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Regularity criterion for 3D nematic liquid crystal flows in terms of finite frequency parts in $\dot{B}^{-1}_{\infty,\infty}$

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Abstract

In this paper, we establish the regularity criterion for the weak solution of nematic liquid crystal flows in three dimensions when the $L^\infty(0,T;\dot{B}^{-1}_{\infty,\infty})$-norm of a suitable low frequency part of $(u,\nabla d)$ is bounded by a scaling invariant constant and the initial data $(u_0,\nabla d_0)$. Our result refines the corresponding one in (Liu and Zhao in J. Math. Anal. Appl. 407:557-566, 2013) and that in (Ri in Nonlinear Anal. TMA 190:111619, 2020).

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1 Introduction

This note focuses on the regularity criteria for the following 3D nematic liquid crystal fluid flow:

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla\pi = -\lambda\nabla\cdot(\nabla d\odot\nabla d), & (x,t)\in\mathbf{R}^3\times(0,+\infty),\\ \partial_t d + (u\cdot\nabla)d = \gamma(\Delta d + |\nabla d|^2 d), & (x,t)\in\mathbf{R}^3\times(0,+\infty),\\ \operatorname{div} u = 0, & (x,t)\in\mathbf{R}^3\times(0,+\infty),\\ (u,d)|_{t=0} = (u_0,d_0), & x\in\mathbf{R}^3, \end{cases} \tag{1.1}$$

where $u(x,t)$ is the unknown velocity field, $d(x,t):\mathbf{R}^3\times(0,+\infty)\to\mathbb{S}^2$, the unit sphere in $\mathbf{R}^3$, is the unknown (averaged) macroscopic/continuum molecule orientation of the nematic liquid crystal flow and π is the scalar pressure. ν, λ, γ are positive constants that represent viscosity, the competition between kinetic energy and potential energy, and the microscopic elastic relaxation time for the molecular orientation field. The notation $\nabla d\odot\nabla d$ denotes the 3×3 matrix whose (i,j) entry is given by $\partial_i d\cdot\partial_j d$ $(1\le i,j\le 3)$.

It is well-known that Ericksen and Leslie ([3–5, 8] established the hydrodynamic theory of liquid crystals in 1960s. Lin [9] first introduced the above liquid crystal flow (1.1). Later Lin and Liu [11] obtained the global existence theorem for a weak solution and the local existence for the strong solution to the system (1.1).

We first introduce the definition of Morrey spaces.

Definition 1.1 For $1 \leq p \leq q \leq \infty$, we call $\dot{M}_{p,q}(\mathbf{R}^3)$ a Morrey space, if and only if

$$\|f\|_{\dot{M}_{p,q}(\mathbf{R}^3)} = \sup_{x\in\mathbf{R}^3, 0<R<\infty} R^{\frac{3}{q}-\frac{3}{p}} \left(\int_{B(x,R)} |f(y)|^p \, dy \right)^{\frac{1}{p}} < +\infty,$$

here $B(x,R)$ denotes the ball in $\mathbf{R}^3$ with center x and radius R.

In 2008, Fan and Guo [4] showed that, if u satisfies one of the following conditions:

$$u \in L^s(0,T;\dot{M}_{p,q}(\mathbf{R}^3)) \quad \text{with } \frac{2}{s} + \frac{3}{p} = 1, p \geq 3, p \geq q \geq 1,$$
$$\nabla u \in L^s(0,T;\dot{M}_{p,q}(\mathbf{R}^3)) \quad \text{with } \frac{2}{s} + \frac{3}{p} = 2, p \geq \frac{3}{2}, p \geq q \geq 1,$$

then (u,d) is extended beyond $t = T$. Later Liu, Zhao and Cui [12] obtained the regularity criterion to the system (1.1) under the assumption that $\partial_3 u \in L^\beta(0,T;L^\alpha)$ with $\frac{2}{\beta} + \frac{3}{\alpha} \leq 1, \alpha > 3$. Recently, Wei, Li and Yao [16] proved that, if the weak solution (u,d) satisfies

$$u_3, \nabla d \in L^\beta(0,T;L^\alpha(\mathbf{R}^3)), \quad \text{with } \frac{2}{\beta} + \frac{3}{\alpha} \leq \frac{3}{4} + \frac{1}{\alpha}, \alpha > \frac{10}{3},$$

then (u,b) can be extended beyond $t = T$. Liu and Zhao [13] proved that the solution (u,d) to (1.1) is smooth up to time T provided that

$$\|(u,\nabla d)\|_{L^\infty(0,T;B^{-1}_{\infty,\infty}(\mathbf{R}^3))} \leq \varepsilon_0.$$

When $d = 0$, the system (1.1) becomes an incompressible Navier–Stokes equation. There is a large literature on the regularity criteria on the Navier–Stokes equation; see [1, 6, 7, 15].

By traditional turbulence theory, viscous incompressible flows develop in such a way that energy is transferred from large scales to neighboring smaller scales. Hence, it is important to study regularity for the Navier–Stokes equation based on various wave-number band parts of weak solutions is important since it reveals in a way the relationship between regularity of weak solutions and turbulent flows. Cheskidov and Shvydkoy [2] proved that a Leray–Hopf weak solution u to the Navier–Stokes equation is regular in $(0,T]$ if

$$\|u^k\|_{B^{-1}_{\infty,\infty}(\mathbf{R}^3)} < C\nu,$$

where u^k is high frequency part of u with Fourier models $|\xi| \geq k$. Kim, Kwak and Yoo [5] proved that, if sufficiently high frequency parts of a weak solution to the Navier–Stokes equation on a torus belong to Serrin's class, then the weak solution is regular. Very recently, Ri [14] proved that a Leray–Hopf weak solution u to 3D Navier–Stokes equations is regular if the $L^\infty(0,T;B^{-1}_{\infty,\infty}(\mathbf{R}^3))$-norm of a suitable low frequency part of u is bounded by a scaling invariant constant depending on the kinematic viscosity ν and initial value u_0. Motivated by [2, 5, 13] and [14], we will investigate the regularity criteria for the weak solution (u,d) to the liquid crystal fluid flows (1.1) in the critical function space $L^\infty(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbf{R}^3))$

based on low and medium frequency parts, respectively. Before stating our result, we shall present some symbols and notations.

Let

$$u_k := \int_0^k u_{[s]}\, ds, \qquad u^k := \int_k^\infty u_{[s]}\, ds, \quad u_{h,k} := u_k - u_h, 0 < h < k < \infty. \tag{1.2}$$

Here

$$u_{[k]}(t,x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{|\xi|=k} \hat{u}(t,\xi) e^{ix\cdot\xi}\, d\sigma_\xi,$$

and $\hat{u}$ denotes Fourier transform of u. Our result is stated as follows.

Theorem 1.2 *Let (u,d) be a weak solution to (1.1) with $(u_0,d_0) \in H^3(\mathbf{R}^3) \times H^4(\mathbf{R}^3)$, $\mathrm{div} u_0 = 0$. Assume that, for $0 < T < \infty$, there exists $\delta \in (0,T)$ such that if (u,d) is regular in $(0,T)$ the inequalities*

$$\left\|(u_{\tilde{k}}, \nabla d_{\tilde{k}})\right\|_{L^\infty(T-\delta,T;\dot{B}^{-1}_{\infty,\infty})} < C_1 \tag{1.3}$$

and

$$\begin{aligned} &\left\|(u_{\tilde{k}/2,\tilde{k}}, \nabla d_{\tilde{k}/2,\tilde{k}})\right\|_{L^\infty(T-\delta,T;\dot{B}^{-1}_{\infty,\infty})} \\ &\quad < C_2\left(\|u_0\|_{L^2} + \|\nabla d_0\|_{L^2}\right)^{-1}\left(\left\|\nabla u_0^{\tilde{k}}\right\|_{L^2} + \left\|\Delta d_0^{\tilde{k}}\right\|_{L^2}\right)^{-1} \end{aligned} \tag{1.4}$$

hold. Then (u,d) is regular on $(0,T]$, where $\tilde{k} > 0$ is defined by

$$\tilde{k} = C_3\left(\left\|\nabla u_0^{\tilde{k}}\right\|_{L^2} + \left\|\Delta d_0^{\tilde{k}}\right\|_{L^2}\right)^2,$$

and the C_i, $i = 1,2,3$, are absolute constants.

Remark 1.1 Theorem 1.2 can be regarded as the generalization of Theorem 1.1 in [13] and Theorem 1.1 in [14].

The rest of this paper is organized as follows. Some useful facts are presented in Sect. 2. The proof of Theorem 1.2 is given in Sect. 3.

2 Preliminaries and some basic facts

In order to define Besov spaces, we first introduce the Littlewood–Paley decomposition theory. Let $\mathcal{S}(\mathbf{R}^n)$ be the Schwartz class of rapidly decreasing functions.

For given $f \in \mathcal{S}(\mathbf{R}^n)$, its Fourier transform $\mathcal{F}(f) = \hat{f}$ and its inverse Fourier transform $\mathcal{F}^{-1}(f) = \check{f}$ are given by

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x)\, dx$$

and

$$\check{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} f(x)\, d\xi,$$

respectively. Let us choose two nonnegative radial functions $\chi,\varphi \in \mathcal{S}(\mathbf{R}^n)$ satisfying $\operatorname{supp}\chi \subset B = \{\xi \in \mathbf{R}^n : |\xi| \le \frac{4}{3}\}$ and $\operatorname{supp}\varphi \subset \mathcal{C} = \{\xi \in \mathbf{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ such that

$$\sum_{j\in\mathbf{Z}} \varphi\big(2^{-j}\xi\big) = 1, \quad \text{for any } \xi \in \mathbf{R}^n\backslash\{0\}$$

and

$$\chi(\xi) + \sum_{j\ge 0} \varphi\big(2^{-j}\xi\big) = 1, \quad \text{for any } \xi \in \mathbf{R}^n.$$

For $j \in \mathbf{Z}$, the homogeneous Littlewood–Paley projection operators S_j and $\dot{\Delta}_j$ are defined by

$$\dot{S}_j f = \chi\big(2^{-j}D\big)f = 2^{nj}\int_{\mathbf{R}^n} \tilde{h}\big(2^j y\big)f(x-y)\,dy, \quad \text{where } \tilde{h} = \mathcal{F}^{-1}\chi,$$

and

$$\dot{\Delta}_j f = \varphi\big(2^{-j}D\big)f = 2^{nj}\int_{\mathbf{R}^n} h\big(2^j y\big)f(x-y)\,dy, \quad \text{where } h = \mathcal{F}^{-1}\varphi.$$

$\dot{\Delta}_j$ is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$, and $\dot{S}_j$ is a frequency projection to the ball $\{|\xi| \le 2^j\}$. Let $s \in \mathbf{R}, p, q \in [1,\infty]$. The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbf{R}^n)$ is presented by the distributions $f \in \mathcal{S}'_h$ such that

$$\Big(\sum_{j\in\mathbf{Z}} 2^{jsq}\|\dot{\Delta}_j f\|^q_{L^p}\Big)^{\frac{1}{q}} < \infty,$$

with the norm

$$\|f\|_{\dot{B}^s_{p,q}(\mathbf{R}^n)} = \begin{cases} (\sum_{j\in\mathbf{Z}} 2^{jsq}\|\dot{\Delta}_j f\|^q_{L^p})^{\frac{1}{q}}, & 1 \le q < \infty, \\ \sup_{j\in\mathbf{Z}}\{2^{js}\|\dot{\Delta}_j f\|_{L^p}\}, & q = \infty. \end{cases} \tag{2.1}$$

On the other hand, we recall some facts that can be found in [14]. If $u \in L^2(\mathbf{R}^3)$, then it follows from the definition of u_k and u^k that

$$\big(u_k, u^k\big) = 0, \quad \forall k > 0. \tag{2.2}$$

Moreover, for $0 \le r < s$, by Plancherel's theorem,

$$\begin{aligned} \|u_k\|_{\dot{H}^s} &= \big\||\xi|^s\hat{u}_k\big\|_{L^2} \le k^{s-r}\big\||\xi|^r\hat{u}_k\big\|_{L^2} = k^{s-r}\|u_k\|_{\dot{H}^r}, \\ \big\|u^k\big\|_{\dot{H}^s} &= \big\||\xi|^s\hat{u}_k\big\|_{L^2} \ge k^{s-r}\big\||\xi|^r\hat{u}_k\big\|_{L^2} = k^{s-r}\|u_k\|_{\dot{H}^r}. \end{aligned} \tag{2.3}$$

Since $\|\Delta u\|_{L^2} \sim \|\nabla^2 u\|_{L^2}, \forall u \in \dot{H}^2(\mathbb{R}^3)$, we have

$$k\big\|\nabla u^k\big\|_{L^2} \le \big\|\nabla^2 u^k\big\|_{L^2} \le c\big\|\Delta u^k\big\|_{L^2}, \quad \forall u \in H^2\big(\mathbb{R}^3\big), \tag{2.4}$$

with some $c > 0$. Moreover, it can be easily seen that

$$(u_k v_l)^m = 0, \quad \forall u, v \in L^2(\mathbb{R}^3), \forall k, l > 0, \forall m > k + l, \tag{2.5}$$

because the Fourier transform of $u_k v_l$ is supported in $\{\xi \in \mathbb{R}^3 : |\xi| \le k + l\}$.

3 Proof of Theorem 1.2

For convenience, we assume $\mu = \lambda = 1$ throughout the proof of Theorem 1.2.

Proof Assume that a weak solution (u, d) of (1.1) is regular in $(0, T)$, but not in $(0, T]$. Then $\lim_{t\to T-0} \|\nabla u(t)\|_{L^2} + \|\Delta d(t)\|_{L^2} = \infty$. Notice that, for all smooth solutions to system (1.1), one has the following basic energy law (see [10]):

$$\begin{aligned} &\|u(\cdot,t)\|_{L^2}^2 + \|\nabla d(\cdot,t)\|_{L^2}^2 + \int_0^t \left(\|\nabla u(\cdot,\tau)\|_{L^2}^2 + \|(\Delta d + |\nabla d|^2)(\cdot,\tau)\|_{L^2}^2\right) d\tau \\ &\quad \le \|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2, \end{aligned} \tag{3.1}$$

for all $0 < t < \infty$. By (1.2), one has

$$\|\nabla u(t)\|_{L^2}^2 + \|\Delta d(t)\|_{L^2}^2 \le k^2 \|u_0\|_{L^2}^2 + k^2 \|\nabla d_0\|_{L^2}^2 + \|\nabla u^k(t)\|_{L^2}^2 + \|\Delta d^k(t)\|_{L^2}^2.$$

Thus,

$$\lim_{t\to T-0} \|\nabla u^k(t)\|_{L^2}^2 + \|\Delta d^k(t)\|_{L^2}^2 = \infty. \tag{3.2}$$

We can see from [13] that, if there exists a positive constant $\varepsilon_0 > 0$ such that

$$\|(u, \nabla d)\|_{L^\infty(0,T;\dot{B}^{-1}_{\infty,\infty})} \le \varepsilon_0,$$

then the solution (u, d) is smooth up to time T.

Now we multiply the first equation of (1.1) with $-\Delta u^k$ and integrate over $\mathbf{R}^3$ to get by (2.2)

$$\frac{1}{2}\frac{d}{dt}\|\nabla u^k\|_{L^2}^2 + \|\Delta u^k\|_{L^2}^2 = (u \cdot \nabla u, \Delta u^k) + \left(\Delta d \cdot \nabla d + \frac{1}{2}\nabla |\nabla d|^2, \Delta u^k\right). \tag{3.3}$$

Applying ∇ to the second equation of (1.1) and making an L^2 inner product with respect to $\nabla \Delta d^k$, we can verify

$$\frac{1}{2}\frac{d}{dt}\|\Delta d^k\|_{L^2}^2 + \|\nabla \Delta d^k\|_{L^2}^2 = (\nabla(u \cdot \nabla d), \nabla \Delta d^k) + (\nabla(|\nabla d|^2 d), \nabla \Delta d^k). \tag{3.4}$$

Adding (3.3) and (3.4) gives rise to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\nabla u^k\|_{L^2}^2 + \|\Delta d^k\|_{L^2}^2\right) + \left(\|\Delta u^k\|_{L^2}^2 + \|\nabla \Delta d^k\|_{L^2}^2\right) \\ &\quad = (u \cdot \nabla u, \Delta u^k) + (\Delta d \cdot \nabla d, \Delta u^k) + \frac{1}{2}(\nabla |\nabla d|^2, \Delta u^k) \\ &\qquad + (\nabla(u \cdot \nabla d), \nabla \Delta d^k) + (\nabla(|\nabla d|^2 d), \nabla \Delta d^k) \\ &\quad := I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{3.5}$$

Next we estimate I_1–I_5, respectively. From [14], we have

$$\begin{aligned} |I_1| &= \left|\left(u\cdot\nabla u, \Delta u^k\right)\right| \\ &\le Ck^2\|u_0\|_{L^2}^2\|u_{\frac{k}{2},k}\|_{L^\infty}^2 + C\|u_k\|_{\dot B^{-1}_{\infty,\infty}}\left\|\Delta u^k\right\|_{L^2}^2 \\ &\quad + Ck^{-\frac{1}{2}}\left\|\nabla u^k\right\|_{L^2}\left\|\Delta u^k\right\|_{L^2}^2 + \frac{1}{4}\|\Delta u_{k,2k}\|_{L^2}^2. \end{aligned} \tag{3.6}$$

Since $d = d_k + d^k$, we write

$$(\Delta d\cdot\nabla)d = (\Delta d_k\cdot\nabla)d_k + \left(\Delta d^k\cdot\nabla\right)d_k + (\Delta d_k\cdot\nabla)d^k + \left(\Delta d^k\cdot\nabla\right)d^k.$$

Then

$$\begin{aligned} I_2 &= \left(\Delta d\cdot\nabla d, \Delta u^k\right) \\ &= \left((\Delta d_k\cdot\nabla)d_k, \Delta u^k\right) + \left(\left(\Delta d^k\cdot\nabla\right)d^k, \Delta u^k\right) + \left((\Delta d_k\cdot\nabla)d^k, \Delta u^k\right) \\ &\quad + \left(\left(\Delta d^k\cdot\nabla\right)d_k, \Delta u^k\right) \\ &:= I_{21} + I_{22} + I_{23} + I_{24}. \end{aligned} \tag{3.7}$$

Note that $d_k = d_{\frac{k}{2}} + d_{\frac{k}{2},k}$ and the Fourier transform of $(\Delta d_k\cdot\nabla)d_k$ is supported in $\{|\xi|\le 2k\}$, thus we deduce

$$\begin{aligned} I_{21} &= \left((\Delta d_k\cdot\nabla)d_k, \Delta u^k\right) \\ &= \left(\left[(\Delta d_k\cdot\nabla)d_k\right]_{k,2k}, \Delta u_{k,2k}\right) \\ &= \left(\left[(\Delta d_k\cdot\nabla)d_{\frac{k}{2}} + (\Delta d_k\cdot\nabla)d_{\frac{k}{2},k}\right]_{k,2k}, \Delta u_{k,2k}\right) \\ &= \left(\left[(\Delta d_{\frac{k}{2}}\cdot\nabla)d_{\frac{k}{2}} + (\Delta d_{\frac{k}{2},k}\cdot\nabla)d_{\frac{k}{2}} + (\Delta d_k\cdot\nabla)d_{\frac{k}{2},k}\right]_{k,2k}, \Delta u_{k,2k}\right) \\ &= \left((\Delta d_k\cdot\nabla)d_{\frac{k}{2},k}, \Delta u_{k,2k}\right) + \left((\Delta d_{\frac{k}{2},k}\cdot\nabla)d_{\frac{k}{2}}, \Delta u_{k,2k}\right) \\ &:= I_{211} + I_{212}, \end{aligned} \tag{3.8}$$

where we used the fact $[(\Delta d_{\frac{k}{2}}\cdot\nabla)d_{\frac{k}{2}}]_{k,2k} = 0$. Thanks to the Hölder inequality, the Young inequality and (3.1), we get

$$\begin{aligned} |I_{211}| &\le \|\Delta d_k\|_{L^2}\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}\|\Delta u_{k,2k}\|_{L^2} \\ &\le Ck\|\nabla d_k\|_{L^2}\|\nabla d_{k/2,k}\|_{L^\infty}\|\Delta u_{k,2k}\|_{L^2} \\ &\le Ck^2\|\nabla d_k\|_{L^2}^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2 + \frac{1}{8}\|\Delta u_{k,2k}\|_{L^2}^2 \\ &\le Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\left(\|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2\right) + \frac{1}{8}\|\Delta u_{k,2k}\|_{L^2}^2. \end{aligned} \tag{3.9}$$

Similarly,

$$\begin{aligned}|I_{212}| &\le \|\Delta d_{\frac{k}{2},k}\|_{L^\infty}\|\nabla d_{\frac{k}{2}}\|_{L^2}\|\Delta u_{k,2k}\|_{L^2}\\ &\le Ck\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}\|\nabla d_{\frac{k}{2}}\|_{L^2}\|\Delta u_{k,2k}\|_{L^2}\\ &\le Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\|\nabla d_{\frac{k}{2}}\|_{L^2}^2+\frac{1}{8}\|\Delta u_{k,2k}\|_{L^2}^2\\ &\le Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\big(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\big)+\frac{1}{8}\|\Delta u_{k,2k}\|_{L^2}^2,\end{aligned}\tag{3.10}$$

which along with (3.9) implies

$$|I_{21}| \le Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\big(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\big)+\frac{1}{4}\|\Delta u_{k,2k}\|_{L^2}^2.\tag{3.11}$$

With the help of Hölder's inequality, (2.4), Gagaliardo–Nirenberg's inequality, Sobolev's embedding and Young's inequality, one has

$$\begin{aligned}|I_{22}| &\le \|\Delta d^k\|_{L^3}\|\nabla d^k\|_{L^6}\|\Delta u^k\|_{L^2}\\ &\le C\|\Delta d^k\|_{L^2}^{\frac12}\|\nabla\Delta d^k\|_{L^2}^{\frac12}\|\nabla^2 d^k\|_{L^2}\|\Delta u^k\|_{L^2}\\ &\le Ck^{-\frac12}\|\nabla\Delta d^k\|_{L^2}\|\Delta d^k\|_{L^2}\|\Delta u^k\|_{L^2}\\ &\le Ck^{-\frac12}\|\Delta d^k\|_{L^2}\big(\|\Delta u^k\|_{L^2}^2+\|\nabla\Delta d^k\|_{L^2}^2\big).\end{aligned}\tag{3.12}$$

By the definition of the $\dot{B}^{-1}_{\infty,\infty}$-norm, we have

$$\|u_k\|_{L^\infty} \le Ck\|u_k\|_{\dot{B}^{-1}_{\infty,\infty}},\quad \forall k>0.\tag{3.13}$$

From the Hölder inequality, (3.13) and the Young inequality, we can conclude that

$$\begin{aligned}|I_{23}| &\le \|\Delta d_k\|_{L^\infty}\|\nabla d^k\|_{L^2}\|\Delta u^k\|_{L^2}\\ &\le Ck\|\nabla d_k\|_{L^\infty}\|\nabla d^k\|_{L^2}\|\Delta u^k\|_{L^2}\\ &\le Ck^2\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla d^k\|_{L^2}\|\Delta u^k\|_{L^2}\\ &\le C\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla\Delta d^k\|_{L^2}\|\Delta u^k\|_{L^2}\\ &\le C\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\big(\|\Delta u^k\|_{L^2}^2+\|\nabla\Delta d^k\|_{L^2}^2\big).\end{aligned}\tag{3.14}$$

Similarly,

$$\begin{aligned}|I_{24}| &\le \|\Delta d^k\|_{L^2}\|\nabla d_k\|_{L^\infty}\|\Delta u^k\|_{L^2}\\ &\le Ck\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\|\Delta d^k\|_{L^2}\|\Delta u^k\|_{L^2}\\ &\le C\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla\Delta d^k\|_{L^2}\|\Delta u^k\|_{L^2}\\ &\le C\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\big(\|\Delta u^k\|_{L^2}^2+\|\nabla\Delta d^k\|_{L^2}^2\big).\end{aligned}\tag{3.15}$$

Combining (3.7), (3.11), (3.12), (3.14) and (3.15), one arrives at

$$\begin{aligned} |I_2| &\leq Ck^{-\frac{1}{2}} \|\Delta d^k\|_{L^2} \left(\|\Delta u^k\|_{L^2}^2 + \|\nabla\Delta d^k\|_{L^2}^2\right) \\ &\quad + C\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}} \left(\|\Delta u^k\|_{L^2}^2 + \|\nabla\Delta d^k\|_{L^2}^2\right) \\ &\quad + Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2 \left(\|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2\right) + \frac{1}{4}\|\Delta u_{k,2k}\|_{L^2}^2. \end{aligned} \tag{3.16}$$

To estimate I_3, we make the following decomposition:

$$\begin{aligned} \frac{1}{2}\nabla|\nabla d|^2 &= \frac{1}{2}\nabla\left|\nabla d^k + \nabla d_k\right|^2 \leq \nabla\left|\nabla d^k\right|^2 + \nabla|\nabla d_k|^2 \\ &= 2\nabla d^k \cdot \nabla^2 d^k + 2\nabla d_k \cdot \nabla^2 d_k. \end{aligned}$$

Then

$$|I_3| \leq 2\left|\left(\nabla d^k \cdot \nabla^2 d^k, \Delta u^k\right)\right| + 2\left|\left(\nabla d_k \cdot \nabla^2 d_k, \Delta u^k\right)\right| := I_{31} + I_{32}. \tag{3.17}$$

Applying the same method to the bound (3.12) gives rise to

$$\begin{aligned} I_{31} &\leq \|\nabla d^k\|_{L^3} \|\nabla^2 d^k\|_{L^6} \|\Delta u^k\|_{L^2} \\ &\leq C\|\nabla d^k\|_{\dot{H}^{\frac{1}{2}}} \|\Delta d^k\|_{\dot{H}^1} \|\Delta u^k\|_{L^2} \\ &\leq C\|\nabla d^k\|_{L^2}^{\frac{1}{2}} \|\nabla d^k\|_{\dot{H}^1}^{\frac{1}{2}} \|\Delta d^k\|_{\dot{H}^1} \|\Delta u^k\|_{L^2} \\ &\leq C\|\nabla d^k\|_{L^2}^{\frac{1}{2}} \|\Delta d^k\|_{L^2}^{\frac{1}{2}} \|\nabla\Delta d^k\|_{L^2} \|\Delta u^k\|_{L^2} \\ &\leq Ck^{-\frac{1}{2}} \|\Delta d^k\|_{L^2} \|\nabla\Delta d^k\|_{L^2} \|\Delta u^k\|_{L^2} \\ &\leq Ck^{-\frac{1}{2}} \|\Delta d^k\|_{L^2} \left(\|\nabla\Delta d^k\|_{L^2}^2 + \|\Delta u^k\|_{L^2}^2\right). \end{aligned} \tag{3.18}$$

Similarly to (3.8), we have

$$\begin{aligned} I_{32} &= 2\left|\left(\nabla d_k \cdot \nabla^2 d_k, \Delta u^k\right)\right| \\ &= 2\left|\left(\nabla d_k \cdot \nabla^2 d_{\frac{k}{2},k}, \Delta u_{k,2k}\right) + \left(\nabla d_{\frac{k}{2},k} \cdot \nabla^2 d_{\frac{k}{2}}, \Delta u_{k,2k}\right)\right| \\ &\leq 2\left|\left(\nabla d_k \cdot \nabla^2 d_{\frac{k}{2},k}, \Delta u_{k,2k}\right)\right| + 2\left|\left(\nabla d_{\frac{k}{2},k} \cdot \nabla^2 d_{\frac{k}{2}}, \Delta u_{k,2k}\right)\right| \\ &:= I_{321} + I_{322}. \end{aligned} \tag{3.19}$$

Using Hölder's inequality, (2.4), Young's inequality and (3.1), one can verify

$$\begin{aligned} I_{321} &\leq 2\|\nabla d_k\|_{L^2} \|\Delta d_{\frac{k}{2},k}\|_{L^\infty} \|\Delta u_{k,2k}\|_{L^2} \\ &\leq Ck\|\nabla d_{\frac{k}{2},k}\|_{L^\infty} \|\nabla d_k\|_{L^2} \|\Delta u_{k,2k}\|_{L^2} \\ &\leq Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2 \left(\|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2\right) + \frac{1}{8}\|\Delta u_{k,2k}\|_{L^2}^2. \end{aligned} \tag{3.20}$$

Similarly,

$$
\begin{aligned}
I_{322} &\le 2\|\Delta d_{\frac{k}{2}}\|_{L^2}\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}\|\Delta u_{k,2k}\|_{L^2} \\
&\le Ck\|\nabla d_{\frac{k}{2}}\|_{L^2}\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}\|\Delta u_{k,2k}\|_{L^2} \\
&\le Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\big(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\big)+\frac{1}{8}\|\Delta u_{k,2k}\|_{L^2}^2,
\end{aligned}
\tag{3.21}
$$

which along with (3.20) implies

$$
I_{32} \le Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\big(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\big)+\frac{1}{4}\|\Delta u_{k,2k}\|_{L^2}^2. \tag{3.22}
$$

From (3.17), (3.18) and (3.22), we can deduce

$$
\begin{aligned}
|I_3| &\le Ck^{-\frac{1}{2}}\|\Delta d^k\|_{L^2}\big(\|\Delta u^k\|_{L^2}^2+\|\nabla\Delta d^k\|_{L^2}^2\big) \\
&\quad + Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\big(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\big)+\frac{1}{4}\|\Delta u_{k,2k}\|_{L^2}^2.
\end{aligned}
\tag{3.23}
$$

We now address the term I_4. We decompose I_4 into the following form:

$$
I_4 = \big((\nabla u\cdot\nabla)d,\nabla\Delta d^k\big)+\big((u\cdot\nabla)\nabla d,\nabla\Delta d^k\big) := I_{41}+I_{42}. \tag{3.24}
$$

Since

$$
(\nabla u\cdot\nabla)d = \big(\nabla u^k\cdot\nabla\big)d^k+\big(\nabla u^k\cdot\nabla\big)d_k+(\nabla u_k\cdot\nabla)d^k+(\nabla u_k\cdot\nabla)d_k,
$$

we can get

$$
\begin{aligned}
I_{41} &= \big(\big(\nabla u^k\cdot\nabla\big)d^k,\nabla\Delta d^k\big)+\big(\big(\nabla u^k\cdot\nabla\big)d_k,\nabla\Delta d^k\big)+\big((\nabla u_k\cdot\nabla)d^k,\nabla\Delta d^k\big) \\
&\quad + \big((\nabla u_k\cdot\nabla)d_k,\nabla\Delta d^k\big) := I_{411}+I_{412}+I_{413}+I_{414}.
\end{aligned}
\tag{3.25}
$$

Similar to the estimate (3.12), one has

$$
\begin{aligned}
|I_{411}| &\le \|\nabla u^k\|_{L^6}\|\nabla d^k\|_{L^3}\|\nabla\Delta d^k\|_{L^2} \\
&\le C\|\nabla u^k\|_{\dot H^1}\|\nabla d^k\|_{L^2}^{\frac{1}{2}}\|\nabla d^k\|_{\dot H^1}^{\frac{1}{2}}\|\nabla\Delta d^k\|_{L^2} \\
&\le Ck^{-\frac{1}{2}}\|\Delta d^k\|_{L^2}\|\Delta u^k\|_{L^2}\|\nabla\Delta d^k\|_{L^2} \\
&\le Ck^{-\frac{1}{2}}\|\Delta d^k\|_{L^2}\big(\|\Delta u^k\|_{L^2}^2+\|\nabla\Delta d^k\|_{L^2}^2\big).
\end{aligned}
\tag{3.26}
$$

The Hölder inequality, the Young inequality and (3.13) imply

$$
\begin{aligned}
I_{412} &\le \|\nabla u^k\|_{L^2}\|\nabla d_k\|_{L^\infty}\|\nabla\Delta d^k\|_{L^2} \\
&\le Ck\|\nabla d_k\|_{\dot B^{-1}_{\infty,\infty}}\|\nabla u^k\|_{L^2}\|\nabla\Delta d^k\|_{L^2} \\
&\le C\|\nabla d_k\|_{\dot B^{-1}_{\infty,\infty}}\|\Delta u^k\|_{L^2}\|\nabla\Delta d^k\|_{L^2} \\
&\le C\|\nabla d_k\|_{\dot B^{-1}_{\infty,\infty}}\big(\|\Delta u^k\|_{L^2}^2+\|\nabla\Delta d^k\|_{L^2}^2\big).
\end{aligned}
\tag{3.27}
$$

Similarly,

$$\begin{aligned} I_{413} &\le \|\nabla u_k\|_{L^\infty} \left\|\nabla d^k\right\|_{L^2} \left\|\nabla\Delta d^k\right\|_{L^2} \\ &\le Ck\|u_k\|_{L^\infty} \left\|\nabla d^k\right\|_{L^2} \left\|\nabla\Delta d^k\right\|_{L^2} \\ &\le Ck^2\|u_k\|_{\dot B^{-1}_{\infty,\infty}} \left\|\nabla d^k\right\|_{L^2} \left\|\nabla\Delta d^k\right\|_{L^2} \\ &\le C\|u_k\|_{\dot B^{-1}_{\infty,\infty}} \left\|\nabla\Delta d^k\right\|^2_{L^2}. \end{aligned} \tag{3.28}$$

Arguing as (3.8), we have

$$I_{414} = \bigl((\nabla u_k\cdot\nabla) d_{\frac{k}{2},k} + (\nabla u_{\frac{k}{2},k}\cdot\nabla) d_{\frac{k}{2}}, \nabla\Delta d_{k,2k}\bigr) := I_{4141} + I_{4142}.$$

By Hölder's inequality, (2.4) and Young's inequality, we get

$$\begin{aligned} |I_{4141}| &\le \|\nabla u_k\|_{L^2}\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}\|\nabla\Delta d_{k,2k}\|_{L^2} \\ &\le Ck\|u_k\|_{L^2}\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}\|\nabla\Delta d_{k,2k}\|_{L^2} \\ &\le Ck^2\|u_k\|^2_{L^2}\|\nabla d_{\frac{k}{2},k}\|^2_{L^\infty} + \frac{1}{16}\|\nabla\Delta d_{k,2k}\|^2_{L^2} \\ &\le Ck^2\|\nabla d_{\frac{k}{2},k}\|^2_{L^\infty}\bigl(\|u_0\|^2_{L^2} + \|\nabla d_0\|^2_{L^2}\bigr) + \frac{1}{16}\|\nabla\Delta d_{k,2k}\|^2_{L^2}. \end{aligned} \tag{3.29}$$

Similarly,

$$\begin{aligned} |I_{4142}| &\le \|\nabla u_{\frac{k}{2},k}\|_{L^\infty}\|\nabla d_{\frac{k}{2}}\|_{L^2}\|\nabla\Delta d_{k,2k}\|_{L^2} \\ &\le Ck\|u_{\frac{k}{2},k}\|_{L^\infty}\|\nabla d_{\frac{k}{2}}\|_{L^2}\|\nabla\Delta d_{k,2k}\|_{L^2} \\ &\le Ck^2\|u_{\frac{k}{2},k}\|^2_{L^\infty}\|\nabla d_{\frac{k}{2}}\|^2_{L^2} + \frac{1}{16}\|\nabla\Delta d_{k,2k}\|^2_{L^2} \\ &\le Ck^2\|u_{\frac{k}{2},k}\|^2_{L^\infty}\bigl(\|u_0\|^2_{L^2} + \|\nabla d_0\|^2_{L^2}\bigr) + \frac{1}{16}\|\nabla\Delta d_{k,2k}\|^2_{L^2}, \end{aligned} \tag{3.30}$$

which together with (3.29) reads

$$|I_{414}| \le Ck^2\bigl(\|u_{\frac{k}{2},k}\|^2_{L^\infty} + \|\nabla d_{\frac{k}{2},k}\|^2_{L^\infty}\bigr)\bigl(\|u_0\|^2_{L^2} + \|\nabla d_0\|^2_{L^2}\bigr) + \frac{1}{8}\|\nabla\Delta d_{k,2k}\|^2_{L^2}. \tag{3.31}$$

Combining (3.26)–(3.28) and (3.31) yields

$$\begin{aligned} |I_{41}| \le{}& \frac{1}{8}\|\nabla\Delta d_{k,2k}\|^2_{L^2} + Ck^{-\frac{1}{2}}\left\|\Delta d^k\right\|_{L^2}\bigl(\left\|\Delta u^k\right\|^2_{L^2} + \left\|\nabla\Delta d^k\right\|^2_{L^2}\bigr) \\ &+ C\bigl(\|u_k\|_{\dot B^{-1}_{\infty,\infty}} + \|\nabla d_k\|_{\dot B^{-1}_{\infty,\infty}}\bigr)\bigl(\left\|\Delta u^k\right\|^2_{L^2} + \left\|\nabla\Delta d^k\right\|^2_{L^2}\bigr) \\ &+ Ck^2\bigl(\|u_{\frac{k}{2},k}\|^2_{L^\infty} + \|\nabla d_{\frac{k}{2},k}\|^2_{L^\infty}\bigr)\bigl(\|u_0\|^2_{L^2} + \|\nabla d_0\|^2_{L^2}\bigr). \end{aligned} \tag{3.32}$$

To handle I_{42}, we split I_{42} into

$$\begin{aligned} I_{42} &= \big((u_k\cdot\nabla)\nabla d^k, \nabla\Delta d^k\big) + \big((u^k\cdot\nabla)\nabla d^k, \nabla\Delta d^k\big) + \big((u^k\cdot\nabla)\nabla d_k, \nabla\Delta d^k\big) \\ &\quad + \big((u_k\cdot\nabla)\nabla d_k, \nabla\Delta d^k\big) \\ &:= I_{421} + I_{422} + I_{423} + I_{424}. \end{aligned} \tag{3.33}$$

By Hölder's inequality and (2.4), we get

$$\begin{aligned} |I_{421}| &\le \|u_k\|_{L^\infty}\big\|\nabla^2 d^k\big\|_{L^2}\big\|\nabla\Delta d^k\big\|_{L^2} \\ &\le \|u_k\|_{L^\infty}\big\|\Delta d^k\big\|_{L^2}\big\|\nabla\Delta d^k\big\|_{L^2} \\ &\le c\|u_k\|_{\dot{B}^{-1}_{\infty,\infty}}\big\|\nabla\Delta d^k\big\|_{L^2}^2. \end{aligned} \tag{3.34}$$

Similarly to (3.12), one has

$$\begin{aligned} |I_{422}| &\le \big\|u^k\big\|_{L^6}\big\|\nabla^2 d^k\big\|_{L^3}\big\|\nabla\Delta d^k\big\|_{L^2} \\ &\le C\big\|u^k\big\|_{L^6}\big\|\Delta d^k\big\|_{L^3}\big\|\nabla\Delta d^k\big\|_{L^2} \\ &\le C\big\|\nabla u^k\big\|_{L^2}\big\|\Delta d^k\big\|_{L^2}^{1/2}\big\|\Delta d^k\big\|_{\dot{H}^1}^{1/2}\big\|\nabla\Delta d^k\big\|_{L^2} \\ &\le C\big\|\nabla u^k\big\|_{L^2}\big\|\Delta d^k\big\|_{L^2}^{1/2}\big\|\nabla\Delta d^k\big\|_{L^2}^{1/2}\big\|\nabla\Delta d^k\big\|_{L^2} \\ &\le Ck^{-1/2}\big\|\nabla u^k\big\|_{L^2}\big\|\nabla\Delta d^k\big\|_{L^2}^2. \end{aligned} \tag{3.35}$$

Hölder's inequality, (2.4), and Young's inequality guarantee

$$\begin{aligned} |I_{423}| &\le \big\|u^k\big\|_{L^2}\big\|\nabla^2 d_k\big\|_{L^\infty}\big\|\nabla\Delta d^k\big\|_{L^2} \\ &\le \big\|u^k\big\|_{L^2}\|\Delta d_k\|_{L^\infty}\big\|\nabla\Delta d^k\big\|_{L^2} \\ &\le ck\big\|u^k\big\|_{L^2}\|\nabla d_k\|_{L^\infty}\big\|\nabla\Delta d^k\big\|_{L^2} \\ &\le c\big\|\Delta u^k\big\|_{L^2}\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\big\|\nabla\Delta d^k\big\|_{L^2} \\ &\le c\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\big(\big\|\Delta u^k\big\|_{L^2}^2 + \big\|\nabla\Delta d^k\big\|_{L^2}^2\big). \end{aligned} \tag{3.36}$$

Similarly to (3.8), we write

$$I_{424} = \big((u_k\cdot\nabla)\nabla d_{\frac{k}{2},k}, \nabla\Delta d_{k,2k}\big) + \big((u_{\frac{k}{2},k}\cdot\nabla)\nabla d_{\frac{k}{2}}, \nabla\Delta d_{k,2k}\big) := I_{4241} + I_{4242}.$$

From the Hölder inequality and the Young inequality, we conclude

$$\begin{aligned} |I_{4241}| &\le \|u_k\|_{L^2}\big\|\nabla^2 d_{\frac{k}{2},k}\big\|_{L^\infty}\|\nabla\Delta d_{k,2k}\|_{L^2} \\ &\le C\|u_k\|_{L^2}\|\Delta d_{\frac{k}{2},k}\|_{L^\infty}\|\nabla\Delta d_{k,2k}\|_{L^2} \\ &\le Ck\|u_k\|_{L^2}\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}\big\|\nabla\Delta d^k\big\|_{L^2} \\ &\le Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\big(\|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2\big) + \frac{1}{16}\big\|\nabla\Delta d^k\big\|_{L^2}^2 \end{aligned} \tag{3.37}$$

and

$$
\begin{aligned}
|I_{4242}| &\le \|u_{\frac{k}{2},k}\|_{L^\infty} \left\| \nabla^2 d_{\frac{k}{2}} \right\|_{L^2} \|\nabla \Delta d_{k,2k}\|_{L^2} \\
&\le C\|u_{\frac{k}{2},k}\|_{L^\infty} \|\Delta d_{\frac{k}{2}}\|_{L^2} \|\nabla \Delta d_{k,2k}\|_{L^2} \\
&\le ck\|u_{\frac{k}{2},k}\|_{L^\infty} \|\nabla d_{\frac{k}{2}}\|_{L^2} \|\nabla \Delta d_{k,2k}\|_{L^2} \\
&\le Ck^2\|u_{\frac{k}{2},k}\|_{L^\infty}^2 \left(\|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2\right) + \frac{1}{16}\left\|\nabla \Delta d^k\right\|_{L^2}^2.
\end{aligned}
\tag{3.38}
$$

Therefore, by (3.24)–(3.38), we have

$$
\begin{aligned}
|I_4| &\le Ck^{-\frac{1}{2}} \left(\left\|\nabla u^k\right\|_{L^2} + \left\|\Delta d^k\right\|_{L^2}\right)\left(\left\|\Delta u^k\right\|_{L^2}^2 + \left\|\nabla \Delta d^k\right\|_{L^2}^2\right) \\
&\quad + C\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}} \left(\left\|\Delta u^k\right\|_{L^2}^2 + \left\|\nabla \Delta d^k\right\|_{L^2}^2\right) \\
&\quad + C\|u_k\|_{\dot{B}^{-1}_{\infty,\infty}} \left\|\nabla \Delta d^k\right\|_{L^2}^2 + Ck^2\left(\|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2\right)\big) \\
&\quad \times \left(\|u_{\frac{k}{2},k}\|_{L^\infty}^2 + \|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\right) + \frac{1}{4}\|\nabla \Delta d_{k,2k}\|_{L^2}^2.
\end{aligned}
\tag{3.39}
$$

It is left to deal with the last term, I_5. Using the fact that

$$
\nabla\left(|\nabla d|^2 d\right) = 2\nabla^2 d \nabla d d + |\nabla d|^2 \nabla d,
$$

we can rewrite I_5 as follows:

$$
I_5 = 2\left(\nabla^2 d \nabla d d, \nabla \Delta d^k\right) + \left(|\nabla d|^2 \nabla d, \nabla \Delta d^k\right) := I_{51} + I_{52}. \tag{3.40}
$$

Since

$$
2\nabla^2 d \nabla d d = \left(2\nabla^2 d_k \nabla d_k + 2\nabla^2 d_k \nabla d^k + 2\nabla^2 d^k \nabla d_k + 2\nabla^2 d^k \nabla d^k\right) d,
$$

we have

$$
\begin{aligned}
I_{51} &= 2\left(\nabla^2 d_k \nabla d_k d, \nabla \Delta d^k\right) + 2\left(\nabla^2 d_k \nabla d^k d, \nabla \Delta d^k\right) + 2\left(\nabla^2 d^k \nabla d_k d, \nabla \Delta d^k\right) \\
&\quad + 2\left(\nabla^2 d^k \nabla d^k d, \nabla \Delta d^k\right) \\
&:= I_{511} + I_{512} + I_{513} + I_{514}.
\end{aligned}
\tag{3.41}
$$

Reasoning as (3.8), one has

$$
\begin{aligned}
I_{511} &= \left(\nabla^2 d_k \nabla d_{\frac{k}{2},k} d, \nabla \Delta d_{k,2k}\right) + \left(\nabla^2 d_{\frac{k}{2},k} \nabla d_{\frac{k}{2}} d, \nabla \Delta d_{k,2k}\right) \\
&:= I_{5111} + I_{5112}.
\end{aligned}
\tag{3.42}
$$

Using $|d| = 1$, Hölder's inequality, inequality (2.4) and Young's inequality, we have

$$\begin{aligned} |I_{5111}| &\le 2\|\nabla^2 d_k\|_{L^2}\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}\|\nabla\Delta d_{k,2k}\|_{L^2} \\ &\le C\|\Delta d_k\|_{L^2}\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}\|\nabla\Delta d_{k,2k}\|_{L^2} \\ &\le Ck\|\nabla d_k\|_{L^2}\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}\|\nabla\Delta d_{k,2k}\|_{L^2} \\ &\le Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\big(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\big)+\frac{1}{8}\|\nabla\Delta d_{k,2k}\|_{L^2}^2. \end{aligned} \tag{3.43}$$

Similarly,

$$\begin{aligned} |I_{5112}| &\le \|\nabla^2 d_{\frac{k}{2},k}\|_{L^\infty}\|\nabla d_{\frac{k}{2}}\|_{L^2}\|\nabla\Delta d_{k,2k}\|_{L^2} \\ &\le Ck\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}\|\nabla d_{\frac{k}{2}}\|_{L^2}\|\nabla\Delta d_{k,2k}\|_{L^2} \\ &\le Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\big(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\big)+\frac{1}{8}\|\nabla\Delta d_{k,2k}\|_{L^2}^2, \end{aligned} \tag{3.44}$$

which all taken together implies

$$|I_{511}| \le Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\big(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\big)+\frac{1}{4}\|\nabla\Delta d_{k,2k}\|_{L^2}^2. \tag{3.45}$$

By the fact $|d| = 1$, the Hölder inequality, (2.4) and (3.13), we can get

$$\begin{aligned} |I_{512}| &\le 2\|\nabla^2 d_k\|_{L^\infty}\|\nabla d^k\|_{L^2}\|\nabla\Delta d^k\|_{L^2} \\ &\le C\|\Delta d_k\|_{L^\infty}\|\nabla d^k\|_{L^2}\|\nabla\Delta d^k\|_{L^2} \\ &\le Ck\|\nabla d_k\|_{L^\infty}\|\nabla d^k\|_{L^2}\|\nabla\Delta d^k\|_{L^2} \\ &\le Ck^2\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla d^k\|_{L^2}\|\nabla\Delta d^k\|_{L^2} \\ &\le C\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla\Delta d^k\|_{L^2}^2. \end{aligned} \tag{3.46}$$

Similarly,

$$\begin{aligned} |I_{513}| &\le 2\|\nabla^2 d^k\|_{L^2}\|\nabla d_k\|_{L^\infty}\|\nabla\Delta d^k\|_{L^2} \\ &\le C\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla\Delta d^k\|_{L^2}^2. \end{aligned} \tag{3.47}$$

Reasoning as (3.12) again, one has

$$\begin{aligned} |I_{514}| &\le 2\|\nabla^2 d^k\|_{L^3}\|\nabla d^k\|_{L^6}\|\nabla\Delta d^k\|_{L^2} \\ &\le C\|\nabla^2 d^k\|_{L^2}^{\frac{1}{2}}\|\nabla^2 d^k\|_{\dot{H}^1}^{\frac{1}{2}}\|\nabla d^k\|_{\dot{H}^1}\|\nabla\Delta d^k\|_{L^2} \\ &\le C\|\Delta d^k\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta d^k\|_{L^2}^{\frac{1}{2}}\|\Delta d^k\|_{L^2}\|\nabla\Delta d^k\|_{L^2} \\ &\le Ck^{-\frac{1}{2}}\|\Delta d^k\|_{L^2}\|\nabla\Delta d^k\|_{L^2}^2. \end{aligned} \tag{3.48}$$

Therefore, inequalities (3.45)–(3.48) yield

$$\begin{aligned}|I_{51}| &\leq Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\left(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\right)+C\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\left\|\nabla\Delta d^k\right\|_{L^2}^2\\&\quad+Ck^{-\frac{1}{2}}\left\|\Delta d^k\right\|_{L^2}\left\|\nabla\Delta d^k\right\|_{L^2}^2+\frac{1}{4}\|\nabla\Delta d_{k,2k}\|_{L^2}^2.\end{aligned}\tag{3.49}$$

It is easy to get $\Delta d\cdot d=-|\nabla d|^2$ due to $|d|=1$. Then $|\nabla d|^2\nabla d=-\Delta d\cdot d\nabla d$. Hence we decompose I_{52} in the following way:

$$\begin{aligned}I_{52}&=-\left(\Delta d\cdot d\nabla d,\nabla\Delta d^k\right)\\&=-\left(\Delta d^k\cdot d\nabla d^k,\nabla\Delta d^k\right)-\left(\Delta d^k\cdot d\nabla d_k,\nabla\Delta d^k\right)\\&\quad-\left(\Delta d_k\cdot d\nabla d^k,\nabla\Delta d^k\right)-\left(\Delta d_k\cdot d\nabla d_k,\nabla\Delta d^k\right)\\&:=I_{521}+I_{522}+I_{523}+I_{524}.\end{aligned}\tag{3.50}$$

Repeating the methods to prove (3.12), we obtain

$$\begin{aligned}|I_{521}|&\leq\left\|\Delta d^k\right\|_{L^3}\left\|\nabla d^k\right\|_{L^6}\left\|\nabla\Delta d^k\right\|_{L^2}\\&\leq C\left\|\Delta d^k\right\|_{L^2}^{\frac{1}{2}}\left\|\Delta d^k\right\|_{\dot{H}^1}^{\frac{1}{2}}\left\|\nabla d^k\right\|_{\dot{H}^1}\left\|\nabla\Delta d^k\right\|_{L^2}\\&\leq C\left\|\Delta d^k\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla\Delta d^k\right\|_{L^2}^{\frac{1}{2}}\left\|\Delta d^k\right\|_{L^2}\left\|\nabla\Delta d^k\right\|_{L^2}\\&\leq Ck^{-\frac{1}{2}}\left\|\Delta d^k\right\|_{L^2}\left\|\nabla\Delta d^k\right\|_{L^2}^2.\end{aligned}\tag{3.51}$$

Similarly to (3.46), we have

$$|I_{522}|+|I_{523}|\leq C\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\left\|\nabla\Delta d^k\right\|_{L^2}^2.\tag{3.52}$$

Similarly to (3.45), one has

$$|I_{524}|\leq Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\left(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\right)+\frac{1}{4}\|\nabla\Delta d_{k,2k}\|_{L^2}^2.\tag{3.53}$$

Thus

$$\begin{aligned}|I_{52}|&\leq Ck^{-\frac{1}{2}}\left\|\Delta d^k\right\|_{L^2}\left\|\nabla\Delta d^k\right\|_{L^2}^2+C\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\left\|\nabla\Delta d^k\right\|_{L^2}^2\\&\quad+Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\left(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\right)+\frac{1}{4}\|\nabla\Delta d_{k,2k}\|_{L^2}^2.\end{aligned}\tag{3.54}$$

From (3.49) and (3.54), we deduce

$$\begin{aligned}|I_5|&\leq Ck^2\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\left(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\right)+C\|\nabla d_k\|_{\dot{B}^{-1}_{\infty,\infty}}\left\|\nabla\Delta d^k\right\|_{L^2}^2\\&\quad+Ck^{-\frac{1}{2}}\left\|\Delta d^k\right\|_{L^2}\left\|\nabla\Delta d^k\right\|_{L^2}^2+\frac{1}{2}\|\nabla\Delta d_{k,2k}\|_{L^2}^2.\end{aligned}\tag{3.55}$$

Combining (3.6), (3.16), (3.23), (3.39) and (3.55), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u^k\|_{L^2}^2+\|\Delta d^k\|_{L^2}^2\right)+\left(\|\Delta u^k\|_{L^2}^2+\|\nabla\Delta d^k\|_{L^2}^2\right)\\
&\quad\le C_1k^2\left(\|u_{\frac{k}{2},k}\|_{L^\infty}^2+\|\nabla d_{\frac{k}{2},k}\|_{L^\infty}^2\right)\left(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\right)\\
&\qquad+C_2\left(\|u_k\|_{\dot B^{-1}_{\infty,\infty}}+\|\nabla d_k\|_{\dot B^{-1}_{\infty,\infty}}\right)\left(\|\Delta u^k\|_{L^2}^2+\|\nabla\Delta d^k\|_{L^2}^2\right)\\
&\qquad+C_3k^{-\frac{1}{2}}\left(\|\nabla u^k\|_{L^2}+\|\Delta d^k\|_{L^2}\right)\left(\|\Delta u^k\|_{L^2}^2+\|\nabla\Delta d^k\|_{L^2}^2\right)+\frac{3}{4}\|\Delta u_{k,2k}\|_{L^2}^2\\
&\qquad+\frac{3}{4}\|\nabla\Delta d_{k,2k}\|_{L^2}^2
\end{aligned}\tag{3.56}$$

and

$$\begin{aligned}
&\frac{d}{dt}\left(\|\nabla u^k\|_{L^2}^2+\|\triangle d^k\|_{L^2}^2\right)\\
&\quad\le\Big[c_1k^2\left(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\right)\left(\|u_{k/2,k}\|_{L^\infty}^2+\|\nabla d_{k/2,k}\|_{L^\infty}^2\right)\\
&\qquad-\frac{1}{8}\left(\|\triangle u^k\|_{L^2}^2+\|\nabla\triangle d^k\|_{L^2}^2\right)\Big]+\left(c_2\left(\|u_k(t)\|_{\dot B^{-1}_{\infty,\infty}}+\|\nabla d_k\|_{\dot B^{-1}_{\infty,\infty}}\right)-\frac{1}{4}\right)\\
&\qquad\times\left(\|\triangle u^k\|_{L^2}^2+\|\nabla\triangle d^k\|_{L^2}^2\right)\\
&\qquad+\left(c_3k^{-1/2}\left(\|\nabla u^k\|_{L^2}+\|\triangle d^k\|_{L^2}\right)-\frac{1}{8}\right)\left(\|\triangle u^k\|_{L^2}^2+\|\nabla\triangle d^k\|_{L^2}^2\right).
\end{aligned}\tag{3.57}$$

Let

$$\tilde k=128\times 4c_3^2\left(\|\nabla u_0^{\tilde k}\|_{L^2}+\|\triangle d_0^{\tilde k}\|_{L^2}\right)^2.\tag{3.58}$$

Then

$$\|\nabla u_0^{\tilde k}\|_{L^2}+\|\Delta d_0^{\tilde k}\|_{L^2}<\frac{\tilde k^{\frac{1}{2}}}{16c_3}.$$

Since $\lim_{t\to T-0}\|\nabla u^{\tilde k}(t)\|_{L^2}+\|\Delta d^{\tilde k}(t)\|_{L^2}=\infty$, there is some $\delta\in(0,T)$ such that

$$\|\nabla u^{\tilde k}(T-\delta)\|_{L^2}+\|\Delta d^{\tilde k}(T-\delta)\|_{L^2}=\frac{\tilde k^{\frac{1}{2}}}{16c_3},\tag{3.59}$$

$$\|\nabla u^{\tilde k}(t)\|_{L^2}+\|\Delta d^{\tilde k}(t)\|_{L^2}>\frac{\tilde k^{\frac{1}{2}}}{16c_3}.\tag{3.60}$$

From (3.60), we get for any $t\in(T-\delta,T)$

$$\begin{aligned}
&c_1\tilde k^2\left(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\right)\left(\|u_{\frac{\tilde k}{2},\tilde k}\|_{L^\infty}^2+\|\nabla d_{\frac{\tilde k}{2},\tilde k}\|_{L^\infty}^2\right)-\frac{1}{8}\left(\|\Delta u^{\tilde k}\|_{L^2}^2+\|\nabla\Delta d^{\tilde k}\|_{L^2}^2\right)\\
&\quad\le\tilde k^2\left[c_1\left(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\right)\left(\|u_{\frac{\tilde k}{2},\tilde k}\|_{L^\infty}^2+\|\nabla d_{\frac{\tilde k}{2},\tilde k}\|_{L^\infty}^2\right)-\frac{1}{8}\left(\|\nabla u^{\tilde k}\|_{L^2}^2+\|\Delta d^{\tilde k}\|_{L^2}^2\right)\right]\\
&\quad\le\tilde k^2\left[c_1\left(\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\right)\left(\|u_{\frac{\tilde k}{2},\tilde k}\|_{L^\infty}^2+\|\nabla d_{\frac{\tilde k}{2},\tilde k}\|_{L^\infty}^2\right)-\frac{1}{16}\frac{\tilde k}{256c_3^2}\right]
\end{aligned}$$

$$\le 0,$$

provided

$$\|u_{\frac{\tilde{k}}{2},\tilde{k}}(t)\|_{L^\infty} + \|\nabla d_{\frac{\tilde{k}}{2},\tilde{k}}(t)\|_{L^\infty} < \frac{\tilde{k}^{\frac{1}{2}}}{32c_3\sqrt{c_1}(\|u_0\|_{L^2} + \|\nabla d_0\|_{L^2})}, \quad \forall t \in (T-\delta, T). \tag{3.61}$$

In view of (3.58), the inequality (3.61) is equivalent to

$$\begin{aligned} &\|u_{\frac{\tilde{k}}{2},\tilde{k}}(t)\|_{B^{-1}_{\infty,\infty}} + \|\nabla d_{\frac{\tilde{k}}{2},\tilde{k}}(t)\|_{B^{-1}_{\infty,\infty}} \\ &\quad < c\frac{1}{\tilde{k}^{\frac{1}{2}}32c_3\sqrt{c_1}(\|u_0\|_{L^2} + \|\nabla d_0\|_{L^2})} \\ &\quad < c\frac{1}{16\sqrt{2}c_3(\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2}) \times 32c_3\sqrt{c_1}(\|u_0\|_{L^2} + \|\nabla d_0\|_{L^2})} \\ &\quad < c\frac{1}{512\sqrt{2}c_3^2\sqrt{c_1}(\|u_0\|_{L^2} + \|\nabla d_0\|_{L^2})(\|\nabla u_0^{\tilde{k}}\|_{L^2} + \|\Delta d_0^{\tilde{k}}\|_{L^2})}. \end{aligned} \tag{3.62}$$

Thus, if (3.62) and

$$c_2(\|u_{\tilde{k}}\|_{L^\infty(T-\delta,T;\dot{B}^{-1}_{\infty,\infty})} + \|\nabla d_{\tilde{k}}\|_{L^\infty(T-\delta,T;\dot{B}^{-1}_{\infty,\infty})} \le \frac{1}{4} \tag{3.63}$$

hold, we can infer from (3.57) that

$$\begin{aligned} &\frac{d}{dt}(\|\nabla u^{\tilde{k}}\|_{L^2}^2 + \|\Delta d^{\tilde{k}}\|_{L^2}^2) \\ &\quad \le \left(c_3\tilde{k}^{-\frac{1}{2}}(\|\nabla u^{\tilde{k}}\|_{L^2} + \|\Delta d^{\tilde{k}}\|_{L^2}) - \frac{1}{8}\right)(\|\Delta u^{\tilde{k}}\|_{L^2}^2 + \|\nabla\Delta d^{\tilde{k}}\|_{L^2}^2). \end{aligned} \tag{3.64}$$

Since $c_3\tilde{k}^{-\frac{1}{2}}(\|\nabla u^{\tilde{k}}(T-\delta)\|_{L^2} + \|\Delta d^{\tilde{k}}(T-\delta)\|_{L^2}) - \frac{1}{8} = c_3\tilde{k}^{-\frac{1}{2}}\frac{\tilde{k}^{\frac{1}{2}}}{16c_3} - \frac{1}{8} < 0$, there is a right neighborhood I of $t = T-\delta$ such that

$$c_3\tilde{k}^{-\frac{1}{2}}(\|\nabla u^{\tilde{k}}(t)\|_{L^2} + \|\Delta d^{\tilde{k}}(t)\|_{L^2}) - \frac{1}{8} < 0, \quad \forall t \in \mathrm{I}.$$

Hence, it follows by (3.64) that the function $t \to \|\nabla u^{\tilde{k}}\|_{L^2} + \|\Delta d^{\tilde{k}}\|_{L^2}$ decreases in I, which contradicts (3.59) and (3.60). Thus, when (3.62) and (3.63) are satisfied, u and ∇d cannot blow up at $t = T$, and u and ∇d are regular in $(0, T]$. The proof of the theorem is completed. □

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Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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