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New multiplicity of positive solutions for some class of nonlocal problems

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Abstract

In this paper, we study the following nonlocal problem:

$$\begin{cases} -(a-b\int_\Omega |\nabla u|^2\,dx)\Delta u = \lambda |u|^{q-2}u, & x\in\Omega, \\ u=0, & x\in\partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N\geq 3$, $a,b>0$, $1<q<2$ and $\lambda>0$ is a parameter. By virtue of the variational method and Nehari manifold, we prove the existence of multiple positive solutions for the nonlocal problem. As a co-product of our arguments, we also obtain the blow-up and the asymptotic behavior of these solutions as $b\searrow 0$.

MSC: 35J20; 35J60

Keywords: Variational method; Nonlocal problem; Multiple positive solutions; Blow up

1 Introduction and main results

In this paper, we are concerned with the multiplicity of positive solutions for the nonlocal problem

$$\begin{cases} -(a-b\int_\Omega |\nabla u|^2\,dx)\Delta u = \lambda |u|^{q-2}u, & x\in\Omega, \\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N\geq 3)$, $a,b>0$, $1<q<2$ and $\lambda>0$ is a parameter.

In the past two decades, the following Kirchhoff type problem on a bounded domain

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2\,dx)\Delta u = f(x,u), & x\in\Omega, \\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.2}$$

has attracted great attention of many researchers. The Kirchhoff type problem is often viewed as nonlocal due to the presence of the term $b\int_\Omega |\nabla u|^2\,dx$ which implies that such a problem is no longer a pointwise identity. By using the variational method, there are many interesting results of positive solutions to (1.2), see e.g. [1, 4, 5, 9, 11] and the references therein.

If we replace $b\int_\Omega |\nabla u|^2\,dx$ with $-b\int_\Omega |\nabla u|^2\,dx$, then (1.2) turns out to be the following new nonlocal one:

$$\begin{cases} -(a-b\int_\Omega |\nabla u|^2\,dx)\Delta u = f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega. \end{cases} \tag{1.3}$$

This kind of problem involving negative nonlocal term not only presents some interesting difficulties different from Kirchhoff type problem but also has its own physical and mechanical motivation, see [8, 14]. Yin and Liu [16] considered problem (1.3) when $f(x,u)=|u|^{p-2}u$ with $2<p<2^*$ and showed the existence of two nontrivial solutions. Based on [16], Wang and Yang [15] further obtained the existence of infinitely many sign-changing solutions. In [17], the authors extended the results of [16] to a general case of nonlinear terms. For $f(x,u)=\lambda|u|^{-\gamma}$ with $0<\gamma<1$, [6] got the multiplicity of positive solutions to (1.3). In [10], we proved that problem (1.3) possesses at least one positive solution when $N=3$, $f(x,u)=\lambda f(x)|u|^{p-2}u$ with $3<p<4$ and $f(x)\in L^{\frac{6}{6-p}}(\Omega)$ may change sign. In particular, Duan et al. [3] and Lei et al. [7] proved that there exists $\lambda_*>0$ such that, for each $\lambda\in(0,\lambda_*)$, problem (1.1) has two positive solutions by using the minimization argument and the mountain pass theorem.

From the works described before, it is important and interesting to ask whether the multiplicity of positive solutions to problem (1.1) can be established by other methods? In the present paper, we shall give an affirmative answer. The main technique applied here is a separation argument for the Nehari-type set of problem (1.1), which has been firstly introduced by Tarantello [13] and later refined by Sun and Li [12].

Let $H:=H_0^1(\Omega)$ and $L^s(\Omega)$ be the standard Sobolev spaces endowed with the standard norms $\|\cdot\|$ and $|\cdot|_p$, respectively. Denote by $\to$ and $\rightharpoonup$ the strong and weak convergence, respectively. We use $o_n(1)$ to denote a quantity such that $o_n(1)\to 0$ as $n\to\infty$. C and C_i denote various positive constants which may vary from line to line. We say that $I\in C^1(H,\mathbb{R})$ satisfies the Palais–Smale condition at level $c\in\mathbb{R}$ ($(\mathrm{PS})_c$ in short) if any sequence $\{u_n\}\subset H$ such that $I(u_n)\to c$ and $I'(u_n)\to 0$ in H^{-1} as $n\to\infty$ has a convergent subsequence. S denotes the best constant in the Sobolev embedding $H\hookrightarrow L^{2^*}(\Omega)$, that is,

$$S=\inf_{u\in H\setminus\{0\}}\frac{\int|\nabla u|^2\,dx}{(\int|u|^{2^*}\,dx)^{2/2^*}}>0.$$

Associated with problem (1.1), we define the energy functional

$$I_b(u)=\frac{a}{2}\|u\|^2-\frac{b}{4}\|u\|^4-\frac{\lambda}{q}\int_\Omega |u|^q\,dx.$$

Then $I_b\in C^1(H,\mathbb{R})$. Recall that a function $u\in H$ is called a weak solution to (1.1) if, for any $\phi\in H$, there holds

$$\left(a-b\|u\|^2\right)\int_\Omega \nabla u\nabla\phi\,dx-\lambda\int_\Omega |u|^{q-2}u\phi\,dx=0.$$

Define the Nehari type set of (1.1)

$$\Lambda_b = \{u \in H : \langle I_b'(u), u\rangle = 0\} = \left\{u \in H : a\|u\|^2 - b\|u\|^4 = \lambda \int_\Omega |u|^q \,dx\right\},$$

and then decompose Λ_b into three subsets:

$$\Lambda_b^0 = \{u \in \Lambda_b : a(2-q)\|u\|^2 - b(4-q)\|u\|^4 = 0\},$$
$$\Lambda_b^+ = \{u \in \Lambda_b : a(2-q)\|u\|^2 - b(4-q)\|u\|^4 > 0\},$$
$$\Lambda_b^- = \{u \in \Lambda_b : a(2-q)\|u\|^2 - b(4-q)\|u\|^4 < 0\}.$$

It is important to notice that there exists a norm gap in Λ_b:

$$\|\tilde{u}\|^2 > \frac{a(2-q)}{b(4-q)} > \|u\|^2 \quad \text{for all } u \in \Lambda_b^+, \tilde{u} \in \Lambda_b^-. \tag{1.4}$$

Set

$$T_b = \frac{2(2-q)}{(4-q)^2} \frac{a^{\frac{4-q}{2}}}{b^{\frac{2-q}{2}}} \frac{S^{\frac{q}{2}}}{|\Omega|^{\frac{2^*-q}{2^*}}}.$$

Our main results are as follows.

Theorem 1.1 *Assume that $\lambda \in (0, T_b)$, then problem (1.1) has at least two positive solutions $u_* \in \Lambda_b^+$, $\tilde{u}_* \in \Lambda_b^-$ with $\|u_*\| < \|\tilde{u}_*\|$.*

Moreover, as a by-product of our arguments, we regard b as a parameter and obtain the blow-up behavior of the solution $\tilde{u}_b \in \Lambda_b^-$ and the asymptotic behavior of the other one $u_b \in \Lambda_b^+$ of problem (1.1) as $b \searrow 0$. Namely, we have the following theorem.

Theorem 1.2 *Assume that $\{b_n\}$ is a sequence satisfying $b_n \searrow 0$ as $n \to \infty$. Then there exists a subsequence, still denoted by $\{b_n\}$, such that*

(i) *$\|\tilde{u}_{b_n}\| \to \infty$ as $n \to \infty$.*

(ii) *$u_{b_n} \to u_0$ in H as $n \to \infty$, where u_0 is a positive solution of the problem*

$$\begin{cases} -a\Delta u = \lambda |u|^{q-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

Remark 1.3 Compared with [3, 7], we adapt a new method to show the existence and multiplicity of positive solutions to problem (1.1). In particular, we obtain the blow-up and the asymptotic behavior of these solutions. As far as we know, such phenomena about the solutions to (1.1) are first observed, which reveals some relationship between the nonlocal problem (1.1) and the classical semilinear problem (1.5).

The paper is organized as follows. In Sect. 2, we present some preliminaries. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2, respectively.

2 Preliminaries

Lemma 2.1 *Let* $\lambda \in (0, T_b)$. *Then* $\Lambda_b^{\pm} \neq \emptyset$ *and* $\Lambda_b^0 = \{0\}$.

Proof For any $u \in H$, $u \neq 0$, we define

$$h(t) = at^{2-q}\|u\|^2 - bt^{4-q}\|u\|^4, \quad \forall\, t > 0.$$

It is easy to see that $g(t)$ attains its maximum value at $t_{\max} = [\frac{a(2-q)}{b(4-q)\|u\|^2}]^{1/2}$ with

$$h(t_{\max}) = \frac{2(2-q)}{(4-q)^2}\frac{a^{\frac{4-q}{2}}}{b^{\frac{2-q}{2}}}\|u\|^q.$$

We note that, by Hölder's inequality, for $\lambda \in (0, T_b)$, there holds

$$\lambda \int_\Omega |u|^q\,dx \leq \lambda|\Omega|^{\frac{2^*-q}{2^*}}S^{-q/2}\|u\|^q < h(t_{\max}).$$

It follows that there are two and only two positive constants $t^+ = t^+(u)$ and $t^- = t^-(u)$ such that

$$h(t^+) = \lambda \int_\Omega |u|^q\,dx = h(t^-) \quad \text{and} \quad h'(t^+) < 0 < h'(t^-).$$

Equivalently, we obtain $t^+u \in \Lambda_b^-$ and $t^-u \in \Lambda_b^+$.

Next, we prove that $\Lambda_b^0 = \{0\}$. Arguing by contradiction, we assume that there exists $w \in \Lambda_b^0$ satisfying $w \neq 0$. Then we have $a(2-q)\|w\|^2 - b(4-q)\|w\|^4 = 0$. This yields $b\|w\|^2 = \frac{a(2-q)}{4-q}$. For $\lambda \in (0, T_b)$, it follows from $w \in \Lambda_b$ and Hölder's inequality that

$$\begin{aligned}
0 &= a\|w\|^2 - b\|w\|^4 - \lambda \int_\Omega |w|^q\,dx \\
&\geq \frac{2a}{4-q}\|w\|^2 - \lambda|\Omega|^{\frac{2^*-q}{2^*}}S^{-q/2}\|w\|^q \\
&= \|w\|^q\left[\frac{2(2-q)}{(4-q)^2}\frac{a^{\frac{4-q}{2}}}{b^{\frac{2-q}{2}}}\left(\frac{4-q}{2-q}\right)^{\frac{q}{2}} - \lambda|\Omega|^{\frac{2^*-q}{2^*}}S^{\frac{-q}{2}}\right] > 0,
\end{aligned}$$

which makes no sense. This ends the proof. □

Lemma 2.2 *Given* $u \in \Lambda_b^{\pm}$, *there exist* $\rho_u > 0$ *and a differential function* $g_{\rho_u} : B_{\rho_u}(0) \to \mathbb{R}^+$ *defined for* $w \in H$, $w \in B_{\rho_u}(0)$ *satisfying*

$$g_{\rho_u}(0) = 1, \quad g_{\rho_u}(w)(u - w) \in \Lambda_b^{\pm},$$

and

$$\langle g'(0), \phi\rangle = \frac{(2a - 4b\|u\|^2)\int_\Omega \nabla u \nabla\phi\,dx - q\lambda\int_\Omega |u|^{q-2}u\phi\,dx}{a(2-q)\|u\|^2 - b(4-q)\|u\|^4}.$$

Proof We only give the proof for the case $u \in \Lambda_b^-$. In a similar way, one can prove the other case $u \in \Lambda_b^+$. Fix $u \in \Lambda_b^-$ and define $F : \mathbb{R}^+ \times H \to \mathbb{R}$ by

$$F(t,w) = at\|u-w\|^2 - bt^3\|u-w\|^4 - \lambda t^{q-1}\int_\Omega |u-w|^q\,dx.$$

By $u \in \Lambda_b^- \subset \Lambda_b$, we easily see that $F(1,0) = 0$ and

$$F_t(1,0) = a(2-q)\|u\|^2 - b(4-q)\|u\|^4 < 0.$$

Then, we are able to use the implicit function theorem for F at the point $(1,0)$ and get $\overline{\rho} > 0$ and a differential functional $g = g(w) > 0$ defined for $w \in H$, $\|w\| < \overline{\rho}$ such that

$$g(0) = 1, \qquad g(w)(u-w) \in \Lambda_b, \quad \forall w \in H, \|w\| < \overline{\rho}.$$

Thanks to the continuity of g, we can take $\rho > 0$ possibly smaller ($\rho < \overline{\rho}$) such that, for any $w \in H$, $\|w\| < \rho$, there holds

$$g(w)(u-w) \in \Lambda_b^-.$$

Moreover, for any $\phi \in H$, $r > 0$, it follows from

$$\begin{aligned}
&F(1,0+r\phi) - F(1,0)\\
&\quad = a\|u-r\phi\|^2 - b\|u-r\phi\|^4 - \lambda\int_\Omega |u-r\phi|^q\,dx - a\|u\|^2 + b\|u\|^4 + \lambda\int_\Omega |u|^p\,dx\\
&\quad = -a\int_\Omega \bigl(2r\nabla u\nabla\phi - r^2|\nabla\phi|^2\bigr)\,dx - \lambda\int_\Omega \bigl(|u-r\phi|^q - |u|^q\bigr)\,dx\\
&\qquad + b\biggl[2\int_\Omega |\nabla u|^2\,dx\int_\Omega \bigl(2r\nabla u\nabla\phi - r^2|\nabla\phi|^2\bigr)\,dx\\
&\qquad - \biggl(\int_\Omega \bigl(2r\nabla u\nabla\phi - r^2|\nabla\phi|^2\bigr)\,dx\biggr)^2\biggr]
\end{aligned}$$

that

$$\begin{aligned}
&\langle F_w,\phi\rangle|_{t=1,w=0}\\
&\quad = \lim_{r\to 0}\frac{F(1,0+r\phi) - F(1,0)}{r}\\
&\quad = -\bigl(2a - 4b\|u\|^2\bigr)\int_\Omega \nabla u\nabla\phi\,dx + q\lambda\int_\Omega |u|^{p-2}u\phi\,dx.
\end{aligned}$$

Consequently, we derive

$$\bigl\langle g'(0),\phi\bigr\rangle = -\frac{\langle F_w,\phi\rangle}{F_t}|_{t=1,w=0} = \frac{(2a-4b\|u\|^2)\int_\Omega \nabla u\nabla\phi\,dx - q\lambda\int_\Omega |u|^{q-2}u\phi\,dx}{a(2-q)\|u\|^2 - b(4-q)\|u\|^4}.$$

This completes the proof. □

Lemma 2.3 *If* $\lambda \in (0, T_b)$, *then we have*

(i) *the functional* I_b *is coercive and bounded from below on* Λ_b;

(ii) $\inf_{\Lambda_b^+ \cup \Lambda_b^0} I_b = \inf_{\Lambda_b^+} I_b \in (-\infty, 0)$.

Proof (i) For $u \in \Lambda_b$, by Hölder's inequality, we have

$$\begin{aligned} I_b(u) &= I_b(u) - \frac{1}{4}\langle I_b'(u), u\rangle \\ &= \frac{a}{4}\|u\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{4}\right)\int_\Omega |u|^q\,dx \\ &\geq \frac{a}{4}\|u\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{4}\right)|\Omega|^{\frac{2^*-q}{2^*}} S^{-q/2}\|u\|^q, \end{aligned}$$

and the conclusion (i) follows.

(ii) For $u \in \Lambda_b^+$, there holds

$$\begin{aligned} I_b(u) &= I_b(u) - \frac{1}{q}\langle I_b'(u), u\rangle \\ &= a\left(\frac{1}{2} - \frac{1}{q}\right)\|u\|^2 - b\left(\frac{1}{4} - \frac{1}{q}\right)\|u\|^4 \\ &< \frac{-a(2-q)\|u\|^2 + b(4-q)\|u\|^4}{4q} < 0. \end{aligned}$$

This together with Lemma 2.1 gives that $\inf_{\Lambda_b^+ \cup \Lambda_b^0} I_b = \inf_{\Lambda_b^+} I_b < 0$. Moreover, from (i) we infer that $\inf_{\Lambda_b^+ \cup \Lambda_b^0} I_b \neq -\infty$. Therefore, $\inf_{\Lambda_b^+ \cup \Lambda_b^0} I \in (-\infty, 0)$. □

Lemma 2.4 *For all* $\lambda > 0$, I_b *satisfies the* $(PS)_c$ *condition at any level* $c < \frac{a^2}{4b}$.

Proof The proof is similar to that of [16, Lemma 2]. We omit the details. □

3 Proof of Theorem 1.1

Lemma 3.1 *Assume that* $\lambda \in (0, T_b)$, *then problem* (1.1) *has a positive solution* u_b *with* $u_b \in \Lambda_b^+$.

Proof It is easily verified that the sets $\Lambda_b^+ \cup \Lambda_b^0$ and Λ_b^- are closed. Applying the Ekeland variational principle, we can derive a minimizing sequence $\{u_n\} \subset \Lambda_b^+ \cup \Lambda_b^0$ satisfying that

$$\lim_{n\to\infty} I_b(u_n) = \inf_{\Lambda_b^+ \cup \Lambda_b^0} I_b < 0 \tag{3.1}$$

and

$$I_b(z) \geq I_b(u_n) - \frac{1}{n}\|z - u_n\| \quad \text{for all } z \in \Lambda_b^+ \cup \Lambda_b^0. \tag{3.2}$$

Noting that $I_b(|u|) = I_b(u)$, we may suppose that $u_n \geq 0$ in Ω. By Lemma 2.3, $\{u_n\}$ is bounded in H, and so we can assume

$$u_n \rightharpoonup u_b \quad \text{in } H,$$

$$u_n \to u_b \quad \text{in } L^s(\Omega), 2 \le s < 2^*,$$
$$u_n \to u_b \quad \text{a.e. in } \Omega.$$

In what follows we prove that u_b is a positive solution to (1.1). The proof will be divided into four steps.

textbfStep 1: $u_b \neq 0$.

By contradiction, we suppose that $u_b = 0$. Since $u_n \in \Lambda_b^+ \cup \Lambda_b^0$, we see that, for n large,

$$a\|u_n\|^2 > \frac{4-q}{2-q} b\|u_n\|^4.$$

As a consequence, we derive

$$I_b(u_n) = \frac{1}{2}a\|u_n\|^2 - \frac{1}{4}b\|u_n\|^4 + o_n(1) \ge \left(\frac{4-q}{2(2-q)} - \frac{1}{4}\right) b\|u_n\|^4 + o_n(1) > 0,$$

which is a contradiction to (3.1). Thus, $u_b \neq 0$.

textbfStep 2: There exists a constant $C_1 > 0$ such that

$$2a\|u_n\|^2 - \lambda(4-q)\int_\Omega |u_n|^q\,dx < -C_1. \tag{3.3}$$

To prove that, it suffices to verify

$$2a \liminf_{n\to\infty} \|u_n\|^2 < \lambda(4-q)\int_\Omega |u_b|^q\,dx.$$

By $u_n \in \Lambda_b^+ \cup \Lambda_b^0$,

$$2a \liminf_{n\to\infty} \|u_n\|^2 \le \lambda(4-q)\int_\Omega |u_b|^q\,dx.$$

Suppose to the contrary that

$$2a \liminf_{n\to\infty} \|u_n\|^2 = \lambda(4-q)\int_\Omega |u_b|^q\,dx.$$

Then we can assume $\|u_n\|^2 \to A > 0$ as $n \to \infty$, where A satisfies

$$\lambda \int_\Omega |u_b|^q\,dx = \frac{2aA}{4-q}.$$

Combining this with $\{u_n\} \subset \Lambda_b$, we have

$$0 = aA - bA^2 - \frac{2aA}{4-q}.$$

It follows that

$$A = \frac{a(2-q)}{b(4-q)},$$

which leads to a contradiction

$$\begin{aligned} 0 &< \frac{2(2-q)}{(4-q)^2}\frac{a^{\frac{4-q}{2}}}{b^{\frac{2-q}{2}}}\|u_n\|^q - \lambda\int_\Omega |u_n|^q\,dx \\ &\to \frac{2(2-q)}{(4-q)^2}\frac{a^{\frac{4-q}{2}}}{b^{\frac{2-q}{2}}}\left[\frac{a(2-q)}{b(4-q)}\right]^{q/2} - \frac{2a}{4-q}\frac{a(2-q)}{b(4-q)} \\ &= \frac{2a^2(2-q)}{b(4-q)^2}\left[\left(\frac{2-q}{4-q}\right)^{q/2} - 1\right] < 0 \end{aligned}$$

when $\lambda \in (0, T_b)$. Thus, (3.3) holds.

textbfStep 3: $I_b'(u_n) \to 0$ in H^{-1}.

Let $0 < \rho < \rho_n \equiv \rho_{u_n}$, $g_n \equiv g_{u_n}$, where ρ_{u_n} and g_{u_n} are given as in Lemma 2.2 with $u = u_n$. Let $w_\rho = \rho u$ with $\|u\| = 1$. Fix n and set $z_\rho = g_n(w_\rho)(u_n - w_\rho)$. By $z_\rho \in \Lambda_b^+$, we have from (3.2) that

$$I_b(z_\rho) - I_b(u_n) \ge -\frac{1}{n}\|z_\rho - u_n\|.$$

Then, by the mean value theorem,

$$\langle I_b'(u_n), z_\rho - u_n\rangle + o\bigl(\|z_\rho - u_n\|\bigr) \ge -\frac{1}{n}\|z_\rho - u_n\|.$$

Hence, we derive

$$\bigl\langle I_b'(u_n), -w_\rho + \bigl(g_n(w_\rho) - 1\bigr)(u_n - w_\rho)\bigr\rangle \ge -\frac{1}{n}\|z_\rho - u_n\| + o\bigl(\|z_\rho - u_n\|\bigr),$$

and thus,

$$-\rho\langle I_b'(u_n), u\rangle + \bigl(g_n(w_\rho) - 1\bigr)\langle I_b'(u_n), u_n - w_\rho\rangle \ge -\frac{1}{n}\|z_\rho - u_n\| + o\bigl(\|z_\rho - u_n\|\bigr),$$

from which it follows that

$$\langle I_b'(u_n), u\rangle \le \frac{1}{n}\frac{\|z_\rho - u_n\|}{\rho} + \frac{o(\|z_\rho - u_n\|)}{\rho} + \frac{g_n(w_\rho) - 1}{\rho}\langle I_b'(u_n), u_n - w_\rho\rangle. \tag{3.4}$$

By Step 2, Lemma 2.2, and the boundedness of $\{u_n\}$, one sees that

$$\|z_\rho - u_n\| = \bigl\|\bigl(g_n(w_\rho) - 1\bigr)(u_n - w_\rho) - w_\rho\bigr\| \le \bigl|g_n(w_\rho) - 1\bigr|C_2 + \rho$$

and

$$\lim_{\rho\to 0}\frac{|g_n(w_\rho) - 1|}{\rho} = \langle g_n'(0), u\rangle \le \bigl\|g_n'(0)\bigr\| \le C_3.$$

Therefore, for fixed n, we deduce by taking $\rho \to 0$ in (3.4) that

$$\langle I_b'(u_n), u\rangle \le \frac{C}{n},$$

which provides that $I_b'(u_n) \to 0$ as $n \to \infty$.

textbfStep 4: u_b is a positive solution of problem (1.1) and $u_b \in \Lambda_b^+$.

It follows from Step 3, Lemmas 2.3 and 2.4 that, along a subsequence, $u_n \to u_b$ in H with $I_b(u_b) < 0$ and $I_b'(u_b) = 0$. Hence, $u_b \geq 0$ is a weak solution to problem (1.1) satisfying $u_b \in \Lambda_b^+$. The standard elliptic regularity argument and the strong maximum principle imply that u_b is positive. Thus we complete the proof of Lemma 3.1. □

Lemma 3.2 *Assume that* $\lambda \in (0, T_b)$, *then problem* (1.1) *has a positive solution* $\tilde{u}_b$ *with* $\tilde{u}_b \in \Lambda_b^-$.

Proof Similar to the proof of Lemma 3.1, one can construct a bounded and nonnegative sequence $\{\tilde{u}_n\} \subset \Lambda_b^-$ satisfying that

$$\text{(i)} \quad \lim_{n\to\infty} I_b(\tilde{u}_n) = \inf_{\Lambda_b^-} I_b,$$

$$\text{(ii)} \quad I_b(z) \geq I_b(\tilde{u}_n) - \frac{1}{n}\|z - u_n\|, \quad \text{for all } z \in \Lambda_b^-,$$

$$\text{(iii)} \quad \tilde{u}_n \rightharpoonup \tilde{u}_b \quad \text{in } H,$$

$$\text{(iv)} \quad \tilde{u}_n \to \tilde{u}_b \quad \text{in } L^s(\Omega), \quad 2 \leq s < 2^*,$$

$$\text{(v)} \quad \tilde{u}_n \to \tilde{u}_b \quad \text{a.e. in } \Omega.$$

Without loss of generality, we may suppose that $0 \in \Omega$. Take a cut-off function $\varphi(x) \in C_0^\infty(\Omega)$ satisfying $0 \leq \varphi \leq 1$ in Ω and $\varphi(x) \equiv 1$ near zero. Define

$$v_\varepsilon(x) = \varphi(x)\frac{(N(N-2))^{(N-2)/4}\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{1/2}}.$$

It is known that (see [2])

$$\|v_\varepsilon\|^2 = S^{N/2} + O\big(\varepsilon^{N-2}\big).$$

Firstly, we prove the following upper bound for $\inf_{\Lambda_b^-} I_b$:

$$\inf_{\Lambda_b^-} I_b^- \leq \sup_{t>0} I_b(u_b + tv_\varepsilon) < \frac{a^2}{4b}, \tag{3.5}$$

where u_b is the first positive solution obtained in the previous subsection. By $u_b \in \Lambda_b^+$ and (1.4), we easily see that $a - b\|u_b\|^2 > 0$. Since u_b is a positive solution of (1.1), we also have

$$0 = \big\langle I_b'(u_b), tv_\varepsilon\big\rangle = t\big(a - b\|u_b\|^2\big)\int_\Omega \nabla u_b \nabla v_\varepsilon\,dx - t\lambda \int_\Omega u_b^{q-1} v_\varepsilon\,dx, \tag{3.6}$$

from which it follows that

$$\int_\Omega \nabla u_b \nabla v_\varepsilon\,dx = \frac{\lambda \int_\Omega u_b^{q-1} v_\varepsilon\,dx}{a - b\|u_b\|^2} > 0. \tag{3.7}$$

To proceed, set $w_\varepsilon = u_b + Rv_\varepsilon$ with $R > 1$. By (3.7), we have

$$\|w_\varepsilon\|^2 = \|u_b\|^2 + 2R\int_\Omega \nabla u_b \nabla v_\varepsilon\,dx + R^2\|v_\varepsilon\|^2 \geq \|u_b\|^2 + R^2 S^{3/2} + O(\varepsilon). \tag{3.8}$$

Let $h(t)$ be defined as in Lemma 2.1. As can be seen from the proof of Lemma 2.1, we have that $h(t_\varepsilon) = \lambda \int_\Omega |\frac{w_\varepsilon}{\|w_\varepsilon\|}|^q \, dx$ and $h'(t_\varepsilon) < 0$, where $t_\varepsilon = t^+(\frac{w_\varepsilon}{\|w_\varepsilon\|})$. From the structure of h and $\int_\Omega |\frac{w_\varepsilon}{\|w_\varepsilon\|}|^q \, dx > 0$, it follows that t_ε is uniformly bounded by a suitable positive constant C_1, $\forall R \geq 1$ and $\forall \varepsilon > 0$.

On the other hand, we can infer from (3.8) that there exists $\varepsilon_1 > 0$ such that

$$\|w_\varepsilon\|^2 \geq \|u_b\|^2 + \frac{1}{2} R^2 S^{3/2}, \quad \forall \varepsilon \in (0, \varepsilon_1).$$

Thus, we can find $R_1 \geq 1$ such that $\|w_\varepsilon\| > C_1$, $\forall R \geq R_1$, and $\forall \varepsilon \in (0, \varepsilon_1)$.

Let

$$E_1 = \left\{ u : u = 0 \text{ or } \|u\| < t^+\left(\frac{u}{\|u\|}\right) \right\} \quad \text{and} \quad E_2 = \left\{ u : \|u\| > t^+\left(\frac{u}{\|u\|}\right) \right\}.$$

Note that $H - \Lambda_b^- = E_1 \cup E_2$ and $\Lambda_b^+ \subset E_1$. Since $u_b \in \Lambda_b^+$, by the continuity of $t^+(u)$, one sees that $u_b + tR_1 v_\varepsilon$ for $t \in (0,1)$ must intersect Λ_b^-, and consequently

$$\inf_{\Lambda_b^-} I_b \leq \sup_{t>0} I_b(u_b + tv_\varepsilon).$$

Hence (3.5) will follow if we show that

$$\sup_{t>0} I_b(u_b + tv_\varepsilon) < \frac{a^2}{4b}.$$

By the mean value theorem, we can get $\delta(x) \in [0,1]$ satisfying

$$\big(u_b(x) + tv_\varepsilon(x)\big)^q - u_b^q(x) = q\big(u_b(x) + \delta(x) tv_\varepsilon(x)\big)^{q-1} tv_\varepsilon(x) \geq qtu_b^{q-1}(x) v_\varepsilon(x) \tag{3.9}$$

for any $x \in \Omega$. By (3.6), (3.7), and (3.9),

$$\begin{aligned}
&I_b(u_b + tv_\varepsilon) \\
&\quad = \frac{a}{2}\|u_b\|^2 + at\int_\Omega \nabla u_b \nabla v_\varepsilon \, dx + \frac{a}{2}t^2\|v_\varepsilon\|^2 - \frac{b}{4}\|u_b\|^4 - bt^2\left(\int_\Omega \nabla u_b \nabla v_\varepsilon \, dx\right)^2 \\
&\qquad - \frac{b}{4}t^4\|v_\varepsilon\|^4 - bt\|u_b\|^2 \int_\Omega \nabla u_b \nabla v_\varepsilon \, dx - \frac{b}{2}t^2\|u_b\|^2\|v_\varepsilon\|^2 \\
&\qquad - bt^3\|v_\varepsilon\|^2 \int_\Omega \nabla u_b \nabla v_\varepsilon \, dx - \frac{\lambda}{q}\int_\Omega (u_b + tv_\varepsilon)^q \, dx \\
&\quad \leq I_b(u_b) + \frac{a}{2}t^2\|v_\varepsilon\|^2 - \frac{b}{4}t^4\|v_\varepsilon\|^4 - \frac{b}{2}t^2\|u_b\|^2\|v_\varepsilon\|^2 \\
&\qquad - \frac{\lambda}{q}\int_\Omega \big[(u_b + tv_\varepsilon)^q - u_b^q - qtu_b^{q-1}v_\varepsilon\big] \, dx \\
&\quad \leq \frac{a}{2}t^2\|v_\varepsilon\|^2 - \frac{b}{4}t^4\|v_\varepsilon\|^4 - \frac{b}{2}t^2\|u_b\|^2\|v_\varepsilon\|^2,
\end{aligned}$$

which implies that there exists $t_1 > 0$ small enough such that

$$\sup_{0<t<t_1} I_b(u_b + tv_\varepsilon) < \frac{a^2}{4b}.$$

Thus, we only need to consider the case of $t \geq t_1$. Since

$$\begin{aligned}\sup_{t\geq t_1} I_b(u_b + tv_\varepsilon) &\leq \sup_{t>0}\left\{\frac{a}{2}t^2\|v_\varepsilon\|^2 - \frac{b}{4}t^4\|v_\varepsilon\|^4\right\} - \frac{b}{2}t_1^2\|u_b\|^2\|v_\varepsilon\|^2 \\ &= \frac{a^2}{4b} - \frac{b}{2}t_1^2\|u_b\|^2\|v_\varepsilon\|^2 < \frac{a^2}{4b},\end{aligned}$$

we deduce that (3.5) holds.

Secondly, we claim that $\tilde{u}_b \neq 0$. If, to the contrary, $\tilde{u}_b = 0$, from $\{\tilde{u}_n\} \subset \Lambda_b^-$ it then follows that

$$a\|\tilde{u}_n\|^2 - b\|\tilde{u}_n\|^4 + o_n(1) = 0.$$

As a consequence, we obtain $\|\tilde{u}_n\|^2 \to \frac{a}{b}$ as $n \to \infty$. Furthermore,

$$\inf_{\Lambda_b^-} I_b = \lim_{n\to\infty} I_b(\tilde{u}_n) = \lim_{n\to\infty}\left[\frac{a}{2}\|\tilde{u}_n\|^2 - \frac{b}{4}\|\tilde{u}_n\|^4 - \lambda\int_\Omega |\tilde{u}_n|^q\,dx\right] = \frac{a^2}{4b},$$

which contradicts (3.5). Hence, the claim holds. This time we can proceed as in the proof of Lemma 3.1 and deduce that $\tilde{u}_b$ is a positive solution of problem (1.1) with $\tilde{u}_b \in \Lambda_b^-$. The proof is complete. □

Proof of Theorem 1.1 This is an immediate consequence of (1.4), Lemmas 3.1 and 3.2. □

4 Proof of Theorem 1.2

Proof of Theorem 1.2 For any sequence $\{b_n\}$ with $b_n \searrow 0$, we can use Theorem 1.1 to obtain sequences $\{u_{b_n}\} \subset \Lambda_{b_n}^+$ and $\{\tilde{u}_{b_n}\} \subset \Lambda_{b_n}^-$ corresponding to positive solutions to problem (1.1) with $b = b_n$ when $\lambda \in (0, T_{b_n})$.

By $\tilde{u}_{b_n} \in \Lambda_{b_n}^-$ and (1.4), we see that

$$\lim_{n\to\infty}\|\tilde{u}_{b_n}\|^2 \geq \lim_{n\to\infty}\frac{a(2-q)}{b_n(4-q)} = \infty,$$

and conclusion (i) follows.

Next, we prove conclusion (ii) of Theorem 1.2. Note that

$$I_b(u_{b_n}) = \inf_{\Lambda_{b_n}^+\cup\Lambda_{b_n}^0} I_{b_n} < 0$$

for all $n \in \mathbb{N}$. Then, by Hölder's inequality, we have

$$\begin{aligned}0 &\geq I_{b_n}(u_{b_n}) - \frac{1}{4}\langle I'_{b_n}(u_{b_n}), u_{b_n}\rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{4}\right)\|u_{b_n}\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{4}\right)|\Omega|^{\frac{2^*-q}{2^*}} S^{\frac{-q}{2}}\|u_{b_n}\|^q.\end{aligned}$$

Since $1 < q < 2$, it follows that $\{u_{b_n}\}$ is bounded in H. As a consequence, there exists a subsequence of $\{b_n\}$ (still denoted by $\{b_n\}$) such that $u_{b_n} \rightharpoonup u_0$ in H as $n \to \infty$. Furthermore, we have that, for all $v \in H$,

$$\begin{aligned} 0 &= \lim_{n\to\infty}\langle I'_{b_n}(u_{b_n}), v\rangle \\ &= \lim_{n\to\infty}\left[\left(a - b_n\|u_{b_n}\|^2\right)\int_\Omega \nabla u_{b_n}\nabla v\,dx - \lambda\int_\Omega u_{b_n}^{q-1}v\,dx\right] \\ &= a\int_\Omega \nabla u_0\nabla v\,dx - \lambda\int_\Omega u_0^{q-1}v\,dx, \end{aligned}$$

which implies that u_0 is a positive solution to problem (1.5). To complete the proof, we only need to show that $u_{b_n} \to u_0$ in H. This follows easily from

$$\begin{aligned} & a\|u_{b_n} - u_0\|^2 \\ &\quad = \langle I'_{b_n}(u_{b_n}) - I'_0(u_0), u_{b_n} - u_0\rangle + b_n\int_\Omega |\nabla u_{b_n}|^2\,dx\int_\Omega \nabla u_{b_n}\nabla(u_{b_n} - u_0)\,dx \\ &\qquad + \lambda\int_\Omega \left(u_{b_n}^{q-1} - u_0^{q-1}\right)(u_{b_n} - u_0)\,dx \\ &\quad \to 0, \end{aligned}$$

as $n \to \infty$. Theorem 1.2 is thus proved. □

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