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Optimal harvesting strategies of a stochastic competitive model with S-type distributed time delays and Lévy jumps

Hong Qiu[1], Wenmin Deng[1] and Mingqi Xiang[1*]

*Correspondence:
xiangmingqi_hit@163.com
[1]Department of Mathematics, College of Science, Civil Aviation University of China, Tianjin, 300300, P.R. China

Abstract

The aim of this paper is to investigate the optimal harvesting strategies of a stochastic competitive Lotka–Volterra model with S-type distributed time delays and Lévy jumps by using ergodic method. Firstly, the sufficient conditions for extinction and stable in the time average of each species are established under some suitable assumptions. Secondly, under a technical assumption, the stability in distribution of this model is proved. Then the sufficient and necessary criteria for the existence of optimal harvesting policy are established under the condition that all species are persistent. Moreover, the explicit expression of the optimal harvesting effort and the maximum of sustainable yield are given.

Keywords: Optimal harvesting; Competitive model; White noise; S-type distributed time delays; Lévy jumps; Stability in distribution

1 Introduction

As is well known, over-harvesting and unreasonable harvesting policies could cause a number of adverse effects, such as ecological destruction, species extinction, and desertification. Therefore, the optimal harvesting problem is a meaningful and significant topic in biology and mathematics (Zou and Wang [1]). In addition, several scholars have paid attention to investigating competitive systems and obtained a lot of successful results (see, e.g., [2–4]) in recent years. Wang et al. [5] have studied stability for the distribution of a stochastic competitive Lotka–Volterra system with S-type distributed time delays. Hence, it is very interesting to study the optimal harvesting of a competitive model. The typical competitive model with harvesting can be described as follows:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)[r_1 - h_1 - c_{11}x_1(t) - c_{12}x_2(t)]\,\mathrm{d}t, \\ \mathrm{d}x_2(t) = x_2(t)[r_2 - h_2 - c_{21}x_1(t) - c_{22}x_2(t)]\,\mathrm{d}t, \end{cases}$$

with initial data

$$x_i(0) = \phi_i(0), \quad i = 1, 2,$$

where x_1 and x_2 stand for the population size of two species, respectively, $r_i > 0$ is the growth rate of x_i, $i = 1,2$. $h_i > 0$ represents the harvesting effort of x_i, $i = 1,2$. $c_{ii} > 0$ is for the intraspecific competition coefficients of x_i, $i = 1,2$; c_{ij} ($i \neq j$; $i,j = 1,2$) denotes the interspecific competition rate.

In fact, the dynamics and optimal harvesting of population are inevitably affected by some environmental perturbations in virtually all ecosystems, which mainly include two types: white noise and jumping noise. White noise describes the continuous noise, such as light, drought, cold wave and so on (see, e.g., [6–9]). Jumping noise describes sudden environment shocks, such as earthquakes, floods, and epidemics (see, e.g., [10, 11]). In recent years, several authors have studied some systems both with white noise and jumping noise and published a number of successful articles (see, e.g., [12–17]).

On the other hand, Gopalsamy [18] have pointed out that "the current growth of a population should also be influenced by the past history of the species". So it is necessary to take time delay into consideration. To the best of our knowledge to date, "systems with discrete time delays and those with continuously distributed time delays do not contain each other. However, systems with S-type distributed time delays contain both."(see Wang, Wang and Wei [19, 20]). And stochastic systems with distributed delays were considered in several publications (see e.g. [21–23]). Qiu and Deng [24] have discussed the optimal harvesting problem of a stochastic delay competitive model with Lévy jumps, which is about discrete time delays, and obtained the result that discrete time delays have no impact on the optimal harvesting policy in some cases. Therefore, an interesting and significant problem arises: how does the S-type distributed time delays affect the population dynamics and the optimal harvesting policy? It is more natural and practical for us to consider. In this paper, we consider the following model:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t^-)[r_1 - h_1 - c_{11}x_1(t^-) - \int_{-\tau_{12}}^{0} x_2(t+\theta)\,\mathrm{d}F_{12}(\theta)]\,\mathrm{d}t \\ \qquad\qquad + \alpha_1 x_1(t^-)\,\mathrm{d}B_1(t) + x_1(t^-)\int_{\mathbb{Z}} \gamma_1(\nu)\widetilde{N}(\mathrm{d}t,\mathrm{d}\nu), \\ \mathrm{d}x_2(t) = x_2(t^-)[r_2 - h_2 - \int_{-\tau_{21}}^{0} x_1(t+\theta)\,\mathrm{d}F_{21}(\theta) - c_{22}x_2(t^-)]\,\mathrm{d}t \\ \qquad\qquad + \alpha_2 x_2(t^-)\,\mathrm{d}B_2(t) + x_2(t^-)\int_{\mathbb{Z}} \gamma_2(\nu)\widetilde{N}(\mathrm{d}t,\mathrm{d}\nu), \end{cases} \tag{1}$$

with initial data

$$x_i(\theta) = \phi_i(\theta), \qquad \theta \in [-\tau, 0], \qquad \tau = \max\{\tau_{12}, \tau_{21}\}, \quad i = 1,2,$$

$x_i(t^-)$ is the left limit of $x_i(t)$, $i = 1,2$. $\tau_i \geq 0$, $i = 1,2$ are time delays. $\phi_i(\theta) > 0$, $i = 1,2$ are continuous functions defined on $[-\tau, 0]$. $\int_{-\tau_{12}}^{0} x_2(t+\theta)\,\mathrm{d}F_{12}(\theta)$ and $\int_{-\tau_{21}}^{0} x_1(t+\theta)\,\mathrm{d}F_{21}(\theta)$ are Lebesgue–Stieltjes integrals. $F_{12}(\theta)$ and $F_{21}(\theta)$ are nondecreasing bounded variation functions defined on $[-\tau, 0]$. $\widetilde{N}(\mathrm{d}t,\mathrm{d}\nu) = N(\mathrm{d}t,\mathrm{d}\nu) - \mu(\mathrm{d}\nu)\mathrm{d}t$, N is a Poisson counting measure, μ is the characteristic measure of N on a measurable subset $\mathbb{Z}$ of $(0,+\infty)$ with $\mu(\mathbb{Z}) < +\infty$. γ_i is the effect of Lévy noises on species i, if $\gamma_i(\nu) > 0$, the jumps represent the increasing of the species; if $\gamma_i(\nu) < 0$, the jumps represent the decreasing of the species; Therefore, it is reasonable to assume that $1 + \gamma_i(\nu) > 0$, $\nu \in \mathbb{Z}$, $i = 1,2$.

We wish to solve the problem above and get the optimal harvesting effort (OHE) $H^* = (h_1^*, h_2^*)$ such that the expectation of sustainable yield (ESY) $Y(H) = \lim_{t\to+\infty} \sum_{i=1}^{2} \mathbb{E}(h_i x_i(t))$ is maximum (all species are persistent). Firstly, we establish the sufficient criteria for the extinction and persistence of each species in Sect. 2. Then in Sect. 3, we prove the stability in distribution of this model. Finally, we establish the sufficient and necessary conditions for the existence of the optimal harvesting policy and obtain the explicit expression of OHE and the maximum of ESY (MESY) in Sect. 4.

2 Extinction and persistence

At first, we define some notations for the sake of convenience,

$$\begin{aligned}
&\beta_i = h_i + \frac{\alpha_i^2}{2} + \int_{\mathbb{Z}} \bigl[\gamma_i(v) - \ln\bigl(1+\gamma_i(v)\bigr)\bigr]\mu(\mathrm{d}v), \qquad b_i = r_i - \beta_i, \quad i = 1,2;\\
&R_+^2 = \bigl\{a = (a_1, a_2) \in R^2 | a_i > 0, i = 1,2\bigr\};\\
&\bigl\langle f(t)\bigr\rangle = t^{-1}\int_0^t f(s)\,\mathrm{d}s, \qquad \langle f\rangle^* = \limsup_{t\to+\infty} t^{-1}\int_0^t f(s)\,\mathrm{d}s;\\
&\langle f\rangle_* = \liminf_{t\to+\infty} t^{-1}\int_0^t f(s)\,\mathrm{d}s.
\end{aligned}$$

Before we state our results, we make some assumptions.

Assumption 1 $c_{11}c_{22} - \int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta) > 0$. This is a standard weak competitive assumption which allows the system not to blow up.

Assumption 2 There exists a constant K_1 such that $\int_{\mathbb{Z}}[\ln(1+\gamma_i(v))]^2\mu(\mathrm{d}v) < K_1$, $i = 1,2$, which means that the jump noise is not too strong.

Lemma 2.1 *For any given initial data* $(\phi_1(\theta), \phi_2(\theta)) \in C([-\tau, 0], R_+^2)$, *model* (1) *has a unique global solution* $x(t) = (x_1(t), x_2(t))^T \in R_+^2$ *almost surely* (*a.s.*). *In particular*,

$$\limsup_{t\to+\infty} \frac{\ln x_i(t)}{\ln t} \le 1, \quad a.s., i = 1,2. \tag{2}$$

Remark 2.1 The proof of Lemma 2.1 is a special case of Theorem 5.1 and Theorem 5.2 in Liu and Wang [25] and hence is omitted.

Lemma 2.2 (Liu, Wang and Wu [26]) *Let* $z(t) \in C[\Omega \times [0, +\infty), R_+]$.

($\mathfrak{A}$) *If there exist some constants* $T > 0$, $\lambda_0 > 0$, λ, σ_i *and* λ_i *such that, for all* $t \ge T$,

$$\ln z(t) \le \lambda t - \lambda_0 \int_0^t z(s)\,\mathrm{d}s + \sum_{i=1}^n \sigma_i B_i(t) + \sum_{i=1}^n \lambda_i \int_0^t \int_{\mathbb{Z}} \ln\bigl(1+\gamma_i(v)\bigr)\widetilde{N}(\mathrm{d}s, \mathrm{d}v), \quad a.s.,$$

then

$$\begin{cases} \langle z\rangle^* = \limsup_{t\to+\infty} t^{-1}\int_0^t z(s)\,\mathrm{d}s \le \lambda/\lambda_0 & a.s.,\ if\ \lambda \ge 0,\\ \lim_{t\to+\infty} z(t) = 0 & a.s.,\ if\ \lambda < 0.\end{cases}$$

($\mathfrak{B}$) *If there exist some constants* $T > 0$, $\lambda_0 > 0$, $\lambda > 0$, σ_i *and* λ_i *such that, for all* $t \ge T$,

$$\ln z(t) \ge \lambda t - \lambda_0 \int_0^t z(s)\,\mathrm{d}s + \sum_{i=1}^n \sigma_i B_i(t) + \sum_{i=1}^n \lambda_i \int_0^t \int_{\mathbb{Z}} \ln\bigl(1+\gamma_i(v)\bigr)\widetilde{N}(\mathrm{d}s, \mathrm{d}v), \quad a.s.,$$

then

$$\langle z\rangle_* = \liminf_{t\to+\infty} t^{-1}\int_0^t z(s)\,\mathrm{d}s \ge \lambda/\lambda_0, \quad \textit{a.s.}$$

Lemma 2.3 *Suppose that Assumption 1 and Assumption 2 hold, for model (1),*

(I) *if $b_1<0$ and $b_2<0$, then both x_1 and x_2 tend to extinction a.s., i.e., $\lim_{t\to+\infty} x_i(t)=0$, a.s., $i=1,2$;*

(II) *if $b_1>0$ and $b_2<0$, then x_2 tends to extinction a.s., and*

$$\lim_{t\to+\infty}\langle x_1(t)\rangle = \frac{b_1}{c_{11}}, \quad \textit{a.s.}$$

(III) *if $b_1<0$ and $b_2>0$, then x_1 tends to extinction a.s., and*

$$\lim_{t\to+\infty}\langle x_2(t)\rangle = \frac{b_2}{c_{22}}, \quad \textit{a.s.};$$

(IV) *if $b_1>0$ and $b_2>0$, then*

(i) *if $b_1c_{22}-b_2\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)>0$ and $b_2c_{11}-b_1\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)<0$, then x_2 tends to extinction a.s., and*

$$\lim_{t\to+\infty}\langle x_1(t)\rangle = \frac{b_1}{c_{11}}, \quad \textit{a.s.};$$

(ii) *if $b_1c_{22}-b_2\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)<0$ and $b_2c_{11}-b_1\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)>0$, then x_1 tends to extinction a.s., and*

$$\lim_{t\to+\infty}\langle x_2(t)\rangle = \frac{b_2}{c_{22}}, \quad \textit{a.s.};$$

(iii) *if $b_1c_{22}-b_2\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)>0$ and $b_2c_{11}-b_1\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)>0$, then both x_1 and x_2 are stable in time average a.s.:*

$$\begin{aligned}
\lim_{t\to+\infty}\langle x_1(t)\rangle &= \frac{b_1c_{22}-b_2\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)}{c_{11}c_{22}-\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)}, \quad \textit{a.s.};\\
\lim_{t\to+\infty}\langle x_2(t)\rangle &= \frac{b_2c_{11}-b_1\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)}{c_{11}c_{22}-\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)}, \quad \textit{a.s.}
\end{aligned} \tag{3}$$

Remark 2.2 It is necessary to point out that if $b_1>0$, $b_2>0$ and $c_{11}c_{22}-\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)\times\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)>0$, then $b_1c_{22}-b_2\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)<0$ and $b_2c_{11}-b_1\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)<0$ cannot hold simultaneously.

Proof of Lemma 2.3 Applying the generalized Itô formula to model (1) yields

$$\begin{aligned}
&\ln x_1(t)-\ln x_1(0)\\
&\quad = r_1t-h_1t-0.5\alpha_1^2t-c_{11}\int_0^t x_1(s)\,\mathrm{d}s-\int_0^t\int_{-\tau_{12}}^0 x_2(s+\theta)\,\mathrm{d}F_{12}(\theta)\,\mathrm{d}s+\alpha_1B_1(t)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{Z}}\bigl[\ln\bigl(x_1(s^-)+x_1(s^-)\gamma_1(\nu)\bigr)-\ln\bigl(x_1(s^-)\bigr)-\gamma_1(\nu)\bigr]\mu(\mathrm{d}\nu)\,\mathrm{d}s\\
&+\int_0^t\int_{\mathbb{Z}}\bigl[\ln\bigl(x_1(s^-)+x_1(s^-)\gamma_1(\nu)\bigr)-\ln\bigl(x_1(s^-)\bigr)\bigr]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu)\\
={}&r_1t-h_1t-0.5\alpha_1^2t+t\int_{\mathbb{Z}}\bigl[\ln\bigl(1+\gamma_1(\nu)\bigr)-\gamma_1(\nu)\bigr]\mu(\mathrm{d}\nu)\\
&-c_{11}\int_0^t x_1(s)\,\mathrm{d}s-\int_{-\tau_{12}}^0\mathrm{d}F_{12}(\theta)\int_0^t x_2(s)\,\mathrm{d}s-\int_{-\tau_{12}}^0\int_\theta^0 x_2(s)\,\mathrm{d}s\,\mathrm{d}F_{12}(\theta)\\
&+\int_{-\tau_{12}}^0\int_{t+\theta}^t x_2(s)\,\mathrm{d}s\,\mathrm{d}F_{12}(\theta)+\alpha_1B_1(t)+\int_0^t\int_{\mathbb{Z}}\bigl[\ln\bigl(1+\gamma_1(\nu)\bigr)\bigr]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu)\\
={}&b_1t-c_{11}\int_0^t x_1(s)\,\mathrm{d}s-\int_{-\tau_{12}}^0\mathrm{d}F_{12}(\theta)\int_0^t x_2(s)\,\mathrm{d}s\\
&-\int_{-\tau_{12}}^0\int_\theta^0 x_2(s)\,\mathrm{d}s\,\mathrm{d}F_{12}(\theta)+\int_{-\tau_{12}}^0\int_{t+\theta}^t x_2(s)\,\mathrm{d}s\,\mathrm{d}F_{12}(\theta)\\
&+\alpha_1B_1(t)+\int_0^t\int_{\mathbb{Z}}\bigl[\ln\bigl(1+\gamma_1(\nu)\bigr)\bigr]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&t^{-1}\ln\frac{x_1(t)}{x_1(0)}\\
&\quad=b_1-c_{11}\bigl\langle x_1(t)\bigr\rangle-\frac{1}{t}\int_0^t\int_{-\tau_{12}}^0 x_2(s+\theta)\,\mathrm{d}F_{12}(\theta)\,\mathrm{d}s+\frac{\alpha_1B_1(t)}{t}\\
&\qquad+\frac{1}{t}\int_0^t\int_{\mathbb{Z}}\bigl[\ln\bigl(1+\gamma_1(\nu)\bigr)\bigr]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu)\\
&\quad=b_1-c_{11}\bigl\langle x_1(t)\bigr\rangle-\int_{-\tau_{12}}^0\mathrm{d}F_{12}(\theta)\bigl\langle x_2(t)\bigr\rangle+\frac{\alpha_1B_1(t)}{t}\\
&\qquad+\frac{1}{t}\int_0^t\int_{\mathbb{Z}}\bigl[\ln\bigl(1+\gamma_1(\nu)\bigr)\bigr]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu)\\
&\qquad-\int_{-\tau_{12}}^0\int_\theta^0 x_2(s)\,\mathrm{d}s\,\mathrm{d}F_{12}(\theta)+\int_{-\tau_{12}}^0\int_{t+\theta}^t x_2(s)\,\mathrm{d}s\,\mathrm{d}F_{12}(\theta).
\end{aligned}
\tag{4}
$$

Similarly, one can also derive that

$$
\begin{aligned}
&t^{-1}\ln\frac{x_2(t)}{x_2(0)}\\
&\quad=b_2-\frac{1}{t}\int_0^t\int_{-\tau_{21}}^0 x_1(s+\theta)\,\mathrm{d}F_{21}(\theta)\,\mathrm{d}s-c_{22}\bigl\langle x_2(t)\bigr\rangle+\frac{\alpha_2B_2(t)}{t}\\
&\qquad+\frac{1}{t}\int_0^t\int_{\mathbb{Z}}\bigl[\ln\bigl(1+\gamma_2(\nu)\bigr)\bigr]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu)\\
&\quad=b_2-c_{22}\bigl\langle x_2(t)\bigr\rangle-\int_{-\tau_{21}}^0\mathrm{d}F_{21}(\theta)\bigl\langle x_1(t)\bigr\rangle+\frac{\alpha_2B_2(t)}{t}\\
&\qquad+\frac{1}{t}\int_0^t\int_{\mathbb{Z}}\bigl[\ln\bigl(1+\gamma_2(\nu)\bigr)\bigr]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu)
\end{aligned}
\tag{5}
$$

$$-\int_{-\tau_{21}}^{0}\int_{\theta}^{0} x_1(s)\,\mathrm{d}s\,\mathrm{d}F_{21}(\theta) + \int_{-\tau_{21}}^{0}\int_{t+\theta}^{t} x_1(s)\,\mathrm{d}s\,\mathrm{d}F_{21}(\theta).$$

First, we prove (I). From the first equality in (4), we can get

$$t^{-1}\ln\frac{x_1(t)}{x_1(0)} \le b_1 - c_{11}\langle x_1(t)\rangle + \frac{\alpha_1 B_1(t)}{t} + t^{-1}\int_0^t\int_{\mathbb{Z}}\bigl[\ln\bigl(1+\gamma_1(\nu)\bigr)\bigr]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu).$$

Since $b_1 < 0$, according to ($\mathfrak{A}$) of Lemma 2.2, we have

$$\lim_{t\to+\infty} x_1(t) = 0, \quad \text{a.s.}$$

In the same way, we can derive that if $b_2 < 0$, then $\lim_{t\to+\infty} x_2(t) = 0$, a.s. by (5).

Second, we prove (II). Since $b_2 < 0$, from (I), we can note that $\lim_{t\to+\infty} x_2(t) = 0$, a.s. Thus, for arbitrary $\varepsilon > 0$, there is a random time $T_1 > 0$ such that, for $t \ge T_1$,

$$-\varepsilon \le \frac{1}{t}\int_0^t\int_{-\tau_{12}}^{0} x_2(s+\theta)\,\mathrm{d}F_{12}(\theta)\,\mathrm{d}s \le \varepsilon.$$

The above inequality can be applied to the first equality in (4), we can get

$$\begin{aligned} t^{-1}\ln\frac{x_1(t)}{x_1(0)} &\le b_1 + \varepsilon - c_{11}\langle x_1(t)\rangle + t^{-1}\alpha_1 B_1(t) \\ &\quad + t^{-1}\int_0^t\int_{\mathbb{Z}}\bigl[\ln\bigl(1+\gamma_1(\nu)\bigr)\bigr]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu), \end{aligned} \tag{6}$$

$$\begin{aligned} t^{-1}\ln\frac{x_1(t)}{x_1(0)} &\ge b_1 - \varepsilon - c_{11}\langle x_1(t)\rangle + t^{-1}\alpha_1 B_1(t) \\ &\quad + t^{-1}\int_0^t\int_{\mathbb{Z}}\bigl[\ln\bigl(1+\gamma_1(\nu)\bigr)\bigr]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu), \end{aligned} \tag{7}$$

Because of the arbitrariness of ε, we can choose ε sufficiently small such that $b_1 - \varepsilon > 0$ ($b_1 > 0$). Applying Lemma 2.2 to (6) and (7), respectively, we can obtain

$$\frac{b_1-\varepsilon}{c_{11}} \le \langle x_1\rangle_* \le \langle x_1\rangle^* \le \frac{b_1+\varepsilon}{c_{11}}.$$

Letting $\varepsilon \to 0$, we get $\lim_{t\to+\infty}\langle x_1(t)\rangle = b_1/c_{11}$, a.s.

Third, we prove (III). The proof of (III) is similar to that of (II) by symmetry and hence is omitted.

Fourth, we prove (IV). Firstly, we consider the following equation:

$$\begin{cases} \mathrm{d}y_1(t) = y_1(t^-)[r_1 - h_1 - c_{11}y_1(t^-)]\,\mathrm{d}t + \alpha_1 y_1(t^-)\,\mathrm{d}B_1(t) \\ \qquad\qquad + y_1(t^-)\int_{\mathbb{Z}}\gamma_1(\nu)\widetilde{N}(\mathrm{d}t,\mathrm{d}\nu), \\ \mathrm{d}y_2(t) = y_2(t^-)[r_2 - h_2 - c_{22}y_2(t^-)]\,\mathrm{d}t + \alpha_2 y_2(t^-)\,\mathrm{d}B_2(t) \\ \qquad\qquad + y_2(t^-)\int_{\mathbb{Z}}\gamma_2(\nu)\widetilde{N}(\mathrm{d}t,\mathrm{d}\nu), \end{cases}$$

where $y_i(\theta) = x_i(\theta)$, $\theta \in [-\tau, 0]$. On the basis of the stochastic comparison theorem in Huang [27], one can obtain

$$x_1(t) \le y_1(t), \qquad x_2(t) \le y_2(t). \tag{8}$$

Note that $b_1 > 0$ and $b_2 > 0$, using a proof like (II), we can get

$$\lim_{t\to+\infty}\langle y_i(t)\rangle = \frac{b_i}{c_{ii}}, \quad \text{a.s.}, i = 1, 2.$$

Hence, for arbitrary $\widetilde{\tau} \geq 0$, we have

$$\lim_{t\to+\infty} t^{-1}\int_{t-\widetilde{\tau}}^{t} y_i(s)\,\mathrm{d}s = \lim_{t\to+\infty} t^{-1}\left[\int_0^t y_i(s)\,\mathrm{d}s - \int_0^{t-\widetilde{\tau}} y_i(s)\,\mathrm{d}s\right] = 0, \quad \text{a.s.},$$

according to (8), so we can see that

$$\lim_{t\to+\infty} t^{-1}\int_{t-\widetilde{\tau}}^{t} x_i(s)\,\mathrm{d}s = 0, \quad \text{a.s.}, i = 1, 2, \widetilde{\tau} \geq 0. \tag{9}$$

Computing (5) $\times$ c_{11} − (4) $\times$ $\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)$, we have

$$\begin{aligned}
&\frac{c_{11}}{t}\ln\frac{x_2(t)}{x_2(0)} \\
&\quad = \frac{1}{t}\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)\ln\frac{x_1(t)}{x_1(0)} + b_2c_{11} - b_1\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta) \\
&\qquad - \left(c_{11}c_{22} - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)\right)\langle x_2(t)\rangle \\
&\qquad + \int_0^t\int_{\mathbb{Z}}\left[\frac{c_{11}}{t}\ln\bigl(1+\gamma_2(v)\bigr) - \frac{1}{t}\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)\ln\bigl(1+\gamma_1(v)\bigr)\right]\widetilde{N}(\mathrm{d}s, \mathrm{d}v) \\
&\qquad + \frac{1}{t}\left(c_{11}\alpha_2 B_2(t) - \int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)\alpha_1 B_1(t)\right) + \Delta(t),
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
\Delta(t) = \frac{1}{t}\biggl(&c_{11}\int_{-\tau_{21}}^{0}\int_{t+\theta}^{t} x_1(s)\,\mathrm{d}s\,\mathrm{d}F_{21}(\theta) - c_{11}\int_{-\tau_{21}}^{0}\int_{\theta}^{0} x_1(s)\,\mathrm{d}s\,\mathrm{d}F_{21}(\theta) \\
&+ \int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)\int_{-\tau_{12}}^{0}\int_{\theta}^{0} x_2(s)\,\mathrm{d}s\,\mathrm{d}F_{12}(\theta) \\
&- \int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)\int_{-\tau_{12}}^{0}\int_{t+\theta}^{t} x_2(s)\,\mathrm{d}s\,\mathrm{d}F_{12}(\theta)\biggr).
\end{aligned}$$

Thanks to (2) and (9), for arbitrary $\varepsilon > 0$, there is a random time $T_2 > 0$ such that, for $t \geq T_2$,

$$\frac{1}{t}\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)\ln\frac{x_1(t)}{x_1(0)} < \frac{\varepsilon}{2},$$

$$\begin{aligned}
&\left|\Delta(t)\right| \\
&\leq \frac{1}{t}\left(c_{11}\int_{-\tau_{21}}^{0}\mathrm{d}F_{21}(\theta)\int_{t-\tau_{21}}^{t}x_1(s)\,\mathrm{d}s - c_{11}\int_{-\tau_{21}}^{0}\mathrm{d}F_{21}(\theta)\int_{-\tau_{21}}^{0}x_1(s)\,\mathrm{d}s\right. \\
&\quad + \int_{-\tau_{21}}^{0}\mathrm{d}F_{21}(\theta)\int_{-\tau_{12}}^{0}\mathrm{d}F_{12}(\theta)\int_{-\tau_{12}}^{0}x_2(s)\,\mathrm{d}s \\
&\quad \left. - \int_{-\tau_{21}}^{0}\mathrm{d}F_{21}(\theta)\int_{-\tau_{12}}^{0}\mathrm{d}F_{12}(\theta)\int_{t-\tau_{12}}^{t}x_2(s)\,\mathrm{d}s\right) < \frac{\varepsilon}{2}.
\end{aligned}$$

Applying the above inequalities to (10), we have

$$\begin{aligned}
&\frac{c_{11}}{t}\ln\frac{x_2(t)}{x_2(0)} \\
&\leq b_2c_{11} - b_1\int_{-\tau_{21}}^{0}\mathrm{d}F_{21}(\theta) + \varepsilon \\
&\quad - \left(c_{11}c_{22} - \int_{-\tau_{12}}^{0}\mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0}\mathrm{d}F_{21}(\theta)\right)\langle x_2(t)\rangle \\
&\quad + \frac{1}{t}\left(c_{11}\alpha_2B_2(t) - \int_{-\tau_{21}}^{0}\mathrm{d}F_{21}(\theta)\alpha_1B_1(t)\right) \\
&\quad + \frac{c_{11}}{t}\int_0^t\int_{\mathbb{Z}}\left[\ln\left(1+\gamma_2(\nu)\right)\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu) \\
&\quad - \frac{1}{t}\int_{-\tau_{21}}^{0}\mathrm{d}F_{21}(\theta)\int_0^t\int_{\mathbb{Z}}\left[\ln\left(1+\gamma_1(\nu)\right)\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu).
\end{aligned} \tag{11}$$

Similarly, computing (4) $\times$ c_{22} – (5) $\times \int_{-\tau_{12}}^{0}\mathrm{d}F_{12}(\theta)$ and by (2) and (9), we can show that

$$\begin{aligned}
&\frac{c_{22}}{t}\ln\frac{x_1(t)}{x_1(0)} \\
&\leq b_1c_{22} - b_2\int_{-\tau_{12}}^{0}\mathrm{d}F_{12}(\theta) + \varepsilon \\
&\quad - \left(c_{11}c_{22} - \int_{-\tau_{12}}^{0}\mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0}\mathrm{d}F_{21}(\theta)\right)\langle x_1(t)\rangle \\
&\quad + \frac{1}{t}\left(c_{22}\alpha_1B_1(t) - \int_{-\tau_{12}}^{0}\mathrm{d}F_{12}(\theta)\alpha_2B_2(t)\right) \\
&\quad + \frac{c_{22}}{t}\int_0^t\int_{\mathbb{Z}}\left[\ln\left(1+\gamma_1(\nu)\right)\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu) \\
&\quad - \frac{1}{t}\int_{-\tau_{12}}^{0}\mathrm{d}F_{12}(\theta)\int_0^t\int_{\mathbb{Z}}\left[\ln\left(1+\gamma_2(\nu)\right)\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu),
\end{aligned} \tag{12}$$

for $t > T$ and arbitrary $\varepsilon > 0$.

(i): Note that $b_2c_{11} - b_1\int_{-\tau_{21}}^{0}\mathrm{d}F_{21}(\theta) < 0$, and then let $\varepsilon \to 0$ such that $b_2c_{11} - b_1 \times \int_{-\tau_{21}}^{0}\mathrm{d}F_{21}(\theta) + \varepsilon < 0$. Applying $(\mathfrak{A})$ in Lemma 2.2 to (11), one can get $\lim_{t\to+\infty}x_2(t) = 0$, a.s. The proof of $\lim_{t\to+\infty}\langle x_1(t)\rangle = b_1/c_{11}$, a.s. is similar to that of (II) and hence is omitted.

(ii): The proof of (ii) is similar to that of (i) by symmetry and hence is omitted.

(iii): Since $b_2c_{11} - b_1\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta) > 0$, by (11), ($\mathfrak{A}$) in Lemma 2.2 we note that

$$\langle x_2\rangle^* \le \frac{b_2c_{11} - b_1\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta) + \varepsilon}{c_{11}c_{22} - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)}, \quad \text{a.s.}$$

We let $\varepsilon \to 0$ so that

$$\langle x_2\rangle^* \le \frac{b_2c_{11} - b_1\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)}{c_{11}c_{22} - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)}, \quad \text{a.s.} \tag{13}$$

In the same way, using (12), ($\mathfrak{A}$) in Lemma 2.2 and the arbitrariness of ε we see that

$$\langle x_1\rangle^* \le \frac{b_1c_{22} - b_2\int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)}{c_{11}c_{22} - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)}, \quad \text{a.s.} \tag{14}$$

Let ε be sufficiently small such that $c_{11}\frac{b_1c_{22} - b_2\int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)}{c_{11}c_{22} - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)} - \varepsilon > 0$. Equations (9) and (13) can be applied to (4), we get

$$\begin{aligned}
t^{-1}\ln\frac{x_1(t)}{x_1(0)} &\ge b_1 - \varepsilon - c_{11}\langle x_1(t)\rangle - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\langle x_2\rangle^* + \frac{\alpha_1B_1(t)}{t} \\
&\quad + \frac{1}{t}\int_0^t\int_{\mathbb{Z}}\bigl[\ln\bigl(1+\gamma_1(\nu)\bigr)\bigr]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu) \\
&\ge b_1 - \varepsilon - c_{11}\langle x_1(t)\rangle \\
&\quad - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\frac{b_2c_{11} - b_1\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)}{c_{11}c_{22} - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)} \\
&\quad + \frac{\alpha_1B_1(t)}{t} + \frac{1}{t}\int_0^t\int_{\mathbb{Z}}\bigl[\ln\bigl(1+\gamma_1(\nu)\bigr)\bigr]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu) \\
&= c_{11}\frac{b_1c_{22} - b_2\int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)}{c_{11}c_{22} - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)} - \varepsilon - c_{11}\langle x_1(t)\rangle \\
&\quad + \frac{\alpha_1B_1(t)}{t} + \frac{1}{t}\int_0^t\int_{\mathbb{Z}}\bigl[\ln\bigl(1+\gamma_1(\nu)\bigr)\bigr]\widetilde{N}(\mathrm{d}s,\mathrm{d}\nu),
\end{aligned}$$

for sufficiently large t. Due to ($\mathfrak{B}$) in Lemma 2.2 and the arbitrariness of ε we see that

$$\langle x_1\rangle_* \ge \frac{b_1c_{22} - b_2\int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)}{c_{11}c_{22} - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)}, \quad \text{a.s.} \tag{15}$$

Similarly, when (9) and (14) are used in (5), we get

$$\langle x_2\rangle_* \ge \frac{b_2c_{11} - b_1\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)}{c_{11}c_{22} - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)}, \quad \text{a.s.}$$

This, together with (13)–(15), means that $\lim_{t\to+\infty}\langle x_1(t)\rangle = \frac{b_1c_{22}-b_2\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)}{c_{11}c_{22}-\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)}$, a.s. and $\lim_{t\to+\infty}\langle x_2(t)\rangle = \frac{b_2c_{11}-b_1\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)}{c_{11}c_{22}-\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)}$, a.s.

The proof is complete. □

3 Stability in distribution

In this section, we study the stability in distribution of model (1). Firstly, we state an assumption and a lemma.

Assumption 3 $c_{11} > \int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)$ and $c_{22} > \int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)$.

Lemma 3.1 *There exists a positive K_2 such that*

$$\limsup_{t\to\infty} E\big(x_i(t)\big) \le K_2, \quad i = 1,2.$$

Remark 3.1 We omit the proof for Lemma 3.1. For details, please refer to Bao and Yuan [28].

Lemma 3.2 *If Assumption 3 holds, then model (1) is said to be stable in distribution, i.e., there is a unique probability measure $\varphi(\cdot)$ such that, for every initial data $x(\theta) \in C([-\tau,0],R_+^2)$, the transition probability $p(t,x(\theta),\cdot)$ of $x(t)$ converges weakly to $\varphi(\cdot)$ as $t \to +\infty$.*

Proof of Lemma 3.2 Suppose that $x(t;x(\theta))$ and $x(t;\widetilde{x}(\theta))$ are two solutions of model (1) with initial data $x(\theta) \in C([-\tau,0];R_+^2)$ and $\widetilde{x}(\theta) \in C([-\tau,0];R_+^2)$, respectively. Set

$$V_1(t) = \big|\ln x_1\big(t;x(\theta)\big) - \ln x_1\big(t;\widetilde{x}(\theta)\big)\big| + \big|\ln x_2\big(t;x(\theta)\big) - \ln x_2\big(t;\widetilde{x}(\theta)\big)\big|.$$

In view of the above equality, by Itô's formula, one can deduce that

$$\begin{aligned}
&\mathrm{d}^+V_1(t)\\
&\quad= \operatorname{sgn}\big(x_1\big(t;x(\theta)\big) - x_1\big(t;\widetilde{x}(\theta)\big)\big)\Big[-c_{11}\big(x_1\big(t;x(\theta)\big) - x_1\big(t;\widetilde{x}(\theta)\big)\big)\\
&\qquad - \int_{-\tau_{12}}^0 \big[x_2\big(t+\theta;x(\theta)\big) - x_2\big(t+\theta;\widetilde{x}(\theta)\big)\big]\,\mathrm{d}F_{12}(\theta)\Big]\,\mathrm{d}t\\
&\qquad + \operatorname{sgn}\big(x_2\big(t;x(\theta)\big) - x_2\big(t;\widetilde{x}(\theta)\big)\big)\Big[-c_{22}\big(x_2\big(t;x(\theta)\big) - x_2\big(t;\widetilde{x}(\theta)\big)\big)\\
&\qquad - \int_{-\tau_{21}}^0 \big[x_1\big(t+\theta;x(\theta)\big) - x_1\big(t+\theta;\widetilde{x}(\theta)\big)\big]\,\mathrm{d}F_{21}(\theta)\Big]\,\mathrm{d}t\\
&\quad\le -\sum_{i=1}^2 c_{ii}\big|x_i\big(t;x(\theta)\big) - x_i\big(t;\widetilde{x}(\theta)\big)\big|\,\mathrm{d}t\\
&\qquad + \int_{-\tau_{12}}^0 \big|x_2\big(t+\theta;x(\theta)\big) - x_2\big(t+\theta;\widetilde{x}(\theta)\big)\big|\,\mathrm{d}F_{12}(\theta)\,\mathrm{d}t\\
&\qquad + \int_{-\tau_{21}}^0 \big|x_1\big(t+\theta;x(\theta)\big) - x_1\big(t+\theta;\widetilde{x}(\theta)\big)\big|\,\mathrm{d}F_{21}(\theta)\,\mathrm{d}t.
\end{aligned}$$

Define

$$V(t) = V_1(t) + V_2(t),$$

where

$$\begin{aligned} V_2(t) = & \int_{-\tau_{12}}^{0}\int_{t+\theta}^{t}\big|x_2\big(s;x(\theta)\big) - x_2\big(s;\widetilde{x}(\theta)\big)\big|\,\mathrm{d}s\,\mathrm{d}F_{12}(\theta) \\ & + \int_{-\tau_{21}}^{0}\int_{t+\theta}^{t}\big|x_1\big(s;x(\theta)\big) - x_1\big(s;\widetilde{x}(\theta)\big)\big|\,\mathrm{d}s\,\mathrm{d}F_{21}(\theta). \end{aligned}$$

From Itô's formula, we have

$$\begin{aligned} \mathrm{d}^+V(t) &= \mathrm{d}^+V_1(t) + \mathrm{d}^+V_2(t) \\ &\le -\sum_{i=1}^{2} c_{ii}\big|x_i\big(t;x(\theta)\big) - x_i\big(t;\widetilde{x}(\theta)\big)\big|dt \\ &\quad + \int_{-\tau_{12}}^{0}\big|x_2\big(t;x(\theta)\big) - x_2\big(t;\widetilde{x}(\theta)\big)\big|\,\mathrm{d}F_{12}(\theta)\,\mathrm{d}t \\ &\quad + \int_{-\tau_{21}}^{0}\big|x_1\big(t;x(\theta)\big) - x_1\big(t;\widetilde{x}(\theta)\big)\big|\,\mathrm{d}F_{21}(\theta)\,\mathrm{d}t \\ &= -\left(c_{11} - \int_{-\tau_{21}}^{0}\mathrm{d}F_{21}(\theta)\right)\big|x_1\big(t;x(\theta)\big) - x_1\big(t;\widetilde{x}(\theta)\big)\big|\,\mathrm{d}t \\ &\quad - \left(c_{22} - \int_{-\tau_{12}}^{0}\mathrm{d}F_{12}(\theta)\right)\big|x_2\big(t;x(\theta)\big) - x_2\big(t;\widetilde{x}(\theta)\big)\big|\,\mathrm{d}t. \end{aligned}$$

Consequently,

$$\begin{aligned} 0 \le \mathbb{E}\big(V(t)\big) \le V(0) &- \left(c_{11} - \int_{-\tau_{21}}^{0}\mathrm{d}F_{21}(\theta)\right)\int_0^t \mathbb{E}\big|x_1\big(s;x(\theta)\big) - x_1\big(s;\widetilde{x}(\theta)\big)\big|\,\mathrm{d}s \\ &- \left(c_{22} - \int_{-\tau_{12}}^{0}\mathrm{d}F_{12}(\theta)\right)\int_0^t \mathbb{E}\big|x_2\big(s;x(\theta)\big) - x_2\big(s;\widetilde{x}(\theta)\big)\big|\,\mathrm{d}s, \end{aligned}$$

which implies that

$$\begin{aligned} &\left(c_{11} - \int_{-\tau_{21}}^{0}\mathrm{d}F_{21}(\theta)\right)\int_0^t \mathbb{E}\big|x_1\big(s;x(\theta)\big) - x_1\big(s;\widetilde{x}(\theta)\big)\big|\,\mathrm{d}s \le V(0) < +\infty, \\ &\left(c_{22} - \int_{-\tau_{12}}^{0}\mathrm{d}F_{12}(\theta)\right)\int_0^t \mathbb{E}\big|x_2\big(s;x(\theta)\big) - x_2\big(s;\widetilde{x}(\theta)\big)\big|\,\mathrm{d}s \le V(0) < +\infty. \end{aligned}$$

Therefore,

$$\mathbb{E}\big|x_i\big(t;x(\theta)\big) - x_i\big(t;\widetilde{x}(\theta)\big)\big| \in L^1[0,+\infty), \quad i = 1,2. \tag{16}$$

From model (1), we note that

$$\mathbb{E}\big(x_1(t)\big) = x_1(0) + \int_0^t \Big[(r_1 - h_1)\mathbb{E}\big(x_1(s)\big) - c_{11}\mathbb{E}\big(x_1(s)\big)^2$$

$$-\int_{-\tau_{12}}^{0} \mathbb{E}\bigl(x_1(s)x_2(s+\theta)\bigr)\,\mathrm{d}F_{12}(\theta)\biggr]\,\mathrm{d}s.$$

This implies the differentiability of $\mathbb{E}(x_1(t))$. According to Lemma 3.1,

$$\begin{aligned}\frac{\mathrm{d}\mathbb{E}(x_1(t))}{\mathrm{d}t} &= (r_1-h_1)\mathbb{E}\bigl(x_1(t)\bigr) - c_{11}\mathbb{E}\bigl(x_1(t)\bigr)^2 - \int_{-\tau_{12}}^{0} \mathbb{E}\bigl(x_1(s)x_2(s+\theta)\bigr)\,\mathrm{d}F_{12}(\theta)\\ &\le r_1\mathbb{E}\bigl(x_1(t)\bigr)\\ &\le r_1 D_1,\end{aligned}$$

where $D_1 > 0$ is a constant. Thus $\mathbb{E}(x_1(t))$ is uniformly continuous. Similarly, $\mathbb{E}(x_2(t))$ is also uniformly continuous. According to Barbalat's lemma (Barbalat [29]) and (16), we can get

$$\lim_{t\to+\infty} \mathbb{E}\bigl|x_i(t)-\widetilde{x}_i(t)\bigr| = 0, \quad i = 1,2. \tag{17}$$

Let $P(t,x(\theta),Q)$ denotes the probability of $x(t;x(\theta)) \in Q$. From Lemma 3.1 and the Chebyshev inequality, we note that $\{p(t,x(\theta),dy) : t \ge 0\}$ is tight. $\mathscr{P}(R_+^2)$ denotes all probability measures defined on R_+^2. For any $P_1, P_2 \in \mathscr{P}$, we define the metric

$$d_U(P_1,P_2) = \sup_{g\in U}\biggl|\int_{R_+^2} g(x)P_1(\mathrm{d}x) - \int_{R_+^2} g(x)P_2(\mathrm{d}x)\biggr|,$$

where

$$U = \bigl\{g : R^2 \to R \,|\, \bigl|g(x)-g(y)\bigr| \le \|x-y\|, \bigl|g(\cdot)\bigr| \le 1\bigr\}.$$

For any $g \in U$ and $t, s > 0$, we have

$$\begin{aligned}&\bigl|\mathbb{E}\bigl(g\bigl(x\bigl(t+s;x(\theta)\bigr)\bigr)\bigr) - \mathbb{E}\bigl(g\bigl(x\bigl(t;x(\theta)\bigr)\bigr)\bigr)\bigr|\\ &\quad= \bigl|\mathbb{E}\bigl[\mathbb{E}\bigl(g\bigl(x\bigl(t+s;x(\theta)\bigr)\bigr)|\mathcal{F}_s\bigr)\bigr] - \mathbb{E}g\bigl(x\bigl(t;x(\theta)\bigr)\bigr)\bigr|\\ &\quad= \biggl|\int_{R_+^2} \mathbb{E}g\bigl(x\bigl(t;\widetilde{x}(\theta)\bigr)\bigr)p\bigl(s,x(\theta),\mathrm{d}\widetilde{x}(\theta)\bigr) - \mathbb{E}g\bigl(x\bigl(t;x(\theta)\bigr)\bigr)\biggr|\\ &\quad\le \int_{R_+^2} \bigl|\mathbb{E}g\bigl(x\bigl(t;\widetilde{x}(\theta)\bigr)\bigr) - \mathbb{E}g\bigl(x\bigl(t;x(\theta)\bigr)\bigr)\bigr|p\bigl(s,x(\theta),\mathrm{d}\widetilde{x}(\theta)\bigr)\\ &\quad\le \int_{\bar{G}_K} \bigl|\mathbb{E}g\bigl(x\bigl(t;\widetilde{x}(\theta)\bigr)\bigr) - \mathbb{E}g\bigl(x\bigl(t;x(\theta)\bigr)\bigr)\bigr|p\bigl(s,x(\theta),\mathrm{d}\widetilde{x}(\theta)\bigr)\\ &\qquad + 2p\bigl(s,x(\theta),G_K^c\bigr),\end{aligned} \tag{18}$$

where $\bar{G}_K = \{x \in R_+^2 : |x| \le K\}$ and $G_K^c = R_+^2 - \bar{G}_K$. Note that $\{p(t,x(\theta),\mathrm{d}y)\}$ is tight, thus there exists a sufficiently large K such that $p(s,x(\theta),G_K^c) < \varepsilon$, $\forall s \ge 0$. According to (17), for arbitrarily $\varepsilon > 0$, there exists a $T > 0$ such that

$$\sup_{g\in U}|\mathbb{E}\bigl(g\bigl(x\bigl(t;\widetilde{x}(\theta)\bigr)\bigr)\bigr) - \mathbb{E}\bigl(g\bigl(x\bigl(t;x(\theta)\bigr)\bigr)\bigr)| \le \varepsilon, \quad t \ge T.$$

This inequality can be applied to (18), we can get

$$|\mathbb{E}\bigl(g\bigl(x\bigl(t+s;x(\theta)\bigr)\bigr)\bigr) - \mathbb{E}\bigl(g\bigl(x\bigl(t;x(\theta)\bigr)\bigr)\bigr)| \le 3\varepsilon, \quad t \ge T.$$

By the arbitrariness of g, we obtain

$$\sup_{g\in U}\left|\mathbb{E}\big(g\big(x\big(t+s;x(\theta)\big)\big)-\mathbb{E}\big(g\big(x\big(t;x(\theta)\big)\big)\big)\right|\le 3\varepsilon,\quad t\ge T.$$

That is to say, for arbitrarily $t\ge T$, $s>0$,

$$d_U\big(p\big(t+s,x(\theta),\cdot\big),p\big(t,x(\theta),\cdot\big)\big)\le 3\varepsilon.$$

This implies that $p(t,x(\theta),\cdot)$ is Cauchy in $\mathscr{P}(R_+^2)$ for any initial data $x(\theta)\in C([-\tau,0];R_+^2)$. Therefore, there is a unique $\varphi\in\mathscr{P}(R_+^2)$ such that

$$\lim_{t\to+\infty}d_U\big(p\big(t,\psi(\theta),\cdot\big),\varphi(\cdot)\big)=0, \tag{19}$$

where $\psi(\theta)=(\psi_1(\theta),\psi_2(\theta))^T$, $\psi_i(\theta)\equiv 0.1$, $\theta\in[-\tau,0]$. From (17), we have

$$\lim_{t\to+\infty}d_U\big(p\big(t,x(\theta),\cdot\big),p\big(t,\psi(\theta),\cdot\big)\big)=0,$$

combining with (19) this implies

$$\begin{aligned}&\lim_{t\to+\infty}d_U\big(p\big(t,x(\theta),\cdot\big),\varphi(\cdot)\big)\\&\quad\le\lim_{t\to+\infty}d_U\big(p\big(t,x(\theta),\cdot\big),p\big(t,\psi(\theta),\cdot\big)\big)+\lim_{t\to+\infty}d_U\big(p\big(t,\psi(\theta),\cdot\big),\varphi(\cdot)\big)\\&\quad=0.\end{aligned}$$

The proof is completed. □

4 Optimal harvesting

In this section, we will state and prove our main results. For the sake of making the proof work, we introduce the following technical assumption.

Assumption 4 $4c_{11}c_{22}-(c_{11}c_{22}-\int_{-\tau_{12}}^0\mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^0\mathrm{d}F_{21}(\theta))^2<0$.

Theorem 4.1 *For model* (1), *suppose that Assumptions* 1–4 *hold. Define*

$$\begin{cases}h_1^*=\frac{2c_{11}\aleph_1}{4c_{11}c_{22}-(\int_{-\tau_{12}}^0\mathrm{d}F_{12}(\theta)+\int_{-\tau_{21}}^0\mathrm{d}F_{21}(\theta))^2}\\\qquad+\frac{(\int_{-\tau_{12}}^0\mathrm{d}F_{12}(\theta)+\int_{-\tau_{21}}^0\mathrm{d}F_{21}(\theta))\aleph_2}{4c_{11}c_{22}-(\int_{-\tau_{12}}^0\mathrm{d}F_{12}(\theta)+\int_{-\tau_{21}}^0\mathrm{d}F_{21}(\theta))^2},\\h_2^*=\frac{2c_{22}\aleph_2}{4c_{11}c_{22}-(\int_{-\tau_{12}}^0\mathrm{d}F_{12}(\theta)+\int_{-\tau_{21}}^0\mathrm{d}F_{21}(\theta))^2}\\\qquad+\frac{(\int_{-\tau_{12}}^0\mathrm{d}F_{12}(\theta)+\int_{-\tau_{21}}^0\mathrm{d}F_{21}(\theta))\aleph_1}{4c_{11}c_{22}-(\int_{-\tau_{12}}^0\mathrm{d}F_{12}(\theta)+\int_{-\tau_{21}}^0\mathrm{d}F_{21}(\theta))^2},\end{cases}$$

where $\aleph_1=(r_1-\frac{\alpha_1^2}{2}-\kappa_1)-(r_2-\frac{\alpha_2^2}{2}-\kappa_2)\int_{-\tau_{12}}^0\mathrm{d}F_{12}(\theta)$, $\aleph_2=(r_2-\frac{\alpha_2^2}{2}-\kappa_2)-(r_1-\frac{\alpha_1^2}{2}-\kappa_1)\int_{-\tau_{21}}^0\mathrm{d}F_{21}(\theta)$, $\kappa_i=\int_{\mathbb{Z}}[\gamma_i(\nu)-\ln(1+\gamma_i(\nu))]\mu(\mathrm{d}\nu)$, $i=1,2$.

($\mathcal{A}$) *If* $b_1|_{h_1=h_1^*,h_2=h_2^*} > 0$, $b_2|_{h_1=h_1^*,h_2=h_2^*} > 0$, $b_1c_{22} - b_2\int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)|_{h_1=h_1^*,h_2=h_2^*} > 0$ *and* $b_2c_{11} - b_1\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)|_{h_1=h_1^*,h_2=h_2^*} > 0$, *then OHE is* $H^* = (h_1^*, h_2^*)$ *and MESY is*

$$\begin{aligned} Y^* = {} & h_1^* \frac{b_1c_{22} - b_2\int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)}{c_{11}c_{22} - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)} \\ & + h_2^* \frac{b_2c_{11} - b_1\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)}{c_{11}c_{22} - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)}. \end{aligned}$$

($\mathcal{B}$) *If the conditions* ($\mathcal{A}$) *do not hold, then the optimal harvesting policy does not exist.*

Proof of Theorem 4.1 Let $G = \{H = (h_1, h_2)^T \in R^2 | b_i > 0, \Phi_i > 0, h_i \geq 0, i = 1, 2\}$, where $\Phi_1 = b_1c_{22} - b_2\int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)|_{h_1=h_1^*,h_2=h_2^*}$, $\Phi_2 = b_2c_{11} - b_1\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)|_{h_1=h_1^*,h_2=h_2^*}$. By Lemma 2.3, we can note that if $H \in G$, then (3) holds; if H^* exists, then $H^* \in G$.

Firstly, we prove ($\mathcal{A}$). Clearly, $A = (h_1^*, h_2^*) \in G$, hence G is not empty. For any $H \in G$,

$$\lim_{t\to+\infty} t^{-1}\int_0^t H^T x(s)\,\mathrm{d}s = \sum_{i=1}^{2} h_i \lim_{t\to+\infty} t^{-1}\int_0^t x_i(s)\,\mathrm{d}s, \quad \text{a.s.} \tag{20}$$

From Lemma 3.2, model (1) has a unique invariant measure $\varphi(\cdot)$. It is according to Corollary 3.4.3 in Prato and Zabczyk [30] that $\varphi(\cdot)$ is strong mixing. Then, as follows from Theorem 3.2.6 in [30], $\varphi(\cdot)$ is ergodic. By (3.3.2) in [30], we have

$$\lim_{t\to+\infty} t^{-1}\int_0^t H^T x(s)\,\mathrm{d}s = \int_{R_+^2} H^T x\varphi(\mathrm{d}x). \tag{21}$$

Let $\rho(x)$ express- the stationary probability density of (1). Therefore,

$$Y(H) = \lim_{t\to+\infty} \sum_{i=1}^{2} \mathbb{E}\big(h_i x_i(t)\big) = \lim_{t\to+\infty} \mathbb{E}\big(H^T x(t)\big) = \int_{R_+^2} H^T x\rho(x)\,\mathrm{d}x. \tag{22}$$

Due to the invariant measure of model (1) is unique, it then follows from the one-to-one correspondence between $\rho(x)$ and its invariant measure (see e.g. [30], p.105) that

$$\int_{R_+^2} H^T x\rho(x)\,\mathrm{d}x = \int_{R_+^2} H^T x\varphi(\mathrm{d}x). \tag{23}$$

From (20)–(23) and (3), we obtain

$$\begin{aligned} Y(H) = {} & h_1 \frac{b_1c_{22} - b_2\int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)}{c_{11}c_{22} - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)} \\ & + h_2 \frac{b_2c_{11} - b_1\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)}{c_{11}c_{22} - \int_{-\tau_{12}}^{0} \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^{0} \mathrm{d}F_{21}(\theta)}. \end{aligned} \tag{24}$$

A direct calculation yields

$$\begin{cases} \frac{\partial Y(H)}{\partial h_1} = \frac{-2c_{22}h_1+(\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)+\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta))h_2+\aleph_1}{c_{11}c_{22}-\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)} \\ \frac{\partial Y(H)}{\partial h_2} = \frac{-2c_{11}h_2+(\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)+\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta))h_1+\aleph_2}{c_{11}c_{22}-\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)} \\ \frac{\partial Y^2(H)}{\partial h_1^2} = \frac{-2c_{22}}{c_{11}c_{22}-\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)}, \\ \frac{\partial Y^2(H)}{\partial h_1\,\partial h_2} = \frac{\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)+\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)}{c_{11}c_{22}-\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)}, \\ \frac{\partial Y^2(H)}{\partial h_2\,\partial h_1} = \frac{\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)+\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)}{c_{11}c_{22}-\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)}, \\ \frac{\partial Y^2(H)}{\partial h_2^2} = \frac{-2c_{11}}{c_{11}c_{22}-\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)}. \end{cases} \tag{25}$$

Let $\frac{\partial Y(H)}{\partial h_1} = 0$ and $\frac{\partial Y(H)}{\partial h_2} = 0$. By (25), we have $h_1 = h_1^*$, $h_2 = h_2^*$. From Assumption 3, we observe that

$$\begin{pmatrix} \frac{\partial Y^2(H)}{\partial h_1^2} & \frac{\partial Y^2(H)}{\partial h_1\,\partial h_2} \\ \frac{\partial Y^2(H)}{\partial h_2\,\partial h_1} & \frac{\partial Y^2(H)}{\partial h_2^2} \end{pmatrix}$$

is negative definite. Therefore, (h_1^*, h_2^*) is the unique maximum point of $Y(H)$, and the MESY is $Y(H^*)$.

Now we prove $(\mathcal{B})$. Suppose that OHE $\widetilde{H}^* = (\widetilde{h}_1^*, \widetilde{h}_2^*)^T$ exists, then $\widetilde{H}^* \in G$, i.e.,

$$\begin{aligned} &b_1|_{h_k=\widetilde{h}_k^*} > 0, \qquad b_1c_{22} - b_2\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)|_{h_k=\widetilde{h}_k^*} > 0, \\ &b_2|_{h_k=\widetilde{h}_k^*} > 0, \qquad b_2c_{11} - b_1\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)|_{h_k=\widetilde{h}_k^*} > 0, \quad k = 1,2. \end{aligned}$$

On the other hand, thanks to $\widetilde{H}^*$ being OHE, then $\widetilde{H}^*$ is the solution of (24). Note that the solution of (24) is unique and H^* is also the solution of (24). Hence, $\widetilde{H}^* = H^*$, i.e., $b_1|_{h_1=h_1^*,h_2=h_2^*} > 0$, $b_2|_{h_1=h_1^*,h_2=h_2^*} > 0$, $b_1c_{22} - b_2\int_{-\tau_{12}}^0 \mathrm{d}F_{12}(\theta)|_{h_1=h_1^*,h_2=h_2^*} > 0$ and $b_2c_{11} - b_1\int_{-\tau_{21}}^0 \mathrm{d}F_{21}(\theta)|_{h_1=h_1^*,h_2=h_2^*} > 0$. The contradiction arises.

The proof is completed. □

Remark 4.1 From Theorem 4.1 we can note that the existence of an optimal harvesting policy has a close relationship with S-type distributed time delays, white noises and Lévy jumps.

5 Conclusions

In this paper, we considered the optimal harvesting of a stochastic competitive Lotka–Volterra model with S-type distributed time delays and Lévy jumps. We established the sufficient and necessary conditions for the existence of optimal harvesting policy, and we also obtained the explicit expression of the optimal harvesting effort and maximum yield by using the ergodic method. Theorem 4.1 indicates that the existence of an optimal harvesting policy has a close relationship with S-type distributed time delays, white noises and Lévy jumps.

Some interesting problems can be further investigated, such as the optimal harvesting problem for N-dimensional stochastic competitive Lotka–Volterra model with S-type distributed time delays and Lévy jumps. We can also study the optimal harvesting problem for some stochastic model with infinite time delays and Lévy jumps. It is necessary for us to work hard on these investigations.

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Authors' contributions
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