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Multiple solutions for a coercive quasilinear elliptic equation via Morse theory

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Abstract

We study the quasilinear elliptic problem which is resonant at zero. By using Morse theory, we obtain five nontrivial solutions for the equation with coercive nonlinearities.

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1 Introduction

Let Ω be a bounded domain in \mathbb{R}^N ($N \geq 1$) with smooth boundary $\partial\Omega$. We study the following quasilinear elliptic problem:

$$\begin{cases} -\Delta_p u - \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $2 < p < \infty$ and Δ_p denotes the p -Laplacian operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

In what follows, we denote by

$$0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots$$

the eigenvalues of $-\Delta$ in $W_0^{1,2}(\Omega)$, and we let $\mu_1 > 0$ be the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ (see [9]). Moreover, we make the following assumptions:

(f₁) $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ with $f(x, 0) = 0$, and satisfies the following condition:

$$|f'(x, t)| \leq c(1 + |t|^{q-2}), \quad \forall t \in \mathbb{R}, x \in \Omega,$$

for some constants $c > 0$ and $q \in [2, p^*)$, where $p^* = Np/(N - p)$ if $p < N$ and $p^* = +\infty$ if $N \leq p$,

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(f_2) there exist $M > 0$ and $\lambda < \frac{\lambda_1}{2}$ such that

$$F(x, t) - \frac{1}{p} \mu_1 |t|^p \leq \lambda |t|^2, \quad \text{for } |t| \geq M, x \in \Omega,$$

where $F(x, t) = \int_0^t f(x, s) ds$,

(f^-) there exist $\alpha > 0$ and $k \geq 3$ such that

$$f'(x, 0) = \lambda_k, F(x, t) \leq \frac{1}{2} \lambda_k t^2, \quad \text{for } |t| \leq \alpha, x \in \Omega,$$

(f^+) there exist $\alpha > 0$, $k \geq 3$, $C > 0$ and $2 < \theta < p$ such that

$$f'(x, 0) = \lambda_k, F(x, t) \geq \frac{1}{2} \lambda_k t^2 + C |t|^\theta, \quad \text{for } |t| \leq \alpha, x \in \Omega.$$

Under the conditions above, from [14, Theorem 1.2] we know that Eq. (1.1) has at least four nontrivial solutions. It is worth pointing out that, using similar conditions, the authors in [10, Theorem 3.2] not only obtain four nontrivial solutions, but also prove that two of them are sign changing. Moreover, when the nonlinearity f is resonant at infinity and non-resonant at zero, using variational methods, together with truncation and comparison techniques and Morse theory, the paper [11] can get the existence of six nontrivial solutions (two of them are sign changing).

The aim of this paper is to obtain the existence of another solution. Specifically, our result reads as follows.

Theorem 1.1 *If (f_1), (f_2) and (f^-) (or (f^+)) hold, then Eq. (1.1) has at least five nontrivial solutions.*

Remark 1.2 (1) In our proof, we first obtain a nontrivial solution near zero inspired by papers [13, 15]. Then we use the estimation of critical groups to distinguish the new solution from the known solutions of [10, 14]. The method of estimating critical groups comes from [5], which has studied the bifurcation problem of semilinear elliptic equations at zero, and obtained six nontrivial solutions of the equation with coercive nonlinearities.

(2) Checking the proof below, our result is also true when $p = 2$. So as far as we know, our theorem is new even for the semilinear elliptic equation.

This paper is organized as follows. In Sect. 2, by Morse theory the existence of a new nontrivial solution and the estimation of its critical groups are given. In Sect. 3, we give the proof of Theorem 1.1. In the sequel, the letter C will be used indiscriminately to denote a suitable positive constant whose value may change from line to line.

2 A solution near zero

For any $\lambda \in \mathbb{R}$, let $f(x, u) = \lambda_k u + g(x, u)$ and $G(x, u) = \int_0^u g(x, s) ds$, then we consider the C^2 functional $I_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by setting

$$I_\lambda(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx + \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2} \int_\Omega |u|^2 dx - \int_\Omega G(x, u) dx, \quad (2.1)$$

where $W_0^{1,p}(\Omega)$ is the Sobolev space endowed with the norm

$$\|u\| = \|\nabla u\|_p = \left(\int_{\Omega} |\nabla u|^p dx \right)^{1/p}.$$

By (f_1) , weak solutions of Eq. (1.1) correspond to critical points of functional I_{λ_k} , which is also defined in the following form:

$$I_{\lambda_k}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} F(x, u) dx.$$

By [6, Page 277], the second order differential of I_{λ_k} in isolated critical point u_0 is given by

$$\begin{aligned} \langle I''_{\lambda_k}(u_0)v, w \rangle &= \int_{\Omega} (1 + |\nabla u_0|^{p-2})(\nabla v \nabla w) dx \\ &\quad + \int_{\Omega} (p-2)|\nabla u_0|^{p-4}(\nabla u_0 \nabla v)(\nabla u_0 \nabla w) dx - \int_{\Omega} f'(x, u_0)vw dx, \end{aligned} \quad (2.2)$$

for any $v, w \in W_0^{1,p}(\Omega)$. In addition, if we assume that $I_{\lambda_k}(u_0) = c \in \mathbb{R}$, and U is an isolated neighborhood of u_0 , then the group

$$C_{\ell}(I_{\lambda_k}, u_0) = H_{\ell}(I_{\lambda_k}^c \cap U, I_{\lambda_k}^c \cap U \setminus \{u_0\}), \quad \ell \in \mathbb{N} = \{0, 1, 2, \dots\},$$

is called the ℓ th critical group of the functional I_{λ_k} at u_0 , where $I_{\lambda_k}^c = \{u \in W_0^{1,p}(\Omega) : I_{\lambda_k}(u) \leq c\}$, and $H_{\ell}(\cdot, \cdot)$ are the singular relative homological groups with a coefficient group \mathbb{F} (see [1, Definition 4.1, Chapter I]).

Before stating our results, we recall the following result concerning critical groups estimates.

Lemma 2.1 ([8, Theorem 3.1]) *Let V be a subspace of $W_0^{1,p}(\Omega)$ of finite dimension m . For critical point u_0 of I_{λ_k} , we assume that:*

- (i) *the function I_{λ_k} is of class C^2 on $u_0 + V$ and for every $v \in V$ the functions $u_0 \mapsto \langle I'_{\lambda_k}(u_0), v \rangle$ and $u_0 \mapsto \langle I''_{\lambda_k}(u_0)v, v \rangle$ are continuous on $W_0^{1,p}(\Omega)$,*
- (ii) *$\langle I''_{\lambda_k}(u_0)v, v \rangle < 0$ for every $v \in V \setminus \{0\}$.*

Then we have $C_{\ell}(I_{\lambda_k}, u_0) = \{0\}$ for every $\ell \leq m - 1$.

Lemma 2.2 *If (f_1) and (f^-) hold, then Eq. (1.1) has a nontrivial solution v_0 such that*

$$C_{\ell}(I_{\lambda_k}, v_0) = \{0\}, \quad \forall \ell \leq d_{k-1} - 1, \quad (2.3)$$

where $d_{k-1} = \dim\{\bigoplus_{i \leq k-1} \ker(-\Delta - \lambda_i)\}$.

Proof The proof will be divided into several steps.

- (1) For some $\delta > 0$ with $\lambda_k < \lambda < \lambda_k + \delta < \lambda_{k+1}$, we assume that

$$K(I_{\lambda}) = \{u \in W_0^{1,p}(\Omega) : I'_{\lambda}(u) = 0\}$$

is the set of critical points of I_λ . By (2.1) and (2.2), we know that

$$\langle I'_\lambda(0)v, w \rangle = \int_\Omega \nabla v \nabla w \, dx - \lambda \int_\Omega vw \, dx, \quad \forall v, w \in W_0^{1,p}(\Omega),$$

then $I'_\lambda(0)$ is injective. So $u = 0$ is an isolated critical point (see [7, Corollary 2.4]). Meanwhile, [6, Theorem 1.1] shows that

$$C_\ell(I_\lambda, 0) = \delta_{\ell, d_k} \mathbb{F}. \tag{2.4}$$

For some small $\rho > 0$, let

$$B_\rho(0) = \{u \in W_0^{1,p}(\Omega) : \|u\| < \rho\} \tag{2.5}$$

be an isolated neighborhood of 0 such that $K(I_\lambda) \cap B_\rho(0) = \{0\}$. If we define $O = \overline{B_\rho(0)}$, then I_λ satisfies the Palais–Smale condition on O . Let β_1, β_2 be regular values of I_λ such that

$$\beta_1 < \inf_O I_\lambda \leq \sup_O I_\lambda < \beta_2.$$

Define

$$W = O_{\beta_1}^{\beta_2} = \tilde{O} \cap I_\lambda^{-1}[\beta_1, \beta_2], \quad W^- = W \cap I_\lambda^{-1}(\beta_1),$$

where $\tilde{O} = \bigcup_{t \in \mathbb{R}} \vartheta(t, O)$ and ϑ is the pseudo–gradient flow of I_λ . Here (W, W^-) is called the Gromoll–Meyer pair for the isolated critical point $u = 0$ (see [1, Definition 5.1, Chapter I]). Meanwhile, using (2.4) and [1, Theorem 5.2, Chapter I] we have

$$C_\ell(I_\lambda, 0) = H_\ell(W, W^-) = \delta_{\ell, d_k} \mathbb{F}. \tag{2.6}$$

(2) For any $u, v \in W_0^{1,p}(\Omega)$ and $\lambda_k < \lambda < \lambda_k + \delta$ with $\delta > 0$, using (2.1) we get

$$\begin{aligned} \|I_\lambda(u) - I_{\lambda_k}(u)\|_{C^1(W)} &= \|I_\lambda - I_{\lambda_k}\|_{C(W)} + \|I'_\lambda - I'_{\lambda_k}\|_{C(W)} \\ &= \sup_{u \in W} |I_\lambda(u) - I_{\lambda_k}(u)| + \sup_{u \in W} \sup_{\|v\| \leq 1} |\langle I'_\lambda(u) - I'_{\lambda_k}(u), v \rangle| \\ &\leq \delta \sup_{u \in W} \int_\Omega u^2 \, dx + \delta \sup_{u \in W} \sup_{\|v\| \leq 1} \left| \int_\Omega uv \, dx \right| \\ &\leq C\delta. \end{aligned} \tag{2.7}$$

Thus, for any $\varepsilon > 0$, by (2.7) there is $\delta > 0$ such that

$$\|I_\lambda - I_{\lambda_k}\|_{C^1(W)} < \varepsilon, \quad \text{for } \lambda \in (\lambda_k, \lambda_k + \delta). \tag{2.8}$$

Using [3, Theorem III.4], we know that Gromoll–Meyer pairs are stable under small perturbation, then (2.8) implies that there exists $\delta > 0$ such that (W, W^-) is still a Gromoll–Meyer pair for I_{λ_k} with the critical set

$$S_{\lambda_k} = B_\rho(0) \cap K(I_{\lambda_k}) = W \cap K(I_{\lambda_k}).$$

Then (2.6) implies that

$$C_\ell(I_{\lambda_k}, S_{\lambda_k}) = H_\ell(W, W^-) = \delta_{\ell, d_k} \mathbb{F}. \tag{2.9}$$

Here more properties of $C_q(I_{\lambda_k}, S_{\lambda_k})$ can be found in [3, Definition II.1, Theorem III.3].

(3) Assume that $S_{\lambda_k} = \{v_1, v_2, \dots, v_n\}$ for some $n \in \mathbb{N}$, (W_j, W_j^-) is the Gromoll–Meyer pair for v_j ($j = 1, \dots, n$), and $Q(t)$ is a formal series with nonnegative integral coefficients. By ([2, Page 414]) we have the Morse relation (or see [13, Proposition 2.7]):

$$\sum_{j=1}^n \sum_{\ell} \text{rank } H_\ell(W_j, W_j^-) t^\ell = \sum_{\ell} \text{rank } H_\ell(W, W^-) t^\ell + (1+t)Q(t), \tag{2.10}$$

this together with (2.9) implies that I_{λ_k} has a critical point $u_0(\rho) \in W$ such that

$$C_{d_k}(I_{\lambda_k}, u_0(\rho)) \neq 0. \tag{2.11}$$

Without losing generality, we assume that $u = 0$ is an isolated solution of Eq. (1.1). Using (f_1) and (f^-) , by [14, Theorem 1.1] or [12, Proposition 2.3] for $p = 2$ we have

$$C_\ell(I_{\lambda_k}, 0) = \delta_{\ell, d_{k-1}} \mathbb{F}. \tag{2.12}$$

Then (2.11) and (2.12) show that Eq. (1.1) has a nontrivial solution $u_0(\rho) \in W$.

(4) By standard elliptic regularity arguments we have $u_0(\rho) \in C_0^1(\overline{\Omega})$ (see [6, Page 277]). Since

$$\|u_0(\rho)\|_{C_0^1(\overline{\Omega})} \rightarrow 0, \quad \text{as } \rho \rightarrow 0,$$

for any $\varepsilon > 0$ there is some $\rho > 0$ small enough such that

$$\|u_0(\rho)\|_{C_0^1(\overline{\Omega})} < \varepsilon. \tag{2.13}$$

Now, we choose constants λ_* and λ^* satisfying

$$\lambda_{k-1} < \lambda_* < \lambda_k < \lambda^* < \lambda_{k+1}.$$

Let $\varepsilon > 0$ to be selected suitably later. By $f'(x, 0) = \lambda_k$, the compactness of $\overline{\Omega}$ and (2.13), we may find a solution $u_0(\rho)$ to our equation such that $\|u_0(\rho)\|_{C_0^1(\overline{\Omega})} < \varepsilon$ and

$$\lambda_{k-1} < \lambda_* \leq f'(x, u_0(\rho)) \leq \lambda^* < \lambda_{k+1}, \quad \text{for all } x \in \overline{\Omega}. \tag{2.14}$$

For $v \in E_{k-1} \setminus \{0\}$ where $E_{k-1} = \bigoplus_{i \leq k-1} \ker(-\Delta - \lambda_i)$, by (2.2), (2.13) and (2.14) we have

$$\begin{aligned}
 & \langle I''_{\lambda_k}(u_0(\rho))v, v \rangle \\
 &= \int_{\Omega} (1 + |\nabla u_0(\rho)|^{p-2}) |\nabla v|^2 dx + \int_{\Omega} (p-2) |\nabla u_0(\rho)|^{p-4} (\nabla u_0(\rho) \nabla v)^2 dx \\
 &\quad - \int_{\Omega} f'(x, u_0(\rho)) v^2 dx \\
 &\leq \int_{\Omega} (1 + C\varepsilon) |\nabla v|^2 dx + C\varepsilon \int_{\Omega} (\nabla u_0(\rho) \nabla v)^2 dx - \int_{\Omega} f'(x, u_0(\rho)) v^2 dx \tag{2.15} \\
 &\leq \int_{\Omega} (1 + C\varepsilon) |\nabla v|^2 dx + C\varepsilon \int_{\Omega} |\nabla u_0(\rho)|^2 |\nabla v|^2 dx - \int_{\Omega} f'(x, u_0(\rho)) v^2 dx \\
 &\leq \int_{\Omega} (1 + C\varepsilon) |\nabla v|^2 dx - \int_{\Omega} f'(x, u_0(\rho)) v^2 dx \\
 &\leq \frac{(1 + C\varepsilon)\lambda_{k-1} - \lambda_*}{\lambda_{k-1}} \int_{\Omega} |\nabla v|^2 dx.
 \end{aligned}$$

Moreover, we can choose $\varepsilon > 0$ such that

$$(1 + C\varepsilon)\lambda_{k-1} - \lambda_* < 0,$$

this together with (2.15) shows that

$$\langle I''_{\lambda_k}(u_0(\rho))v, v \rangle < 0, \quad \text{for } v \in E_{k-1} \setminus \{0\},$$

then from Lemma 2.1 we get (2.3). Let $v_0 = u_0(\rho)$, then we complete the proof. □

Lemma 2.3 *If (f_1) and (f^+) hold, then Eq. (1.1) has a nontrivial solution v_0 such that (2.3) holds.*

Proof For some $\delta > 0$ small such that $\lambda_k - \delta < \lambda < \lambda_k$, by [6, Theorem 1.1] we get

$$C_{\ell}(I_{\lambda}, 0) = \delta_{\ell, d_{k-1}} \mathbb{F}.$$

Using (f_1) and (f^+) , from [14, Theorem 1.1], we have

$$C_{\ell}(I_{\lambda_k}, 0) = \delta_{\ell, d_k} \mathbb{F}.$$

Similar to Lemma 2.2, Eq. (1.1) has a nontrivial solution v_0 such that (2.3) holds. We complete the proof. □

Remark 2.4 (1) According to our proof, it is not difficult to find that any critical point in W satisfies the critical group estimation (2.3).

(2) In the lemmas above, the method for the existence of a nontrivial solution is the same as that in [13, 15], but our result also has a new content: the estimation of the critical groups for this solution.

3 Proof of theorem

Now we can give the proof of our theorem as follows.

Proof of Theorem 1.1 Under our assumptions, Ref. [14] has proved that I_{λ_k} satisfies the Palais–Smale condition, and there are three nontrivial solutions u_i ($i = 1, 2, 3$). Moreover, two of them are local minima such that

$$C_\ell(I_{\lambda_k}, u_1) = C_\ell(I_{\lambda_k}, u_2) = \delta_{\ell,0}\mathbb{F}, \quad (3.1)$$

and u_3 is the mountain pass solution such that (see [11] or [10, Page 412])

$$C_\ell(I_{\lambda_k}, u_3) = \delta_{\ell,1}\mathbb{F}. \quad (3.2)$$

From Lemma 2.2 and Lemma 2.3, we know that

$$C_\ell(I_{\lambda_k}, S_{\lambda_k}) = H_\ell(W, W^-) = \delta_{\ell,d}\mathbb{F}, \quad (3.3)$$

where $S_{\lambda_k} = B_\rho(0) \cap K(I_{\lambda_k}) = W \cap K(I_{\lambda_k})$, $d = d_k$ for (f^-) and $d = d_{k-1}$ for (f^+) .

Claim: $u_i \notin W$ for $i = 1, 2, 3$.

Reasoning by contradiction, when $u_i \in W$, from Lemma 2.2, Lemma 2.3 and Remark 2.4, there is $\rho > 0$ small enough such that

$$C_\ell(I_{\lambda_k}, u_i) = \{0\}, \quad \forall \ell \leq d_{k-1} - 1, i = 1, 2, 3,$$

which is in contradiction with (3.1) and (3.2) because of $k \geq 3$. Then the claim holds.

If I_{λ_k} has only four nontrivial critical points: v_0 and u_i for $i = 1, 2, 3$, then, for $a < \inf I_{\lambda_k}(K(I_{\lambda_k}))$, [14, Lemma 4.1] gives the ℓ th critical group of I_{λ_k} at infinity:

$$C_\ell(I_{\lambda_k}, \infty) = H_\ell(W_0^{1,p}(\Omega), I_{\lambda_k}^a) = \delta_{\ell,0}\mathbb{F}, \quad (3.4)$$

where $(W_0^{1,p}(\Omega), I_{\lambda_k}^a)$ is the Gromoll–Meyer pair for $K(I_{\lambda_k})$ (see [4, Theorem 2.2]).

Now from (3.1) to (3.4), using the Gromoll–Meyer pairs (W, W^-) and $(W_0^{1,p}(\Omega), I_{\lambda_k}^a)$, the Morse relation (2.10) gives

$$(-1)^d + (-1)^0 + (-1)^0 + (-1)^1 = (-1)^0,$$

this contradiction implies that Eq. (1.1) has another nontrivial solution $u_4 \notin W$. The proof is completed. \square

4 Conclusions

Using the critical groups estimates, our theorem can get more nontrivial solutions. The main results presented in this paper improve and generalize the results in [10, 14].

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Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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