_n|^q\,dx - \int_\Omega G(x,u_n)\,dx \to \frac{a^2}{2pb}.$$

This is a contradiction since

$$J_\lambda(u_n) \to c < \frac{a^2}{2pb}.$$

Then (2.8) is not true and $d \neq \frac{a}{b}$, this means that any subsequence of $\{a - b\|u_n\|^p\}$ does not converge to zero. Therefore there exists $\delta > 0$ such that $|a - b\|u_n\|^p| > \delta$ when n is large enough. It is clear that $\{a - b\|u_n\|\}$ is bounded.

Therefore, it follows from (2.7) that

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u - u_n)\,dx \to 0 \quad \text{as } n \to \infty.$$

Thus by the (S_+) property, $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$. The proof is complete. □

3 Proof of Theorem 1.1

In this section, we first begin by giving the following general mountain pass theorem (see [17]).

Theorem 3.1 *Let E be a Banach space and $I_\lambda \in C^1(E,\mathbb{R})$ with $J_\lambda(0) = 0$. Suppose that*

(A_1) *there exist $\rho, \alpha > 0$ such that $J_\lambda \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$;*

(A_2) *there exists $e \in E$ satisfying $\|e\|_E > \rho$ such that $I_\lambda(e) < 0$.*

Define $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$.

$$c = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} J_\lambda\bigl(\gamma(t)\bigr) \geq \alpha$$

and there exists a $(PS)_c$ sequence $\{u_n\}_n \subset E$.

Now, we begin proving that J_λ satisfies the assumptions of the mountain pass geometry.

Lemma 3.1 *Under conditions* (g_1)–(g_3), *the functional* J_λ *satisfies the mountain pass geometry, that is,*

(i) *there exist* $\alpha, \rho > 0$ *such that* $J_\lambda(u) \geq \alpha$ *for any* $u \in W_0^{1,p}(\Omega)$ *such that* $\|u\| = \rho$;

(ii) *there exists* $e \in W_0^{1,p}(\Omega)$ *with* $\|e\| > \rho$ *such that* $J_\lambda(e) < 0$.

Proof First, by conditions (g_1) and (g_2), we have that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left|G(x,u)\right| \leq \varepsilon |u|^p + C_\varepsilon |u|^r.$$

By (2.1) and (2.3), we have

$$\begin{aligned} J_\lambda(u) &\geq \frac{a}{p}\|u\|^p - \frac{b}{2p}\|u\|^{2p} - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \varepsilon\int_\Omega |u|^p\,dx - C_\varepsilon\int_\Omega |u|^r\,dx \\ &\geq \left(\frac{a}{2} - \varepsilon S_p^{-p}\right)\|u\|^p - \frac{b}{2p}\|u\|^{2p} - \lambda S_q^{-q}\|u\|^q - C_\varepsilon S_r^{-r}\|u\|^r. \end{aligned} \tag{3.1}$$

Taking $\varepsilon > 0$ satisfies $\frac{a}{2} - \varepsilon S_p^{-p} > 0$. Then, we can choose ρ, $\alpha > 0$ such that $J_\lambda(u) \geq \alpha$ for $\|u\| = \rho$, since $p < q$ and $p < r$. Hence (A_1) in Theorem 3.1 holds.

Next, we verify condition (A_2) of Theorem 3.1. By (g_3) we know that, for all $T > 0$, there exists $C_T > 0$ such that

$$G(x,u) \geq T|u|^\theta - C_T \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}. \tag{3.2}$$

Set $v \in C_0^\infty(\Omega)$ and $v \neq 0$. It follows from (3.2) that

$$J_\lambda(tv) \leq \frac{a}{p}t^p\|v\|^p - \frac{b}{2p}t^{2p}\|v\|^{2p} - \frac{\lambda}{q}t^q\int_\Omega |v|^q\,dx - t^\theta T\int_\Omega |v|^\theta\,dx + C_T\Omega. \tag{3.3}$$

From the fact that $q, \theta > p$, we deduce that $J_\lambda(t_0 v) < 0$ and $t_0\|v\| > \rho$ for t_0 large enough. Set $e = t_0 v$. Hence e is the required function and (A_2) in Theorem 3.1 is valid. This completes the proof. □

Proof of Theorem 1.1 Now, we claim that

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} J_\lambda\bigl(\gamma(t)\bigr) < \frac{a^2}{2pb} \tag{3.4}$$

for all sufficiently large λ.

In fact, in order to prove (3.4), we choose $v \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that $\lim_{t\to\infty} J_\lambda(tv) = -\infty$. Then

$$\sup_{t\geq 0} J_\lambda(tv) = J_\lambda(t_\lambda v)$$

for some $t_\lambda > 0$. Hence t_λ satisfies

$$a t_\lambda^p\|v\|^p - b t^{2p}\|v\|^{2p} = \lambda t_\lambda^q\int_\Omega |v|^q\,dx + \int_\Omega g(x, t_\lambda v)t_\lambda v\,dx. \tag{3.5}$$

Then, by (g_3), we have

$$at_\lambda^p\|v\|^p \geq bt^{2p}\|v\|^{2p} + \lambda t_\lambda^q \int_\Omega |v|^q\,dx - C|\Omega|. \tag{3.6}$$

This fact implies that $\{t_\lambda\}_{\lambda>0}$ is bounded since $q > p$.

In the following, we prove that

$$t_\lambda \to 0 \quad \text{as } \lambda \to \infty. \tag{3.7}$$

Arguing by contradiction, we can assume that there exist $t_0 > 0$ and a sequence $\{\lambda_n\}_n$ with $\lambda_n \to \infty$ as $n \to \infty$ such that $t_{\lambda_n} \to t_0$ as $n \to \infty$. By the Lebesgue dominated convergence theorem, we deduce that

$$\int_\Omega |t_{\lambda_n} v|^q\,dx \to \int_\Omega |t_0 v|^q\,dx \quad \text{as } n \to \infty,$$

from which it follows that

$$\lambda_n \int_\Omega |t_{\lambda_n} v|^q\,dx \to +\infty \quad \text{as } n \to \infty.$$

However, (3.5) implies that this fact is absurd. Hence (3.7) holds.

From (3.7) and the definition of J_λ, we get

$$\lim_{\lambda\to\infty}\Bigl(\sup_{t\geq 0} J_\lambda(tv)\Bigr) = \lim_{\lambda\to\infty} J_\lambda(t_\lambda v) = 0.$$

So, for any $\lambda > \lambda_0 > 0$, we have

$$\sup_{t\geq 0} J_\lambda(tv) < \frac{a^2}{2pb}.$$

If we take $e = \tau v$ with τ large enough to verify $J_\lambda(e) < 0$, then we obtain

$$c_\lambda \leq \max_{t\in[0,1]} J_\lambda\bigl(\gamma(t)\bigr) \quad \text{by taking } \gamma(t) = t\tau\omega.$$

Therefore

$$c_\lambda \leq \sup_{t\geq 0} J_\lambda(t\omega) < \frac{a^2}{2pb} \quad \text{for } \lambda \text{ large enough.}$$

Clearly, Lemmas 2.1, 3.1, and Theorem 3.1 give the existence of nontrivial critical points of J_λ, and this concludes the proof. □

4 Proof of Theorems 1.2 and 1.3

To prove Theorems 1.2 and 1.3, we shall use the following symmetric mountain pass theorem in [17].

Theorem 4.1 (see [17]) *Let $X = Y \oplus Z$ be an infinite dimensional Banach space, where Y is finite dimensional, and let $J \in C^1(X,\mathbb{R})$. Suppose that*

(I_1) *J satisfies the $(PS)_c$ condition for all $c > 0$;*
(I_2) *$J(0) = 0$, $J(-u) = J(u)$ for all $u \in X$;*
(I_3) *There exist constants $\rho, \alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in \partial B_\rho \bigcap Z$;*
(I_4) *For any finite dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{X}) > 0$ such that $J(u) \leq 0$ on $\widetilde{X} \setminus B_R$.*

Then J possesses an unbounded sequence of critical values.

Proof of Theorem 1.2 We shall apply Theorem 4.1 to J_λ. On the one hand, we know that $W_0^{1,p}(\Omega)$ is a Banach space and $J_\lambda \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$. The functional J_λ satisfies $J_\lambda(0) = 0$, $J_\lambda(-u) = J_\lambda(u)$. So, ($I_2$) of Theorem 4.1 is satisfied.

On the other hand, using a similar discussion as in (3.4), there exists $\lambda_2 > 0$ such that

$$c_\lambda < \frac{a^2}{2pb} \quad \text{for all } \lambda > \lambda_2.$$

By Lemma 2.1, one knows that J_λ satisfies the $(PS)_c$ condition. Then (I_1) of Theorem 4.1 is satisfied.

In the following, we will prove that assumptions (I_3) and (I_4) of Theorem 4.1 are satisfied.

In fact, the proof of (I_3) is similar to the proof of (i) in Lemma 3.1. So, we omit the proof here.

In order to prove (I_4) of Theorem 4.1, we take $\widetilde{X}$ is the finite dimensional subspace of Banach space X. By (3.2), we deduce that

$$J_\lambda(u) \leq \frac{a}{p}\|u\|^p - \frac{b}{2p}\|u\|^{2p} - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - T\int_\Omega |u|^\theta\,dx + C_T\Omega. \tag{4.1}$$

Because all the norms on the finite dimensional subspace $\widetilde{X}$ are equivalent, there exists $C > 0$ such that

$$\int_\Omega |u|^q\,dx \geq C_{\widetilde{X}}\|u\|^q \quad \text{and} \quad \int_\Omega |u|^\theta\,dx \geq C_{\widetilde{X}}\|u\|^\theta.$$

From these and (4.1), we have

$$J_\lambda(u) \leq \frac{a}{p}\|u\|^p - \frac{b}{2p}\|u\|^{2p} - \frac{\lambda}{q}C_{\widetilde{X}}\|u\|^q - TC_{\widetilde{X}}\|u\|^\theta + C_T|\Omega|.$$

Thus, there exists $R = R(\widetilde{X})$ such that we can derive $J_\lambda(u) < 0$ for all $u \in \widetilde{X}$ with $\|u\| \geq R$. Consequently, (I_4) of Theorem 4.1 is satisfied.

Since all the assumptions of Theorem 4.1 are satisfied, problem (1.1) possesses infinitely many nontrivial solutions with unbounded energy. □

Proof of Theorem 1.3 If g satisfies conditions (g_1)–(g_2), (g_3^*), and (g_3^{**}). First, we claim that the functional J_λ satisfies the $(PS)_c$ condition, where $c \in (-\infty, \frac{a^2}{2pb})$.

Indeed, we take $\{u_n\} \subset W_0^{1,p}(\Omega)$ to be a $(PS)_c$ sequence associated with the functional J_λ, that is, $J_\lambda(u_n) \to c$ and $J'_\lambda(u_n) \to 0$ as $n \to \infty$. By using (g_3^*) and (g_3^{**}), we have

$$\begin{aligned} c + o(1)\|u_n\| &= J_\lambda(u_n) - \frac{1}{p}\langle J'_\lambda(u_n), u_n\rangle \\ &= \frac{b}{2p}\|u_n\|^{2p} + \left(\frac{1}{p} - \frac{1}{q}\right)\lambda \int_\Omega |u_n|^q \, dx + \frac{1}{p}\int_\Omega \mathcal{G}(x, u_n) \, dx \\ &\geq \frac{b}{2p}\|u_n\|^{2p} + \int_\Omega \big(\mathcal{G}(x,0) - \sigma(x)\big) \, dx. \end{aligned}$$

This fact implies that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ since $1 < p$, $\sigma \in L^1(\Omega)$ and $\mathcal{G}(x,0) \in L^1(\Omega)$. Arguing as in the proof of Lemma 2.1, we can also prove $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$, we omit the details here.

Next, we claim that assumptions (I_3) and (I_4) of Theorem 4.1 are also satisfied.

In fact, it is easy to prove that (I_3) of Theorem 4.1 is satisfied.

In the following, we prove that (I_4) of Theorem 4.1 is satisfied. On the one hand, by (g_3^*), we have

$$G(x,u) \geq C|u|^p - C^*, \quad \forall (x,u) \in \Omega \times \mathbb{R}.$$

Hence, we deduce that

$$J_\lambda(u) \leq \frac{a}{p}\|u\|^p - \frac{b}{2p}\|u\|^{2p} - \frac{\lambda}{q}\int_\Omega |u|^q \, dx - C\int_\Omega |u|^p \, dx + C|\Omega|. \tag{4.2}$$

Similarly, arguing as in the proof of Theorem 1.2, we know that (I_4) of Theorem 4.1 is satisfied. Thus, problem (1.1) possesses infinitely many nontrivial solutions with unbounded energy. □

Acknowledgements
The authors are very grateful to the editors for the careful reading of this manuscript and offering valuable comments for this manuscript.

Funding
Z. Zhang was supported by 2020 Nanjing Vocational College of Information Technology Doctoral special fund project "Study on the Climate Ensemble Forecast Model Based on the East Asian Monsoon Region" (YB20200902). Y.Q. Song was supported by the National Natural Science Foundation of China (No. 12001061), the Research Foundation of Department of Education of Jilin Province (JJKH20200821KJ).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in this paper. Both authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 2 July 2021 Accepted: 12 October 2021 Published online: 02 December 2021

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