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Existence and uniqueness of a finite energy solution for the mixed value problem of porous thermoelastic bodies

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Abstract

We consider the mixed problem with boundary and initial data in thermoelasticity of porous bodies with dipolar structure. By generalizing some known results developed by Dafermos in a more simple case of the classical theory of elasticity, we prove new theorems in which we address the issues regarding the uniqueness and existence of a solution with finite energy of the respective problem after we define this type of solution.

Keywords: Dipolar bodies; Pores; Solution with finite energy; Existence of solution; Uniqueness of solution

1 Introduction

Specialists know that the intrinsic properties of different media can influence the amplitude of the thermal type stress. Considering that our work is dedicated to the porous thermoelastic bodies, we believe that our approach can help the engineers specializing in porous materials, at least in the applications that concern the geological layers or in the production of granular bodies. The research in the field of theory of bodies with pores began with the publication of the study [1] by Goodman and Cowin. In this paper, as in the study Cowin and Nunziato [2], the new theory is based on adding an extra degree of freedom which is meant to describe the mechanical behavior of solids with pores. For this kind of solids, the interstices are voids of material, and the material itself is elastic. There are many concrete situations in which this theory is useful, such as pressed powders, ceramics, in manufacture of porous materials (such as polystyrene), in the study of geological materials, such as soils and rocks. At the beginning, in the works [1–3], this theory referred to elastic environments in which the presence of temperature was not taken into account. It was natural for the analysis to extend to address thermoelastic bodies with pores, and this generalization has been approached in many studies, of which we mention only the work [4] of Iesan. Then, the specialists extended the theory taking into account other effects. It is worth mentioning in this sense the pioneering works of Eringen dedicated to a new and very general concept, that of the microstructure of media, see, for instance, the studies [5] and [6]. Several particular cases of this general theory have been considered, first by Erin-

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gen, then by a very large number of researchers. This includes the theory of micropolar media, microstretch media, media with dipolar structure, and so on. The number of studies dedicated to these types of structures has continuously increased. We suggest only some of them [7–19]. Given that our study addresses environments with dipolar structure, it can be deduced that it falls within the line mentioned above. On the other hand, we have other generalizations in our study, namely the results obtained by Dafermos [20] and Fichera [21] in the much simpler context of classical elastic media. We prove some theorems in which we deduce the uniqueness of a finite energy solution, the existence of such kind of solution, but also some estimates from which the stability of asymptotic type of the solution with finite energy is obtained. The plane of our study is as follows. After we highlight the basic equations and the specific initial and boundary relations for the mixed problem from the context formulated above, we define the solutions with finite energy and propose a method to obtain other types of solutions with finite energy. It is important to specify that we prove the uniqueness of a solution in the most general situation in which both the initial data and the boundary conditions are inhomogeneous. In the last theorem we obtain the existence of at least one finite solution in the most general case of inhomogeneous initial relations, starting from the particular previously demonstrated case.

2 Basic equations and conditions

Suppose that a regular region Ω from the Euclidian space is occupied at the initial time $t = 0$ by a thermoelastic porous body having a dipolar structure. The boundary surface of Ω is denoted by Σ , and suppose that this surface is regular enough, so we can apply the theorem of divergence. We also have that $\bar{\Omega} = \Omega \cup \Sigma$, $\bar{\Omega}$ being the closure of Ω . The evolution of our solid is described by reference to the usual system of axes Ox_i ($i = 1, 2, 3$). We adopt the usual notation for tensors and vectors. A superposed dot on a function is used for the derivative of a respective function with respect to a time variable. A comma followed by an index designates the derivative (partial) regarding the corresponding with respect to the respective spatial variable. The rule of Einstein summation is used whenever the index is repeated.

The density of mass in the initial state (undeformed) is denoted by ρ_0 . Also, the volume fraction in the initial configuration is denoted by ν_0 , while the density of a material matrix has the notation γ_0 . These two quantities are constant regarding the spatial variables, but they depend on the temporal variable and satisfy the following relation:

$$\rho_0 = \gamma_0 \nu_0.$$

To characterize the evolution of a thermoelastic porous body having a dipolar structure, the following independent functions are used:

- $v_i(x, t)$ – components of the vector of displacement, regarding the initial state;
- $\phi_{ij}(x, t)$ – components of the dipolar displacement tensor;
- ϑ – the variation of temperature, between its present value T and the value in the initial state T_0 , that is,

$$\vartheta = T - T_0;$$

- ϕ – the difference between the present v and the initial volume fraction v_0 , that is,

$$\phi = v - v_0.$$

The components of the tensors of strain, namely ε_{ij} and γ_{ij} , are functions of the above variables $v_i(x, t)$, $\phi_{jk}(x, t)$, namely

$$e_{ij} = \frac{1}{2}(v_{j,i} + v_{i,j}), \quad \varepsilon_{ij} = v_{j,i} - \phi_{ij}, \quad \gamma_{ijk} = \phi_{jk,i}. \tag{1}$$

We suppose that our media has zero flux rate and zero intrinsic equilibrated volume forces, and also it has no initial tension and couple stress.

Our subsequent considerations are made only in a linear context, therefore we must assume that the internal density of energy is a form of quadratic order, regarding all its independent functions. As a consequence, the energy conservation principle is useful to make explicit the internal density of energy in the form that follows:

$$\begin{aligned} \Psi = & \frac{1}{2}A_{ijmn}e_{ij}e_{mn} + G_{ijmn}e_{ij}\varepsilon_{mn} + F_{ijmnr}e_{ij}\gamma_{mnr} \\ & + \frac{1}{2}B_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + D_{ijmnr}\varepsilon_{ij}\gamma_{mnr} + \frac{1}{2}C_{ijkmnr}\gamma_{ijk}\gamma_{mnr} \\ & + a_{ijk}e_{ij}\varphi_{,k} + b_{ijk}\varepsilon_{ij}\varphi_{,k} + c_{ijkm}\gamma_{ijk}\varphi_{,m} - a_i\vartheta_{,i} - \frac{1}{2}c\vartheta^2 \\ & - \alpha_{ij}e_{ij}\vartheta - \beta_{ij}\varepsilon_{ij}\vartheta - \delta_{ijk}\gamma_{ijk}\vartheta + \frac{1}{2}d_{ij}\varphi_{,i}\varphi_{,j} + \frac{1}{2}\kappa_{ij}\vartheta_{,i}\vartheta_{,j}. \end{aligned} \tag{2}$$

Using a suggestion given by Nunziato and Cowin [3], we can deduce

$$\begin{aligned} \tau_{ij} &= \frac{\partial \Psi}{\partial e_{ij}}, & t_{ij} &= \frac{\partial \Psi}{\partial \varepsilon_{ij}}, & m_{ijk} &= \frac{\partial \Psi}{\partial \gamma_{ijk}}, \\ h_i &= \frac{\partial \Psi}{\partial \varphi_{,i}}, & S &= -\frac{\partial \Psi}{\partial \vartheta}, & q_i &= \frac{\partial \Psi}{\partial \vartheta_{,i}}, \end{aligned}$$

and we deduce the relations between the tensors of stress and the tensors of strain, in other words, the constitutive relations

$$\begin{aligned} \tau_{ij} &= A_{ijmn}e_{mn} + G_{mnij}\varepsilon_{mn} + F_{mnrj}\gamma_{mnr} + a_{ijk}\varphi_{,k} - \alpha_{ij}\vartheta, \\ t_{ij} &= G_{ijmn}e_{mn} + B_{ijmn}\varepsilon_{mn} + D_{ijmnr}\gamma_{mnr} + b_{ijk}\varphi_{,k} - \beta_{ij}\vartheta, \\ m_{ijk} &= F_{ijkmn}e_{mn} + D_{mnijk}\varepsilon_{mn} + C_{ijkmnr}\gamma_{mnr} + c_{ijk}\varphi_{,r} - \delta_{ijk}\vartheta, \\ h_i &= a_{ijk}e_{jk} + b_{ijk}\varepsilon_{jk} + c_{ijk}\gamma_{jkr} + d_{ij}\varphi_{,j} - a_i\vartheta, \\ S &= \alpha_{ij}e_{ij} + \beta_{ij}\varepsilon_{ij} + c_{ijk}\gamma_{ijk} + a_i\varphi_{,i} + c\vartheta, \\ q_i &= \kappa_{ij}\vartheta_{,j}. \end{aligned} \tag{3}$$

Taking into account that the strain tensor e_{ij} is symmetric (see Eq. (1)₁), we easily deduce that the symmetry relations satisfied by the above constitutive coefficients are

$$A_{jkmn} = A_{kijm} = A_{mnjk}, \quad G_{jkmn} = G_{kjmn}, \quad F_{jkmnr} = F_{kjmnr},$$

$$C_{ijklmnr} = C_{mnrjkl}, \quad d_{mn} = d_{nm}, \quad \alpha_{mn} = \alpha_{nm}, \quad \kappa_{mn} = \kappa_{nm}. \tag{4}$$

Using the same procedure of Nunziato and Cowin [3], we can obtain the following main equations (see also [22]):

- the equations of motion:

$$\begin{aligned} (t_{ij} + \tau_{ij})_{,j} + \varrho f_i &= \varrho \ddot{v}_i, \\ m_{ijk,i} + \tau_{jk} + \varrho g_{jk} &= I_{jm} \ddot{\phi}_{km}. \end{aligned} \tag{5}$$

- the balance of the equilibrated forces:

$$h_{i,i} + \varrho l = \varrho k \ddot{\varphi}; \tag{6}$$

- the balance of the energy:

$$\varrho T_0 \dot{S} = q_{i,i} + \varrho r. \tag{7}$$

In the previous relations, the significance of the following notations remained unspecified: S – the entropy per unit mass, k – the balancing inertia, I_{ij} – the tensor of inertia, h_i – an equilibrium vector of stress, q_i – the vector of heat flux, f_i, g_i, l – body forces, r – the supply of heat. In our following approaches we consider anisotropic material which is assumed to be inhomogeneous.

The entropy production inequality can be used in order to obtain the positive semi-definition of the thermal conductivity tensor κ_{ij} , namely

$$\kappa_{ij} \vartheta_{,i} \vartheta_{,j} \geq 0. \tag{8}$$

It is easy to see that the main equations (5) and (7) are analogous to those from the theory of classical thermoelasticity.

In [23], by using a variational approach, the authors gave a motivation for the new relation (6), the balance of equilibrated forces.

We find similar considerations in [22] and [24].

Considering the constitutive relations (3), we can transform the basic relations (5), (6), and (7) into the next system of equations with partial derivatives with respect to the variables v_m, ϕ_{mn}, φ , and ϑ :

$$\begin{aligned} &(A_{ijmn} + E_{ijmn})v_{n,mj} + (E_{mnij} + B_{ijmn})(v_{n,mj} - \phi_{mn,j}) \\ &\quad + (F_{ijklm} + D_{ijklm})\phi_{lm,kj} + (a_{ijk} + b_{ijk})\varphi_{,kj} - (\alpha_{ij} + \beta_{ij})\vartheta_{,j} + \rho f_i = \rho \ddot{v}_i, \\ &F_{klmnn}v_{n,mj} + D_{mnijkl}(v_{n,mj} - \phi_{mn,j}) + C_{kljmnr}\phi_{nr,mj} + c_{ijkl}\varphi_{,ij} - \delta_{klj}\vartheta_{,j} \\ &\quad + E_{klmnn}v_{m,n} + B_{klmn}(v_{n,m} - \phi_{mn}) + D_{klmnr}\phi_{nr,m} + b_{jkl}\varphi_{,j} - \beta_{kl}\vartheta = I_{kr}\ddot{\phi}_{lr}, \\ &a_{ijk}v_{j,ki} + b_{ijk}(\dot{v}_{j,ki} - \dot{\phi}_{jk,i}) + c_{ijk}\phi_{jk,ri} + d_{ij}\phi_{,ij} - a_i\vartheta_i + \rho l = \rho k \ddot{\varphi}, \\ &\alpha_{ij}\dot{v}_{i,j} + \beta_{ij}(\dot{v}_{i,j} - \dot{\phi}_{ij}) + \delta_{ijk}\dot{\phi}_{ij,k} + a_i\dot{\varphi}_i + c\dot{\vartheta} = \frac{1}{\rho}\kappa_{ij}\vartheta_{,ij} + r, \end{aligned} \tag{9}$$

which are satisfied for any $(x) \in \Omega \times (0, \infty)$.

We intend to construct a mixed problem attached to the system of equations (6). To this aim we need to prescribe some initial and boundary relations. For $t = 0$, we have the initial data:

$$\begin{aligned} v_m(0) &= v_m^0, & \dot{v}_m(0) &= v_m^1, & \phi_{kl}(0) &= \phi_{kl}^0, & \dot{\phi}_{kl}(0) &= \phi_{kl}^1, \\ \varphi(0) &= \varphi^0, & \dot{\varphi}(0) &= \varphi^1, & \vartheta(0) &= \vartheta^0, & & \text{in } \bar{B}. \end{aligned} \tag{10}$$

Denoting by $n = (n_i)$ the normal of the surface Σ which is a unit vector, outward oriented regarding Σ , we can consider the following surface tractions:

$$t_i = (\tau_{ij} + t_{ij})n_j, \quad m_{jk} = m_{ijk}n_i, \quad h = h_i n_i, \quad q = q_i n_i,$$

such that, for $t \in [0, t_0)$, time t_0 can be ∞ , we can prescribe the boundary data as follows:

$$\begin{aligned} u_i &= \bar{u}_i \quad \text{on } \Sigma_1, & t_i &= \bar{t}_i \quad \text{on } \Sigma_1^c, \\ \phi_{jk} &= \bar{\phi}_{jk} \quad \text{on } \Sigma_2, & m_{jk} &= \bar{m}_{jk} \quad \text{on } \Sigma_2^c, \\ \varphi &= \bar{\varphi} \quad \text{on } \Sigma_3, & h &= \bar{h} \quad \text{on } \Sigma_3^c, \\ \vartheta &= \bar{\vartheta} \quad \text{on } \Sigma_4, & q &= \bar{q} \quad \text{on } \Sigma_4^c. \end{aligned} \tag{11}$$

Here $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ and their respective complements $\Sigma_1^c, \Sigma_2^c, \Sigma_3^c, \Sigma_4^c$ are subsurface of the surface Σ such that

$$\begin{aligned} \Sigma_1 \cup \Sigma_1^c &= \Sigma_2 \cup \Sigma_2^c = \Sigma_3 \cup \Sigma_3^c = \Sigma_4 \cup \Sigma_4^c = \Sigma, \\ \Sigma_1 \cap \Sigma_1^c &= \Sigma_2 \cap \Sigma_2^c = \Sigma_3 \cap \Sigma_3^c = \Sigma_4 \cap \Sigma_4^c = \emptyset. \end{aligned}$$

Also, $v_m^0, v_m^1, \phi_{jk}^0, \phi_{jk}^1, \varphi^0, \varphi^1, \vartheta^0, \bar{v}_m, \bar{t}_i, \bar{\phi}_{jk}, \bar{m}_{jk}, \bar{\varphi}, \bar{\vartheta}, \bar{q}$, and \bar{h} are given and regular functions.

Let us denote by \mathcal{P} the mixed problem in our context of thermoelastic dipolar bodies with pores, consisting of the basic equations (9), the initial conditions (10), and the boundary conditions (11). As such, an ordered array $(v_m, \phi_{jk}, \varphi, \vartheta)$ is a solution of this problem if it satisfies equations (9) and conditions (10) and (11).

3 Problem formulation and basic results

Let us denote by \mathcal{P} the mixed problem in our context of thermoelastic dipolar bodies with pores, consisting of the basic equations (9), the initial conditions (10), and the boundary conditions (11). As such, an ordered array $(v_m, \phi_{jk}, \varphi, \vartheta)$ is a solution of this problem if it satisfies equations (9) and conditions (10) and (11).

We now present the regularity conditions necessary to obtain our subsequent results. As usual, we will use the notation $C^n(\bar{D})$ for the set of all functions defined in all points of the domain Ω having a derivative of order n on Ω , and this is a continuous function on the domain $\bar{\Omega}$.

The norm of any function u from the space $C^n(\bar{\Omega})$ is defined by

$$\|u\|_{C^n(\bar{\Omega})} = \sum_{m=0}^n \sum_{k_1, k_2, \dots, k_m} \max |u_{,k_1 k_2 \dots k_m}|.$$

In the case of vector functions (in our situation, with thirteen components), we consider the notation $C^n(\bar{\Omega})$, and any element from this set is from $C^n(\bar{\Omega})$. As a consequence, the norm of a vector function $f = (f_i), i = \overline{1, 13}$, from the space $C^n(\bar{\Omega})$ is defined by

$$\|f\|_{C^n(\bar{\Omega})} = \sum_{i=1}^{13} \|f_i\|_{C^n(\bar{\Omega})}.$$

Let us denote by $H_n(\Omega)$ the completion of the space $C^n(\bar{\Omega})$, and consider the scalar product

$$(f, g)_{H_n(\Omega)} = \sum_{k=0}^n \int_{\Omega} f_{i_1 i_2 \dots i_k} g_{i_1 i_2 \dots i_k} dV,$$

which induces a Hilbert space structure on the space $H_n(\Omega)$ (see, for instance, [25]).

Analogously, if we denote by $\mathbf{H}_n(\Omega)$ the completion of the space $C^n(\bar{\Omega})$ and consider the scalar product

$$(\mathbf{f}, \mathbf{g})_{\mathbf{H}_n(\Omega)} = \sum_{k=1}^{13} (f_k, g_k)_{H_n(\Omega)}, \quad \mathbf{f} = (f_i), \quad \mathbf{g} = (g_i),$$

we obtain a Hilbert space structure on the set $\mathbf{H}_n(\Omega)$.

Other sets of functions that we will use have the following meanings:

$$\begin{aligned} \hat{C}^1(\Omega) &= \{ \vartheta, \vartheta \in C^1(\bar{\Omega}) : \vartheta = 0 \text{ on } \bar{\Sigma}_4 \}, \\ \hat{C}^1(\Omega) &= \{ \mathbf{u} = (\mathbf{v}, \phi, \varphi), \mathbf{u} \in C^1(\bar{\Omega}) : v_i = 0 \text{ on } \bar{\Sigma}_1, \\ &\quad \phi_{ij} = 0 \text{ on } \bar{\Sigma}_2, \varphi = 0 \text{ on } \bar{\Sigma}_3 \}, \\ \tilde{C}^0(\Omega) &= \left\{ (f_m, g_{jk}, l, r) : (f_m, g_{jk}, l) \in C^0(\bar{\Omega}), r \in C^0(\bar{\Omega}); \right. \\ &\quad \text{if } \text{meas}(\bar{\Sigma}_1) = \text{meas}(\bar{\Sigma}_2) = \text{meas}(\bar{\Sigma}_3) = 0, \\ &\quad \text{then } \int_{\Omega} \rho F dV = 0, \int_{\Omega} \rho(x \times F) dV = 0; \\ &\quad \text{if } \text{meas}(\bar{\Sigma}_1) = \text{meas}(\bar{\Sigma}_2) = 0, \text{meas}(\bar{\Sigma}_3) \neq 0, \text{ then } \int_{\Omega} \rho F dV = 0; \\ &\quad \left. \text{if } \text{meas}(\bar{\Sigma}_4) = 0, \text{ then } \int_{\Omega} \rho r dV = 0 \right\}; \\ \tilde{C}^1(\Omega) &= \left\{ (u = (v_m, \phi_{jk}, \phi), \vartheta) : (u, \vartheta) \in \hat{C}^1(\Omega) \times \hat{C}^1(\Omega) \text{ and} \right. \\ &\quad \left. \text{if } \text{meas}(\bar{\Sigma}_4) = 0, \text{ then } \int_{\Omega} [\alpha_{ij} e_{ij} + \beta_{ij} \varepsilon_{ij} + c_{ijk} \gamma_{ijk} + a_i \varphi_{,i} + c \vartheta] dV = 0 \right\}. \end{aligned}$$

We will now introduce some functionals that are necessary both for defining a finite energy solution and for demonstrating the previously mentioned results.

$$\begin{aligned} E_1((\mathbf{v}, \phi, \varphi), (\mathbf{w}, \psi, \chi)) \\ = \frac{1}{2} \int_{\Omega} \{ A_{ijmn} e_{ij}(\mathbf{v}, \phi) e_{mn}(\mathbf{w}, \psi) + G_{ijmn} [e_{ij}(\mathbf{v}, \phi) \varepsilon_{mn}(\mathbf{w}, \psi) \end{aligned}$$

$$\begin{aligned}
 & + e_{ij}(\mathbf{w}, \boldsymbol{\psi})\varepsilon_{mn}(\mathbf{v}, \boldsymbol{\phi})] + F_{ijmnr}[e_{ij}(\mathbf{v}, \boldsymbol{\phi})\gamma_{mnr}(\mathbf{w}, \boldsymbol{\psi}) + e_{ij}(\mathbf{w}, \boldsymbol{\psi})\gamma_{mnr}(\mathbf{v}, \boldsymbol{\phi})] \\
 & + B_{ijmn}\varepsilon_{ij}(\mathbf{v}, \boldsymbol{\phi})\varepsilon_{mn}(\mathbf{w}, \boldsymbol{\psi}) + D_{ijmnr}[\varepsilon_{ij}(\mathbf{v}, \boldsymbol{\phi})\gamma_{mnr}(\mathbf{w}, \boldsymbol{\psi}) + \varepsilon_{ij}(\mathbf{w}, \boldsymbol{\psi})\gamma_{mnr}(\mathbf{v}, \boldsymbol{\phi})] \\
 & + C_{ijkmnr}\gamma_{ijk}(\mathbf{v}, \boldsymbol{\phi})\gamma_{mnr}(\mathbf{w}, \boldsymbol{\psi}) + d_{ij}\varphi_j\chi_{,i} + a_{ijk}[e_{ij}(\boldsymbol{\phi})\varphi_{,k} + e_{ij}(\boldsymbol{\psi})\chi_{,k}] \\
 & + b_{ijk}[\varepsilon_{ij}(\boldsymbol{\phi})\varphi_{,k} + \varepsilon_{ij}(\boldsymbol{\psi})\chi_{,k}] + c_{ijkm}[\gamma_{ijk}(\boldsymbol{\phi})\varphi_{,m} + \gamma_{ijk}(\boldsymbol{\psi})\chi_{,m}] \} dV, \\
 E_2(\vartheta, T) &= \int_{\Omega} \kappa_{ij}\vartheta_{,i}T_{,j} dV, \\
 E_3(z, w) &= \int_0^{t_0} \int_{\Omega} \left\{ (t - t_0)[\varrho\dot{v}_m\ddot{w}_m + I_{jk}\dot{\phi}_{jm}\ddot{\psi}_{km} + \varrho\kappa\dot{\phi}\ddot{\chi} - A_{ijmn}e_{mn}(y)\dot{e}_{ij}(w) \right. \\
 & - G_{ijmn}(e_{ij}(z)\dot{e}_{mn}(w) + \dot{e}_{ij}(w)\varepsilon_{mn}(y)) - F_{ijmnr}(e_{ij}(z)\dot{\gamma}_{mnr}(w) \\
 & + \dot{e}_{ij}(w)\gamma_{mnr}(z)) \\
 & - B_{ijmn}\varepsilon_{ij}(z)\varepsilon_{mn}(w) - D_{ijmnr}(\varepsilon_{ij}(z)\gamma_{mnr}(w) + \varepsilon_{ij}(w)\gamma_{mnr}(z)) \\
 & - C_{ijkmnr}\gamma_{ijk}(z)\gamma_{mnr}(w) - a_{ijk}(\dot{e}_{ij}(w)\varphi_{,k} + e_{ij}(y)\dot{\chi}_{,k}) \\
 & - b_{ijk}(\dot{e}_{ij}(w)\varphi_{,k} + \varepsilon_{ij}(y)\dot{\chi}_{,k}) - c_{ijkm}(\dot{\gamma}_{ijk}(w)\varphi_{,m} + \gamma_{ijk}(y)\dot{\chi}_{,m}) \\
 & + d_{ij}(\varphi_i\dot{\chi}_{,j} + \dot{\varphi}_{,i}\chi_{,j}) + \alpha_{ij}(\vartheta\dot{e}_{ij}(w) + \dot{T}e_{ij}(y)) + \beta_{ij}(\vartheta\dot{e}_{ij}(w) + \dot{T}\dot{e}_{ij}(y)) \\
 & + c(\vartheta\dot{T} + \dot{\vartheta}T) + \delta_{ijk}(\vartheta\dot{\gamma}_{ijk}(w) + \dot{T}\gamma_{ijk}(y)) + a_i(\dot{\varphi}_{,i}\vartheta + \dot{T}\chi_{,i}) \\
 & \left. + \varrho\dot{v}_m\ddot{w}_m + I_{jk}\dot{\phi}_{jm}\dot{\psi}_{km} + \varrho\kappa\dot{\phi}\dot{\chi} + \frac{1}{T_0} \int_0^t \kappa_{ij}\vartheta_{,j}T_{,i} d\tau \right\} dV dt, \\
 E_4(y, w) &= - \int_0^{t_0} \int_{\Omega} (t - t_0) \left(\varrho f_m\dot{w}_m + I_{jk}g_{jm}\dot{\psi}_{km} + \varrho kl\dot{\chi} + \frac{\rho r}{T_0} T \right) dV dt, \\
 E_5(\zeta, w) &= t_0 \int_{\Omega} [\varrho v_m^0 w_m|_{t=0} + I_{jk}\phi_{jm}^0 \psi_{km}|_{t=0} + \varrho k\chi|_{t=0}\phi^0 + cT|_{t=0}\vartheta^0 \\
 & + (\alpha_{mn}e_{mn}(v^0, \vartheta^0) + \beta_{mn}\varepsilon_{mn}(v^0, \vartheta^0) + \delta_{mnr}\gamma_{mnr}(v^0, \vartheta^0) \\
 & + a_m\varphi_{,m})T|_{t=0}] dV.
 \end{aligned} \tag{12}$$

The first functional is defined for any pairs of functions $(\mathbf{u}, \boldsymbol{\varphi}, \phi) \in \hat{C}^1(\Omega)$, $(\mathbf{v}, \boldsymbol{\psi}, \chi) \in \hat{C}^1(\Omega)$, and the second functional corresponds to the pair of functions (T, ϑ) , both from $\hat{C}^1(\Omega)$.

The functionals $E_3(z, w)$, $E_4(y, w)$, and $E_5(\zeta, w)$ are defined for the elements

$$\begin{aligned}
 z &= (v_m, \phi_{jk}, \varphi, \vartheta) \in C^\infty([0, t_0]; \tilde{C}^1(\Omega)), \\
 w &= (w_m, \psi_{jk}, \chi, T) \in C^\infty([0, t_0]; \tilde{C}^1(\Omega)), \\
 y &= (f_m, g_{jk}, l, r) \in C^\infty([0, t_0]; \tilde{C}^0(\Omega)), \\
 \zeta &= (v_m^0, \phi_{jk}^0, \varphi^0, \vartheta^0), \quad v_m^0, \phi_{jk}^0 \in \hat{C}^1(\Omega), \varphi^0, \vartheta^0 \in \hat{C}^1(\Omega),
 \end{aligned}$$

respectively.

To obtain the proposed outcomes, we must force the functional E_1 to satisfy the following condition.

There is a positive constant c_1 so that, for any $(\mathbf{w}, \boldsymbol{\psi}, \chi) \in \hat{\mathbf{C}}^1(\Omega)$, we have

$$\begin{aligned}
 & E_1((\mathbf{w}, \boldsymbol{\psi}, \chi), (\mathbf{w}, \boldsymbol{\psi}, \chi)) \\
 & \geq c_1 \int_{\Omega} \sum_{m,n,r=1}^3 [e_{mn}^2(\mathbf{v}, \boldsymbol{\psi}, \chi) + \varepsilon_{mn}^2(\mathbf{v}, \boldsymbol{\psi}, \chi) + \gamma_{mnr}^2(\mathbf{v}, \boldsymbol{\psi}, \chi) + \chi_{,i}\chi_{,j}] dV. \tag{13}
 \end{aligned}$$

Also, for any $\vartheta \in \hat{\mathbf{C}}^1(\Omega)$, there is positive constant c_2 so that E_2 satisfies the following inequality:

$$E_2(\vartheta, \vartheta) \geq c_2 \int_{\Omega} \kappa_{ij} \vartheta_{,i} \vartheta_{,j} dV. \tag{14}$$

Regarding the material properties, we must impose the conditions

$$c > 0, \quad T_0 > 0, \quad \rho > 0, \quad I_{jk} > 0. \tag{15}$$

With a suggestion from [26, 27], we deduce that there is a positive constant c_3 such that E_1 satisfies the following inequality:

$$\begin{aligned}
 & E_1((\mathbf{v}, \boldsymbol{\phi}, \varphi), (\mathbf{v}, \boldsymbol{\phi}, \varphi)) \\
 & \geq c_3 \int_D [v_m v_m + \phi_{jk} \phi_{jk} + \varphi_{,m} \varphi_{,m} + v_{m,n} v_{m,n} + \phi_{jk,l} \phi_{jk,l} + \varphi_{,mn} \varphi_{,mn}] dV, \tag{16}
 \end{aligned}$$

so that we are led to the conclusion that $E_1((\mathbf{v}, \boldsymbol{\psi}, \chi), (\mathbf{v}, \boldsymbol{\psi}, \chi))$ is a coercive functional on the Hilbert space $\hat{\mathbf{H}}_1(\Omega)$.

In what follows the next identity will be useful which is satisfied by $E_3(z, w)$, by replacing z with w :

$$\begin{aligned}
 E_3(w, w) = & \int_0^{t_0} \int_{\Omega} \left\{ (t - t_0) \left[\varrho \dot{w}_m \dot{w}_m + I_{jk} \dot{\psi}_{jm} \dot{\psi}_{km} + \rho k \dot{\chi}^2 \right. \right. \\
 & + A_{ijmn} e_{mn}(w) \dot{e}_{ij}(w) + 2G_{ijmn} e_{ij}(w) \varepsilon_{mn}(w) \\
 & + 2F_{ijmnr} e_{ij}(w) \gamma_{mnr}(w) + B_{ijmn} \varepsilon_{ij}(w) \varepsilon_{mn}(w) \\
 & + D_{ijmnr} \varepsilon_{ij}(w) \gamma_{mnr}(w) + C_{ijkmnr} \gamma_{ijk}(w) \gamma_{mnr}(w) \\
 & + 2a_{ijk} e_{ij}(w) \varphi_{,k} + 2b_{ijk} \varepsilon_{ij}(w) \varphi_{,k} + 2c_{ijkm} \gamma_{ijk}(w) \varphi_{,m} \\
 & \left. + d_{ij} \chi_{,i} \chi_{,j} + cT^2 + \frac{1}{T_0} \int_0^t \kappa_{ij} T_{,j} T_{,i} d\tau \right] dV \\
 & \left. + \frac{t_0}{2} \int_{\Omega} (\rho \dot{w}_m \dot{w}_m + I_{jk} \dot{\psi}_{jm} \dot{\psi}_{km} + \rho k \dot{\chi}^2 + aT^2)_{t=0} dV \right\} dt. \tag{17}
 \end{aligned}$$

A solution with finite energy, which we will introduce further, is more rigorously defined in the context of Hilbert spaces.

In this regard, we introduce the following scalar product:

$$\begin{aligned}
 & \langle (v_m, \phi_{jk}, \varphi, \vartheta), (w_m, \psi_{jk}, \chi, T) \rangle_1 \\
 & = \int_0^{t_0} \int_{\Omega} [v_m \dot{w}_m + \phi_{jk} \dot{\psi}_{jk} + \varphi \dot{\chi} + \dot{v}_m \dot{w}_m]
 \end{aligned}$$

$$+ \dot{\phi}_{jk} \dot{\psi}_{jk} + \dot{\varphi} \dot{\chi} + v_{m,n} w_{m,n} + \phi_{jk,i} \psi_{jk,i} + \varphi_{,m} \chi_{,m} + \vartheta T + \int_0^t \vartheta_{,m} T_{,m} \Big] dV dt, \tag{18}$$

and denote by H_0^1 the Hilbert space endowed with the norm induced by the scalar product (18). The following scalar product defined by

$$\begin{aligned} & \langle (v_m, \phi_{jk}, \varphi, \vartheta), (w_m, \psi_{jk}, \chi, T) \rangle_2 \\ &= \langle (v_m, \phi_{jk}, \varphi, \vartheta), (w_m, \psi_{jk}, \chi, T) \rangle_1 + \langle (v_m, \phi_{jk}, \varphi, \vartheta), (w_m, \psi_{jk}, \chi, T) \rangle_1 \end{aligned}$$

induces a norm, and we denote by H_0^2 the Hilbert space endowed with this norm.

Finally, on the set

$$\{ (v, w, \vartheta) : (v = (v_m, \phi_{jk}, \varphi), \vartheta) \in \tilde{C}^1(\Omega), (w = (w_m, \psi_{jk}, \chi), \vartheta) \in \hat{C}^1(\Omega) \},$$

we define a scalar product which induces the norm

$$\begin{aligned} |(v, w, \vartheta)|_0 = & \left\{ \frac{1}{2} \int_{\Omega} [\rho w_m w_m + I_{jk} \psi_{jm} \psi_{km} + \rho k \chi^2 \right. \\ & + A_{ijmn} e_{mn}(v) \dot{e}_{ij}(w) + G_{ijmn} (e_{ij}(v) \varepsilon_{mn}(w) + e_{ij}(w) \varepsilon_{mn}(v)) \\ & + F_{ijmnr} (e_{ij}(v) \gamma_{mnr}(w) + e_{ij}(w) \gamma_{mnr}(v)) + B_{ijmn} \varepsilon_{ij}(v) \varepsilon_{mn}(w) \\ & + D_{ijmnr} (\varepsilon_{ij}(v) \gamma_{mnr}(w) + \varepsilon_{ij}(w) \gamma_{mnr}(v)) + C_{ijkmnr} \gamma_{ijk}(v) \gamma_{mnr}(w) \\ & + a_{ijk} (e_{ij}(v) \varphi_{,k} + e_{ij}(w) \chi_{,k}) + b_{ijk} (\varepsilon_{ij}(v) \varphi_{,k} + \varepsilon_{ij}(w) \chi_{,k}) \\ & \left. + c_{ijkm} (\gamma_{ijk}(v) \varphi_{,m} + \gamma_{ijk}(w) \chi_{,m}) + d_{ij} \varphi_{,i} \chi_{,j} + c \vartheta^2 \right] dV \Big\}^{1/2}, \end{aligned}$$

and the spaces endowed with this norm will be denoted by H_0^3 .

Considering hypotheses (13)–(16) and taking into account the previous identity (17), we can deduce that there is a constant $c_4 > 0$, which depends only on c, k , and T_0 so that, for all $\omega \in H_0^2$, the next inequality occurs:

$$|\omega|^2 \leq c_4 E_3(\omega, \omega). \tag{19}$$

For the mixed problem \mathcal{P} , we consider an arbitrary solution $u = (v_m, \phi_{jk}, \varphi)$ and use the notation $y = (u, \vartheta)$ in order to introduce the notion of finite energy solution by means of the following definition.

Definition 1 We assume that the following two conditions are met:

$$E_3(z, w) = E_4(y, w) + E_5(\delta, w), \quad \forall w = ((w_m, \psi_{jk}, \chi), T) \in H_0^2(\Omega), \tag{20}$$

$$\lim_{s \rightarrow 0} u(s) = u_0, \quad \text{in } \mathbf{H}_0(\Omega). \tag{21}$$

Then, we call the solution with finite energy of the problem \mathcal{P} as being the ordered array $((v_m, \phi_{jk}, \varphi), \vartheta)$ corresponding to the initial conditions $\zeta = ((v_m^0), (\phi_{jk}^0), \varphi^0, \vartheta^0)$ and to the following loads $z = ((f_m), (g_{jk}), l, r)$.

Let us suppose that we have given such kind of solution of the above defined problem \mathcal{P} . Then we will be able to determine other solutions with finite energy.

In this sense, we introduce the following notations:

$$\begin{aligned} \hat{f}_m(x, s) &= w_m^0(x) + \int_0^s f_m(x, \tau) d\tau, \\ \hat{g}_{jk}(x, s) &= \psi_{jk}^0(x) + \int_0^s g_{jk}(x, \tau) d\tau, \\ \hat{l}(x, s) &= \chi^0(x) + \int_0^s l(x, \tau) d\tau, \\ \hat{r}(x, s) &= \frac{T_0}{\rho} [\alpha_{mn}(x)e_{mn}(v^0) + \beta_{mn}(x)\varepsilon_{mn}(v^0) \\ &\quad + \delta_{mnr}(x)\gamma_{mnr}(v^0) + c(x)\vartheta^0 + a_m(x)\varphi_{,m}^0] + \int_0^s r(x, \tau) d\tau. \end{aligned}$$

With the help of the finite energy solution $((v_m, \phi_{jk}, \varphi), \vartheta)$, we can obtain a new finite energy solution. To this aim we will use the procedure proposed by Dafermos in the paper [20] and obtain the following result.

Theorem 1 *If $y = ((v_m, \phi_{jk}, \varphi), \vartheta)$ is a solution with finite energy for the problem \mathcal{P} corresponding to the sources $z = (f_m, g_{jk}, l, r)$ and to the inhomogeneous initial relations $\delta = (v_m^0, \phi_{jk}^0, \varphi^0, \vartheta^0)$, then the ordered array $\hat{y} = ((\hat{v}_m, \hat{\phi}_{jk}, \hat{\varphi}), \hat{\vartheta})$ defined by*

$$\hat{y} = \int_0^s y(x, \tau) d\tau$$

satisfies the problem \mathcal{P} as a solution with finite energy, but corresponding to the above defined loads $\hat{z} = (\hat{f}_m, \hat{g}_{jk}, \hat{l}, \hat{r})$ and to the particular initial relations $\hat{\delta} = (0, 0, \varphi^0, 0)$.

If we adapt the procedure proposed by Dafermos in [20], we can prove that the mixed problem \mathcal{P} cannot admit more than one finite energy solution. This uniqueness result is included in the following theorem.

Theorem 2 *The mixed problem in the context of thermoelasticity of dipolar porous bodies cannot admit more than one finite energy solution corresponding to some prescribed sources and to some prescribed initial data.*

In the next theorem we obtain a result regarding the existence for a finite energy solution and, also, an estimation of the respective solution, considering the simple situation of homogeneous initial conditions. The demonstration is made following step by step the procedure proposed by Dafermos in [20].

Theorem 3 *Let us consider mixed problem \mathcal{P} in the context of thermoelasticity of dipolar porous bodies which corresponds to the particular case of null initial conditions, the loads remaining arbitrary and of class L_2 . Then the existence of at least one finite energy solution $y = (v_m, \phi_{jk}, \varphi, \vartheta)$ is ensured. Moreover, it can find a constant $c_5 > 0$ that depends only on t_0, ρ , and T_0 so that the solution satisfies the following inequality:*

$$|y| \leq c_5 \|z\|_{L_2(\Omega_0) \times L_2(\Omega_0)},$$

where we used the notations

$$z = ((f_m), (g_{jk}), l, r),$$

$$\Omega_0 = [0, t_0] \times \Omega.$$

A final result of our study is the proof of a theorem on the existence of at least one solution in the situation where initial relations have their most general form.

In other words, we will expand the existence result included in Theorem 2, starting from the simple situations of the homogeneous initial conditions to the case of initial data in their most general inhomogeneous form.

Let us consider the following system:

$$E_1(v, \omega) = \int_{\Omega} [(\alpha_{mn}e_{mn}(\omega) + \beta_{mn}\varepsilon_{mn}(\omega) + \delta_{mnr}\gamma_{mnr}(\omega) + a_m\psi_{,m} + cT)\vartheta - \rho v_m w_m - I_{jk}\phi_{jr}\psi_{kr} - \rho k\chi + \rho f_m w_m + \rho g_{jk}\psi_{jk} + \rho l\chi] dV, \quad \forall \omega = (w_m, \psi_{jk}, \chi, T) \in \hat{W}_1(\Omega), \tag{22}$$

$$E_2(T, \vartheta) = - \int_{\Omega} \{T_0[\alpha_{mn}e_{mn}(u) + \beta_{mn}\varepsilon_{mn} + \delta_{mnr}\gamma_{mnr}(u) + a_m\psi_{,m} + aT] - \varrho r\} \vartheta dV, \quad \forall T \in \hat{W}_1(\Omega),$$

and the map

$$\mathcal{T}(\cdot) : H_0(\Omega) \rightarrow H(\Omega),$$

$$(w_m, \psi_{jk}, \chi, T) \xrightarrow{\mathcal{T}} (v_m, \phi_{jk}, \varphi, \vartheta).$$

Then the ordered array $(w_m, \psi_{jk}, \chi, T)$ is named the solution for the above system (22).

This kind of solution will be used to obtain the existence result in the case of initial data in their most general inhomogeneous form. But, to obtain this general result, we need the auxiliary result included in the next theorem.

Theorem 4 Consider an arbitrary system of sources $z = (f_m, g_{jk}, l, r)$, $z \in H_0(\Omega)$ and the general initial data $\delta = (v_m^0, \phi_{jk}^0, \varphi^0, \vartheta^0)$. Then the map $\mathcal{T}(z)$ is well defined and is a one-to-one application. As such, it admits the inverse map $\mathcal{T}^{-1}(z)$ which transfers any element z from the set $\mathcal{T}(H_0(\Omega)) \subset H(\Omega)$, that is, the co-domain of \mathcal{T} , to an element from the space $H_0(\Omega)$.

Moreover, for the solution obtained with the help of the map \mathcal{T} , a positive constant c_6 can be found such that the next inequality is satisfied:

$$\|\mathcal{T}(z)\|_{\mathbf{H}_1(\Omega) \times \mathbf{H}_1(\Omega) \times H_1(\Omega)} \leq c_6 \{|\delta| + \|z\|_{\mathbf{H}_0(\Omega) \times H_0(\Omega)}\}.$$

Proof First, we have to consider the fact that E_1 is subject to inequality (13). Then, we take into account that $E_1(w, w)$ is a coercive application regarding the norm $\|\cdot\|_{\mathbf{H}_1(\Omega)}$ from the space $\mathbf{H}_1(\Omega)$. This affirmation can be deduced by using inequality (16). Next we can follow step by step the procedure used by Fichera in his work [21]. □

Finally, we will prove our main result on existence. To this aim, we generalize the map \mathcal{T} to the map \mathcal{T}_v as follows:

$$\begin{aligned} \mathcal{T}_v(z_0, z_1, \dots, z_{v-1}) &= \mathcal{T}(z_0) \circ \mathcal{T}(z_1) \circ \dots \circ \mathcal{T}(z_{v-1}), \\ \mathcal{T}_v(\cdot, \cdot, \dots, \cdot) &: H_0(\Omega) \rightarrow H(\Omega), \end{aligned}$$

which is defined for some arbitrary sources z_0, z_1, \dots, z_{v-1} .

Let us denoted by $H_m(B; y_0, y_1, \dots, y_{m-1})$ the co-domain of the map

$$\mathcal{T}_v(y_0, y_1, \dots, y_{v-1}) : H_0(\Omega) \rightarrow H(\Omega),$$

and we use abbreviated writing

$$H_m(\Omega) = H_m(\Omega; y_0, y_1, \dots, y_{m-1}),$$

so that for an arbitrary element $\delta \in H_m(\Omega)$, we can consider the new norm

$$|\delta|_v = |\mathcal{T}_v^{-1}(0, 0, \dots, 0)\delta|_0.$$

Taking into account the above new approaches, we can address the issue of the existence of at least one solution having an energy finite in the situation in which the initial relations have a nonhomogeneous initial data.

Theorem 5 *In thermoelasticity of dipolar porous bodies, the mixed problem \mathcal{P} corresponding to boundary conditions (11), to the sources*

$$z = (f_m, g_{jk}, l, r) \in C^{v-1}([0, t_0]; H(\Omega)), \quad z^{(v)} \in L_1((0, t_0); H(\Omega)),$$

and to the following initial data:

$$\delta = (v_m^0, \phi_{jk}^0, \varphi^0, \vartheta^0) \in H_m(\Omega; z(0), z^{(1)}(0), \dots, z^{(v-1)}(0)) \quad \text{for } v = 1, 2, \dots$$

has at least one finite energy solution, $y \in H_0^1(\Omega_0)$, where we used the notation $u^{(v)} = \frac{\partial^v u}{\partial z_i^v}$ to designate the generalized derivative of a fixed order v for the function $u = u(z_1, z_2, \dots, z_m)$ regarding its variables (z_1, z_2, \dots, z_m) .

Proof To obtain this result, we must follow step by step the procedure used by Fichera in his work [21]. □

4 Conclusions

After we highlight the basic equations and the specific initial and boundary relations for the mixed problem from the context formulated above, we define the solutions with finite energy and propose a method to obtain other types of solutions with finite energy. It is important to specify that we prove the uniqueness of a solution in the most general situation in which both the initial data and the boundary conditions are inhomogeneous. In the last

theorem we obtain the existence of at least one finite solution in the most general case of inhomogeneous initial relations, starting from the particular previously demonstrated case. We emphasize once again that our results are generalizations of those obtained by Fichera and Dafermos in the simple context of classical linear elasticity. It is interesting to note that the results are similar to those of classical elasticity, even though in our context the basic equations and conditions are much more complicated, because we considered the presence of the temperature, the pores contribution, and the dipolar structure contribution.

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