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A novel method in determining a layered periodic structure

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Abstract
This paper is concerned with the inverse scattering of time-harmonic waves by a penetrable structure. By applying the integral equation method, we establish the uniform L^p_α ($1 < p \le 2$) estimates for the scattered and transmitted wave fields corresponding to a series of incident point sources. Based on these a priori estimates and a mixed reciprocity relation, we prove that the penetrable structure can be uniquely identified by means of the scattered field measured only above the structure induced by a countably infinite number of quasi-periodic incident plane waves.

Keywords: Inverse scattering; Uniqueness; L^p_α estimate; Periodic structures

1 Introduction

In this paper, we consider the inverse problem of determining a penetrable periodic structure in $\mathbb{R}^3$ from the scattered data measured only above the structure. This kind of problem occurs in various applications such as in radar imaging, modern diffractive optics, and non-destructive testing. For convenience, we write a point x in $\mathbb{R}^3$ for $(\widetilde{x}, x_3)$ with $\widetilde{x} := (x_1, x_2) \in \mathbb{R}^2$. Assume that the penetrable profile is described by

$$\Gamma := \{x \in \mathbb{R}^3 : x_3 = f(\widetilde{x})\},$$

where f is a periodic function with respect to the variable $\widetilde{x}$, that is, $f(\widetilde{x}) = f(\widetilde{x} + 2n\pi)$ for $n := (n_1, n_2) \in \mathbb{Z}^2$. Assume further that the homogeneous media above and below Γ are described by

$$\Omega_+ := \{x \in \mathbb{R}^3 : x_3 > f(\widetilde{x})\} \quad \text{and} \quad \Omega_- := \{x \in \mathbb{R}^3 : x_3 < f(\widetilde{x})\}$$

with the wave numbers k_1 and k_2, respectively.

Consider the incident plane waves in the form of

$$u^i(x) = \exp\big(i\alpha_j \cdot \widetilde{x} - i\beta_j^+ x_3\big), \quad j \in \mathbb{Z}^2, \text{with } \alpha_j = \alpha + j, \tag{1.1}$$

which propagate downward from Ω_+ with $\alpha = (\alpha_1, \alpha_2) := k_1(\sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2)$ with the incident angle $\theta_1 \in [0, \pi/2), \theta_2 \in [0, 2\pi)$, and $\beta_j^+ \in \mathbb{C}$ is given by

$$\beta_j^+ = \sqrt{k_1^2 - |\alpha_j|^2} \quad \text{if } |\alpha_j| \le k_1, \qquad \beta_j^+ = i\sqrt{|\alpha_j|^2 - k_1^2} \quad \text{if } |\alpha_j| > k_1.$$

Then the scattering of the incident u^i by the periodic structure can be formulated in determining the total field $u_1 := u^i + u^s$ with the scattered field u^s and the transmitted field u_2 to the following problem:

$$\triangle u_1 + k_1^2 u_1 = 0 \quad \text{in } \Omega_+, \tag{1.2}$$

$$\triangle u_2 + k_2^2 u_2 = 0 \quad \text{in } \Omega_-, \tag{1.3}$$

$$u_1 = u_2, \qquad \frac{\partial u_1}{\partial \nu} = \lambda \frac{\partial u_2}{\partial \nu} \quad \text{on } \Gamma, \tag{1.4}$$

$$u^s(x) = \sum_{n\in\mathbb{Z}^2} u_n^+ \exp\bigl(i\alpha_n \cdot \widetilde{x} + i\beta_n^+ x_3\bigr), \quad x_3 > A_1 := \max_{t\in\mathbb{R}^2} f(t), \tag{1.5}$$

$$u_2(x) = \sum_{n\in\mathbb{Z}^2} u_n^- \exp\bigl(i\alpha_n \cdot \widetilde{x} - i\beta_n^- x_3\bigr), \quad x_3 < A_2 := \min_{t\in\mathbb{R}^2} f(t). \tag{1.6}$$

Here, $u_n^\pm \in \mathbb{C}$ are the solution sequences, λ is the transmission coefficient and the unit normal vector ν on Γ is directed into the interior of Ω_-. Notice that the incident wave $u^i(\cdot)$ satisfies such an α-quasi-periodic condition $u^i(\widetilde{x} + 2n\pi, x_3) = e^{i2\alpha\cdot n\pi} u^i(\widetilde{x}, x_3)$ for all $n \in \mathbb{Z}^2$. Then the solution $u_l, l = 1, 2$, is also required to satisfy the same α-quasi-periodic condition, i.e., $u_l(\widetilde{x} + 2n\pi, x_3) = e^{i2\alpha\cdot n\pi} u_l(\widetilde{x}, x_3)$ in $\mathbb{R}^3$. Conditions (1.5) and (1.6) are known as the Rayleigh expansion conditions of the scattered field u^s in Ω_+ and the transmitted field u_2 in Ω_-, respectively, with β_n^- defined similarly as β_n^+ by the wave number k_2.

The well-posedness of problem (1.2)–(1.6) can be established by the variational method (cf. [31]) or the integral equation method (cf. [32, 33]). In the current paper we first establish the L_α^p $(1 < p \le 2)$ estimates for the scattered field u^s and the transmitted field u_2. Based on these a priori estimates, we focus on the unique identification of the penetrable periodic structure from the scattered field u^s measured only on a straight line above the periodic structure induced by a countably infinite number of quasi-periodic incident plane waves.

There are lots of results concerning the uniqueness issue for the inverse periodic transmission problems (cf. [5, 7, 12, 13, 18, 19, 23, 24, 33, 34]) and for the inverse scattering by the polygonal periodic structure (cf. [6, 11, 14]). For the special case when the medium has the energy absorption property, a uniqueness theorem was obtained in [5] from the measured scattered field for one incident plane wave in a two-dimensional space. The result of [5] was then extended to the three-dimensional case in [2]. It should be remarked that the uniqueness with one incident wave does not hold true for the inverse periodic problem for a real wave number case, that is, the medium does not has a property of energy absorption. See also [7] for a uniqueness theorem on the recovery of a smooth periodic structure with one incident plane wave under some a priori assumptions on the structure. For the case when a priori restrictions on the height of the grating surface are known in advance, a uniqueness result can be found in [18] on the identification of a smooth perfectly reflecting periodic structure from many measurements corresponding to a finite number of incident plane waves. The method of [18] was extended to the periodic transmission problem [13]. There also exist some numerical methods in reconstructing periodic structures. For example, a linear sampling method was developed in [20, 22] for determining the shape of partially coated bi-periodic structures, and in [35] a novel linear sampling method was introduced for simultaneously reconstructing dielectric grating structures in an inhomogeneous periodic medium. See also [10] for a finite element method or [3, 4, 17] for the factorization method in determining the periodic structures, or [30] for the uniquely reconstruction of a locally perturbed infinite plane. Recently, by making use of the differential sampling method, the anisotropic periodic layer can be uniquely determined in [25] under the assumption that the complement of the periodic layer in one period is connected. The analysis of sampling methods for the recovery of a local perturbation in a periodic layer can be found in [16].

For the scattering by general periodic structures case, there are several uniqueness results. We refer to [23] for a uniqueness theorem for the inverse Dirichlet problem, and to [21, 24, 32] for uniqueness results for the inverse transmission problem by means of all quasi-periodic incident plane waves. The reader is referred to [19] for a partially coated perfectly grating case with respect to infinitely many point sources, and to [34] for uniqueness results for both the partially coated perfectly reflecting grating and the periodic transmission case in a two-dimensional space, corresponding to a countably infinite number of quasi-periodic incident plane waves. In this paper we intend to develop a novel method, which differs from the approach used in [34], to prove the uniqueness on the identification of the penetrable periodic structure in the three-dimensional space from the measured data only above the structure with respect to a countably infinite number of quasi-periodic incident plane waves. The technique developed in this paper can date back to the work [27, 36] on the inverse scattering problems of determining the support of penetrable electromagnetic obstacles or to [28] for the fluid-solid interaction problem of identifying the bounded solid obstacle, [29] for the cavity scattering case.

The paper is organized as follows. In Sect. 2, the a priori estimates in the sense of L^p_α $(1 < p \le 2)$ norm for the solution of the direct scattering problem in $\mathbb{R}^3$ are established by applying the integral equation method. Section 3 is devoted to the inverse problem of uniquely determining the periodic structure from the measured data only above the structure produced by a countably infinite number of quasi-periodic incident plane waves.

2 A priori estimates

In this section we establish some a priori estimates for the solution of the direct scattering problem by employing the integral equation method. Eliminating the incident field u^i, it is easily found that the scattered field $w_1 := u_1 - u^i$ in Ω_+ and the transmitted field $w_2 := u_2$ in Ω_- satisfy the following boundary value problem:

$$\triangle w_1 + k_1^2 w_1 = 0 \quad \text{in } \Omega_+, \tag{2.1}$$

$$\triangle w_2 + k_2^2 w_2 = 0 \quad \text{in } \Omega_-, \tag{2.2}$$

$$w_1 - w_2 = f_1, \qquad \frac{\partial w_1}{\partial \nu} - \lambda \frac{\partial w_2}{\partial \nu} = f_2 \quad \text{on } \Gamma, \tag{2.3}$$

$$w_1(x) = \sum_{n\in\mathbb{Z}^2} w_n^+ \exp\bigl(i\alpha_n \cdot \widetilde{x} + i\beta_n^+ x_3\bigr), \quad x_3 > A_1, \tag{2.4}$$

$$w_2(x) = \sum_{n\in\mathbb{Z}^2} w_n^- \exp\bigl(i\alpha_n \cdot \widetilde{x} - i\beta_n^- x_3\bigr), \quad x_3 < A_2 \tag{2.5}$$

in the general case $f_1, f_2 \in L^p_\alpha(\Gamma)$ with $1 < p \le 2$. *Here*, $L^p_\alpha(\Gamma)(p \ge 1)$ denotes the Sobolev space of scalar functions on Γ which is assumed to be α-quasi-periodic with respect to the variable $\widetilde{x}$, equipped with the norm in the usual Sobolev space $L^p(\Gamma)$.

Before going further we first introduce the basic notations that are used in the rest of this paper. For simplicity, we use $\Omega_\pm$ and Γ again to denote the same sets restricted to one period $0 < x_1, x_2 < 2\pi$. For each $h > 0$, denote by $\Omega_+(h) := \{x \in \Omega_+ : x_3 < A_1 + h\}$, $\Omega_-(h) := \{x \in \Omega_- : x_3 > A_2 - h\}$, $\Gamma_+(h) := \{x \in \Omega_+ : x_3 = A_1 + h\}$, and $\Gamma_-(h) := \{x \in \Omega_- : x_3 = A_2 - h\}$, respectively. Then, let $H^1_\alpha(\Omega_\pm(h))$ and $L^p_\alpha(\Omega_\pm(h))(p \ge 1)$ denote the Sobolev spaces of scalar functions on $\Omega_\pm(h)$ which are assumed to be α-quasi-periodic with respect to the variable $\widetilde{x}$, equipped with the norms in the usual Sobolev spaces $H^1(\Omega_\pm(h))$ and $L^p(\Omega_\pm(h))$, respectively. Let $H^{1/2}_\alpha(\Gamma_\pm(h))$ denote the trace space of $H^1_\alpha(\Omega_\pm(h))$, and $H^{-1/2}_\alpha(\Gamma_\pm(h))$ is the dual space of $H^{1/2}_\alpha(\Gamma_\pm(h))$.

We introduce the free space α-quasi-periodic Green function

$$G_1(x,y;k_1) = \frac{i}{8\pi^2} \sum_{n\in\mathbb{Z}^2} \frac{1}{\beta_n^+} \exp\bigl(i\alpha_n \cdot (\widetilde{x} - \widetilde{y}) + i\beta_n^+ |x_3 - y_3|\bigr), \quad x \ne y \tag{2.6}$$

and the α-quasi-periodic layer-potential operators S_1, K_1, K_1', and T_1 defined by

$$S_1\xi(x) = \int_\Gamma G_1(x,y;k_1)\xi(y)\,ds(y), \quad x \in \Gamma, \tag{2.7}$$

$$K_1\xi(x) = \int_\Gamma \frac{\partial}{\partial\nu(y)} G_1(x,y;k_1)\xi(y)\,ds(y), \quad x \in \Gamma, \tag{2.8}$$

$$K_1'\xi(x) = \frac{\partial}{\partial\nu(x)} \int_\Gamma G_1(x,y;k_1)\xi(y)\,ds(y), \quad x \in \Gamma, \tag{2.9}$$

$$T_1\xi(x) = -\frac{\partial}{\partial\nu(x)} \int_\Gamma \frac{\partial}{\partial\nu(y)} G_1(x,y;k_1)\xi(y)\,ds(y), \quad x \in \Gamma. \tag{2.10}$$

Noting that $G_1(x,y;k_1) - \Phi(x,y;k_1)$ is smooth, it follows from [8] that the operators $S_1 : H_\alpha^{-\frac{1}{2}}(\Gamma) \to H_\alpha^{\frac{1}{2}}(\Gamma)$, $K_1 : H_\alpha^{\frac{1}{2}}(\Gamma) \to H_\alpha^{\frac{1}{2}}(\Gamma)$, $K_j' : H_\alpha^{-\frac{1}{2}}(\Gamma) \to H_\alpha^{-\frac{1}{2}}(\Gamma)$, and $T_1 : H_\alpha^{\frac{1}{2}}(\Gamma) \to H_\alpha^{-\frac{1}{2}}(\Gamma)$ are all bounded, where $\Phi(x,y;k_1) = \frac{1}{4\pi}\frac{e^{ik_1|x-y|}}{|x-y|}$ is the fundamental solution of the Helmholtz equation $\triangle\Phi + k_1^2\Phi = -\delta_y$ in the free space $\mathbb{R}^3$.

Theorem 2.1 *For $f_1, f_2 \in L^p_\alpha(\Gamma)$ with $1 < p \le 2$, there exists a unique solution $(w_1, w_2) \in L^p_\alpha(\Omega_+(h)) \times L^p_\alpha(\Omega_-(h))$ to the transmission problem (2.1)–(2.5) satisfying the estimate*

$$\|w_1\|_{L^p_\alpha(\Omega_+(h))} + \|w_2\|_{L^p_\alpha(\Omega_-(h))} \le C\bigl(\|f_1\|_{L^p_\alpha(\Gamma)} + \|f_2\|_{L^p_\alpha(\Gamma)}\bigr), \tag{2.11}$$

where $C > 0$ is a constant independent of f_1, f_2, and depending on $G_j(\cdot,y;k_j), \Omega_+(h)$ with $j = 1,2$ and the boundedness of the operators $S_j, K_j, K_j', j = 1,2$, and $T_2 - T_1$ in $L^p_\alpha(\Gamma)$.

Moreover, if $f_1, f_2 \in L^p_\alpha(\Gamma)$ with $\frac{4}{3} < p \le 2$, we have

$$\|w_1\|_{L^2_\alpha(\Omega_+(h))} + \|w_2\|_{L^2_\alpha(\Omega_-(h))} \le C\bigl(\|f_1\|_{L^p_\alpha(\Gamma)} + \|f_2\|_{L^p_\alpha(\Gamma)}\bigr) \tag{2.12}$$

with a positive constant $C > 0$, *which is independent of* f_1, f_2, *and depending on* $G_j(\cdot, y; k_j)$, $\Omega_+(h)$ *with* $j = 1, 2$ *and the boundedness of the operators* $S_j, K_j, K_j', j = 1, 2$ *and* $T_2 - T_1$ *in* $L^p_\alpha(\Gamma)$.

Proof We seek a solution of problem (2.1)–(2.5) in the form of combined single- and double-layer potential

$$w_1(x) = \int_\Gamma G_1(x, y; k_1)\varphi_1(y)\,ds(y) + \lambda \int_\Gamma \frac{\partial G_1(x, y; k_1)}{\partial \nu(y)}\varphi_2(y)\,ds(y), \tag{2.13}$$

$$w_2(x) = \int_\Gamma G_2(x, y; k_2)\varphi_1(y)\,ds(y) + \int_\Gamma \frac{\partial G_2(x, y; k_2)}{\partial \nu(y)}\varphi_2(y)\,ds(y), \tag{2.14}$$

where $G_2(x, y; k_2)$ is defined as (2.6) with the wave number k_1 replaced by k_2.

With the aid of the jump relations of the layer potentials (see [26] for the case in the L^p norm), we obtain that the transmission problem (2.1)–(2.5) can be reduced to the system of integral equations

$$\begin{pmatrix} \varphi_2 \\ \varphi_1 \end{pmatrix} + L \begin{pmatrix} \varphi_2 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} \frac{2}{1+\lambda} f_1 \\ -\frac{2}{1+\lambda} f_2 \end{pmatrix} \quad \text{in } L^p_\alpha(\Gamma) \times L^p_\alpha(\Gamma), \tag{2.15}$$

where the operator L is given by

$$L := \begin{pmatrix} \frac{2}{1+\lambda}(\lambda K_1 - K_2) & \frac{2}{1+\lambda}(S_1 - S_2) \\ \frac{2\lambda}{1+\lambda}(T_2 - T_1) & \frac{2}{1+\lambda}(\lambda K_2' - K_1') \end{pmatrix}.$$

It is easily shown that (2.15) is of Fredholm type due to the compactness of the operators $S_j, K_j, K_j', j = 1, 2$, and $T_2 - T_1$ in $L^p_\alpha(\Gamma)$. This, together with the uniqueness of the scattering problem (1.2)–(1.6), implies that (2.15) has a unique solution $(\varphi_2, \varphi_1)^T \in L^p_\alpha(\Gamma) \times L^p_\alpha(\Gamma)$ with the estimate

$$\|\varphi_2\|_{L^p_\alpha(\Gamma)} + \|\varphi_1\|_{L^p_\alpha(\Gamma)} \le C\big(\|f_1\|_{L^p_\alpha(\Gamma)} + \|f_2\|_{L^p_\alpha(\Gamma)}\big). \tag{2.16}$$

We next prove the $L^p_\alpha, 1 < p \le 2$ estimates for the solution of the transmission problem (2.1)–(2.5). In fact, it can be checked that

$$\begin{aligned}
&\left\| \int_\Gamma \Omega_+(h) G_1(\cdot, y; k_1)\varphi_1(y)\,ds(y) \right\|_{L^p_\alpha(\Omega_+(h))} \\
&\quad = \sup_{g \in L^q_\alpha, \|g\|_{L^q_\alpha(\Omega_+(h))} = 1} \left| \int_{\Omega_+(h)} \int_\Gamma G_1(x, y; k_1)\varphi_1(y)\,ds(y) g(x)\,dx \right| \\
&\quad = \sup_{g \in L^q_\alpha, \|g\|_{L^q_\alpha(\Omega_+(h))} = 1} \left| \int_\Gamma \int_{\Omega_+(h)} G_1(x, y; k_1) g(x)\,dx \varphi_1(y)\,ds(y) \right| \\
&\quad \le |\Gamma|^{\frac{1}{q}} \sup_{g \in L^q, \|g\|_{L^q_\alpha(\Omega_+(h))} = 1} \sup_{y \in \Gamma} \big\| G_1(\cdot, y; k_1) \big\|_{L^p_\alpha(\Omega_+(h))} \|g\|_{L^q_\alpha(\Omega_+(h))} \|\varphi_1\|_{L^p_\alpha(\Gamma)} \\
&\quad = |\Gamma|^{\frac{1}{q}} \sup_{y \in \Gamma} \big\| G_1(\cdot, y; k_1) \big\|_{L^p_\alpha(\Omega_+(h))} \|\varphi_1\|_{L^p_\alpha(\Gamma)} \le C\|\varphi_1\|_{L^p_\alpha(\Gamma)}
\end{aligned} \tag{2.17}$$

and

$$
\begin{aligned}
&\left\| \int_\Gamma \frac{\partial G_1(\cdot,y;k_1)}{\partial \nu(y)} \varphi_2(y)\,ds(y) \right\|_{L^p_\alpha(\Omega_+(h))} \\
&\quad = \sup_{g\in L^q_\alpha, \|g\|_{L^q_\alpha(\Omega_+(h))}=1} \left| \int_{\Omega_+(h)} \int_\Gamma \frac{\partial G_1(x,y;k_1)}{\partial \nu(y)} \varphi_2(y)\,ds(y) g(x)\,dx \right| \\
&\quad = \sup_{g\in L^q_\alpha, \|g\|_{L^q_\alpha(\Omega_+(h))}=1} \left| \int_\Gamma \frac{\partial}{\partial \nu(y)} \int_{\Omega_+(h)} G_1(x,y;k_1) g(x)\,dx \varphi_2(y)\,ds(y) \right| \\
&\quad \le \sup_{g\in L^q_\alpha, \|g\|_{L^q_\alpha(\Omega_+(h))}=1} \left\| \frac{\partial}{\partial \nu(y)} \int_{\Omega_+(h)} G_1(x,\cdot;k_1) g(x)\,dx \right\|_{L^q_\alpha(\Gamma)} \|\varphi_2\|_{L^p_\alpha(\Gamma)} \\
&\quad \le \sup_{g\in L^q_\alpha, \|g\|_{L^q_\alpha(\Omega_+(h))}=1} C\|g\|_{L^q_\alpha(\Omega_+(h))} \cdot \|\varphi_2\|_{L^p_\alpha(\Gamma)} = C\|\varphi_2\|_{L^p_\alpha(\Gamma)}
\end{aligned} \tag{2.18}
$$

with $\frac{1}{p}+\frac{1}{q}=1$. Here, we have used the fact that the volume potential operator is bounded from $L^q_\alpha(\Omega_+(h))$ into $W^{2,q}_\alpha(\Omega_+(h))$ with $2\le q\le 4$ (see [15, Theorem 9.9]), and the boundary trace operator is bounded from $W^{1,q}_\alpha(\Omega_+(h))$ into $L^q_\alpha(\Gamma)$ with $2\le q\le 4$ (see [1, Theorem 5.36]). It is noted that (2.17)–(2.18) still holds true, with $G_1(x,\cdot;k_1)$ replaced by $G_2(x,\cdot;k_2)$ and $\Omega_+(h)$ replaced by $\Omega_-(h)$, respectively. Now the desired estimate (2.11) follows from (2.13)–(2.14) and (2.16)–(2.18). Furthermore, if $f_1, f_2\in L^p_\alpha(\Gamma)$ with $\frac{4}{3}<p\le 2$, by the similar arguments as those in (2.17)–(2.18), one can derive the required result (2.13). This completes the proof of the theorem. □

Corollary 2.2 *For $y_0\in\Gamma$, define the sequence $y_j := y_0 - \frac{1}{j}\nu(y_0)\in\Omega_+$, $j\in\mathbb{N}$. Let (u_{1j},u_{2j}) be the solution of the scattering problem (1.2)–(1.6) with the incident point source $u^i = G_1(x,y_j;k_1)$. Then, for any $h\in\mathbb{R}$, we have*

$$
\|u_{1j}\|_{L^2_\alpha(\Omega_+(h))} + \|u_{2j}\|_{L^2_\alpha(\Omega_-(h))} \le C \tag{2.19}
$$

uniformly for $j\in\mathbb{N}_+$, where $C>0$ is a constant depending on $G_j(\cdot,y;k_j)$, $\Omega_+(h)$ with $j=1,2$.

Proof It is obvious that (u^s_{1j},u_{2j}) satisfies problem (2.1)–(2.5) with the boundary data

$$
f_1(j) := -G_1(x,y_j;k_1), \qquad f_2(j) := -\frac{\partial G_1(x,y_j;k_1)}{\partial \nu} \quad j\in\mathbb{N}.
$$

It is easy to see that $f_1(j), f_2(j)\in L^p_\alpha(\Gamma)$ are uniformly bounded for $j\in\mathbb{N}$ with $\frac{4}{3}<p<\frac{3}{2}$. Then the required result (2.19) follows from Theorem 2.1. This proves the corollary. □

Theorem 2.3 *Let (u_{1j},u_{2j}) be the solution of the scattering problem (1.2)–(1.6) corresponding to the incident point source $u^i = G_1(x,y_j;k_1)$ with y_j defined in Corollary 2.2. Then, for any $h\in\mathbb{R}$, it holds that*

$$
\|u_{2j}\|_{H^1_\alpha(\Omega_-(h)\setminus\overline{B})} \le C \tag{2.20}
$$

uniformly for $j \in \mathbb{N}_+$. Here, $C > 0$ is a constant depending on $G_j(\cdot, y; k_j), \Omega_+(h)$ with $j = 1, 2$ and the uniform boundedness of $S_{\Gamma\setminus B}(j)$ and $K_{\Gamma\setminus B}(j)$ in the corresponding Hilbert spaces, B is a ball satisfying that $B \supset B_\delta$, and B_δ is a small ball centered at y_0 with the radius $\delta > 0$.

Proof Define $\tilde{y}_j := y_0 + \frac{1}{j}\nu(y_0) \in \Omega_-$, let $w_1(j) := u^s_{1j} - G_1(x, \tilde{y}_j; k_1)$ and $w_2(j) := u_{2j}$, it follows that $(w_1(j), w_2(j))$ satisfies problem (2.1)–(2.5) with the boundary data

$$f_1(j) := -G_1(x, y_j; k_1) - G_1(x, \tilde{y}_j; k_1),$$
$$f_2(j) := -\frac{\partial G_1(x, y_j; k_1)}{\partial \nu} - \frac{\partial G_1(x, \tilde{y}_j; k_1)}{\partial \nu}.$$

Obviously, $f_1(j) \in L^p_\alpha(\Gamma)$ is uniformly bounded for $j \in \mathbb{N}$, where $1 < p < 2$. Furthermore, it is seen from [9, Lemma 4.2] that $f_2(j) \in C(\Gamma)$ is uniformly bounded for $j \in \mathbb{N}$. So $f_2(j) \in L^p_\alpha(\Gamma)$ is uniformly bounded for $j \in \mathbb{N}$, where $1 < p < 2$. Then, by (2.16) in Theorem 2.1, one obtains that the solution $(\varphi_1, \varphi_2)^T$ of (2.15) satisfies

$$\|\varphi_1\|_{L^p_\alpha(\Gamma)} + \|\varphi_2\|_{L^p_\alpha(\Gamma)} \le C\big(\|f_{1j}\|_{L^p_\alpha(\Gamma)} + \|f_{2j}\|_{L^p_\alpha(\Gamma)}\big), \quad 1 < p < 2. \tag{2.21}$$

We next prove that the operator $S_{1j} : L^p_\alpha(\Gamma) \to L^2_\alpha(\Gamma \setminus B)$ is uniformly bounded for $j \in \mathbb{N}$, where $1 < p < 2$. Indeed, by direct calculations, we can deduce that

$$\begin{aligned}
&\left\| \int_\Gamma G_1(\cdot, y; k_1)\varphi_1(y)\, ds(y) \right\|_{L^2(\Gamma\setminus B)} \\
&= \sup_{\psi \in L^2_\alpha, \|\psi\|_{L^2_\alpha(\Gamma\setminus B)}=1} \left| \int_{\Gamma\setminus B} \int_\Gamma G_1(x, y; k_1)\varphi_1(y)\, ds(y)\psi(x)\, dx \right| \\
&= \sup_{\psi \in L^2_\alpha, \|\psi\|_{L^2_\alpha(\Gamma\setminus B)}=1} \left| \int_\Gamma \int_{\Gamma\setminus B} G_1(x, y; k_1)\psi(x)\, dx\varphi_1(y)\, ds(y) \right| \\
&\le |\Gamma|^{\frac{1}{q}} \sup_{\psi \in L^2_\alpha, \|\psi\|_{L^2_\alpha(\Gamma\setminus B)}=1} \sup_{y\in\Gamma\setminus B} \big\| G_1(\cdot, y; k_1)\big\|_{L^2_\alpha(\Gamma\setminus B)} \|\psi\|_{L^2_\alpha(\Gamma\setminus B)} \|\varphi_1\|_{L^p_\alpha(\Gamma)} \\
&= |\Gamma|^{\frac{1}{q}} \sup_{y\in\Gamma\setminus B} \big\| G_1(\cdot, y; k_1)\big\|_{L^2_\alpha(\Gamma\setminus B)} \|\varphi_1\|_{L^p_\alpha(\Gamma)} \le C\|\varphi_1\|_{L^p_\alpha(\Gamma)}.
\end{aligned} \tag{2.22}$$

Here, we have used the fact that $G_1(\cdot, y; k_1)$ is smooth on the boundary $\Gamma \setminus B$ in the first inequality. Then we have that $S_{1j} : L^p_\alpha(\Gamma) \to L^2_\alpha(\Gamma \setminus B)$ is uniformly bounded for $j \in \mathbb{N}_+$. Moreover, by using similar arguments as those in the proof of (2.22), it is seen that the operators S_{ij}, K_{ij}, K'_{ij}, and T_{ij} are all uniformly bounded from $L^p_\alpha(\Gamma)$ into $L^2_\alpha(\Gamma \setminus B)$ for $j \in \mathbb{N}_+$, $i = 1, 2$. Also notice that $f_1(j), f_2(j) \in L^2_\alpha(\Gamma \setminus B)$ are uniformly bounded for $j \in \mathbb{N}_+$. This, combined with equation (2.15), gives that the unique solution $(\varphi_1, \varphi_2)^T$ of (2.15) satisfies that $(\varphi_1, \varphi_2)^T \in L^2_\alpha(\Gamma \setminus B) \times L^2_\alpha(\Gamma \setminus B)$. It is noted from (2.14) that the solution u_{2j} of the transmission problem (2.1)–(2.5) can be rewritten in the form of

$$\begin{aligned}
u_{2j}(x) = {}& \int_{\Gamma\setminus B} G_2(x, y; k_2)\varphi_1(y)\, ds(y) + \int_{\Gamma\cap B} G_2(x, y; k_2)\varphi_1(y)\, ds(y) \\
&+ \int_{\Gamma\setminus B} \frac{\partial G_2(x, y; k_2)}{\partial \nu(y)}\varphi_2(y)\, ds(y) + \int_{\Gamma\cap B} \frac{\partial G_2(x, y; k_2)}{\partial \nu(y)}\varphi_2(y)\, ds(y).
\end{aligned} \tag{2.23}$$

Define

$$S_{\Gamma\setminus B}(j)\varphi_1 := \int_{\Gamma\setminus B} G_2(x,y;k_2)\varphi_1(y)\,ds(y).$$

It is easily seen that $S_{\Gamma\setminus B}(j) : H_\alpha^{-\frac{1}{2}}(\Gamma\setminus B) \to H_\alpha^{\frac{1}{2}}(\Gamma\setminus B)$ is uniformly bounded for $j\in\mathbb{N}$. This in combination with the fact that $\varphi_1 \in L^2_\alpha(\Gamma\setminus B)$ implies that $q_{1j}(x) := S_{\Gamma\setminus B}(j)\varphi_1$ satisfies the following Dirichlet problem:

$$\begin{cases} \triangle w + k_2^2 w = 0 & \text{in } \Omega_-\setminus B, \\ w = q_{1j} \in H_\alpha^{\frac{1}{2}}(\tilde{\Gamma}) & \text{on } \tilde{\Gamma}, \\ w(x) = \sum_{n\in\mathbb{Z}^2} w_n^- \exp(i\alpha_n\cdot\widetilde{x} - i\beta_n^- x_3) & x_3 < A_2, \end{cases} \tag{2.24}$$

where $\tilde{\Gamma} = (\Gamma\setminus B)\cup(\partial B\cap\Omega_-)$. Then the well-posedness of the Dirichlet problem (2.24) yields that, for any $h\in\mathbb{R}$, $q_{1j}\in H^1(\Omega_-(h)\setminus\overline{B})$ uniformly for $j\in\mathbb{N}_+$.

We now define

$$q_{2j}(x) := \int_{\Gamma\cap B} G_2(x,y;k_2)\varphi_1(y)\,ds(y).$$

Since the region $\Omega_-\setminus B$ has a positive distance from y_0, it is found that $q_{2j}(x)\in H^1(\Omega_-(h)\setminus\overline{B})$ uniformly for $j\in\mathbb{N}_+$. We further define

$$K_{\Gamma\setminus B}(j)\varphi_2 := \int_{\Gamma\setminus B} \frac{\partial G_2(x,y;k_2)}{\partial\nu(y)}\varphi_2(y)\,ds(y).$$

Obviously, $K_{\Gamma\setminus B}(j) : H_\alpha^{-\frac{1}{2}}(\Gamma\setminus B)\to H_\alpha^{\frac{1}{2}}(\Gamma\setminus B)$ is uniformly bounded for $j\in\mathbb{N}_+$. Then, by the fact that $\varphi_2\in L^2_\alpha(\Gamma\setminus B)$, we obtain that $q_{3j}(x) := K_{\Gamma\setminus B}(j)\varphi_2$ satisfies the Dirichlet problem (2.24), with the boundary data $w = q_{1j}$ replaced by $w = q_{3j}$ on $\tilde{\Gamma}$. Then using similar arguments as those in the proof of $q_{1j}\in H^1_\alpha(\Omega_-(h)\setminus\overline{B})$ yields that $q_{3j}\in H^1_\alpha(\Omega_-(h)\setminus\overline{B})$ uniformly for $j\in\mathbb{N}_+$. We also define

$$q_{4j}(x) := \int_{\Gamma\cap B} \frac{\partial G_2(x,y;k_2)}{\partial\nu(y)}\varphi_2(y)\,ds(y).$$

The uniform boundedness of $q_{4j}\in H^1_\alpha(\Omega_-(h)\setminus\overline{B})$ for $j\in\mathbb{N}_+$ can be concluded from the positive distance between the region $(\Omega_-(h)\setminus\overline{B})$ and y_0. Finally, the desired result (2.20) follows from the discussions below (2.24). The proof of the theorem is thus completed. □

3 Uniqueness of the inverse problem

In this section we mainly focus on the inverse problem of determining the periodic interface by means of the near-field data measured from one side of the periodic interface. To address this issue, we first introduce a mixed-reciprocity relation between the incident plane wave (1.1) and the incident point source (2.6). To accomplish this, we let $\hat{\alpha} := -\alpha$ and consider an incident point source located at $z\in\Omega_+$ taking the form

$$G_1(x,z;k_1) = \frac{i}{8\pi^2}\sum_{n\in\mathbb{Z}^2}\frac{1}{\hat{\beta}_n^+}\exp\bigl(i\hat{\alpha}_n\cdot(\widetilde{x}-\widetilde{z}) + i\hat{\beta}_n^+|x_3 - z_3|\bigr),\quad x\neq z \tag{3.1}$$

with the coefficients $\hat{\alpha}_n, \hat{\beta}_n^+$ defined by α_n, β_n^+ with α replaced by $\hat{\alpha}$, respectively. Then the inverse scattering of the incident point source $G_1(\cdot, z; k_1)$ by the two-layered periodic interface can be formulated as the following $\hat{\alpha}$-quasi-periodic problem:

$$\triangle \hat{v}_1 + k_1^2 \hat{v}_1 = 0 \quad \text{in } \Omega_+ \setminus \{z\}, \tag{3.2}$$

$$\triangle \hat{v}_2 + k_2^2 \hat{v}_2 = 0 \quad \text{in } \Omega_-, \tag{3.3}$$

$$\hat{v}_1 = \hat{v}_2, \qquad \frac{\partial \hat{v}_1}{\partial \nu} = \lambda \frac{\partial \hat{v}_2}{\partial \nu} \quad \text{on } \Gamma, \tag{3.4}$$

$$\hat{v}^s(x) = \sum_{n \in \mathbb{Z}^2} \hat{v}_n^+ \exp\bigl(i\hat{\alpha}_n \cdot \widetilde{x} + i\hat{\beta}_n^+ x_3\bigr), \quad x_3 > A_1, \tag{3.5}$$

$$\hat{v}_2(x) = \sum_{n \in \mathbb{Z}^2} \hat{v}_n^- \exp\bigl(i\hat{\alpha}_n \cdot \widetilde{x} - i\hat{\beta}_n^- x_3\bigr), \quad x_3 < A_2. \tag{3.6}$$

Here, both $\hat{v}_1$ in Ω_+ and $\hat{v}_2$ in Ω_- satisfy the $\hat{\alpha}$-quasi-periodic condition

$$\hat{v}_j(\widetilde{x} + 2n\pi, x_3) = e^{i2\hat{\alpha}\cdot n\pi} \hat{v}_j(\widetilde{x}, x_3), \quad j = 1, 2.$$

Moreover, we write the scattered field $\hat{v}^s(\cdot, z) := \hat{v}_1(\cdot, z) - G_1(\cdot, z; k_1)$ indicates the dependance of the wave field on the location of the point source, and let $v(\cdot; m)$ and $u^s(\cdot; m)$ be the scattered solution to problem (1.2)–(1.6) with respect to the incident wave $u^i(x; m) = \exp(i\alpha_m \cdot \widetilde{x} - i\beta_m^+ x_3), m \in \mathbb{Z}^2$. Therefore, we have the following mixed-reciprocity relation (for a proof, we refer to [34, Lemma 4.1]).

Lemma 3.1 *For $z_0 \in \Omega_+$, let $\hat{v}_n^+(z_0)$ be the Rayleigh coefficients of $\hat{v}_1^s(\cdot; z_0)$. Then it holds that*

$$u_1^s(z_0; m) = -8\pi^2 i\hat{\beta}_{-m}^+ \hat{v}_{-m}^+(z_0) \quad \textit{for all } m \in \mathbb{Z}^2. \tag{3.7}$$

Now we are in a position to present a uniqueness theorem for our inverse problem. The proof mainly depends on the a priori estimates established in Sect. 2 and a construction of a well-posed transmission problem in a sufficiently small domain.

Theorem 3.2 *Let $u_1^s(\cdot; m)$ and $\widetilde{u}_1^s(\cdot; m)$ be the scattered fields corresponding to problem (1.2)–(1.6) with respect to the different bi-periodic interfaces Γ and $\widetilde{\Gamma}$, respectively, induced by the same incident field $u^i(x; m) = \exp(i\alpha_m \cdot \widetilde{x} - i\beta_m^+ x_3), m \in \mathbb{Z}^2$. If $u_1^s(\cdot; m)|_{\Gamma_+(h)} = \widetilde{u}_1^s(\cdot; m)|_{\Gamma_+(h)}$ for all incident fields $u^i(x; m) m \in \mathbb{Z}^2$, then we have $\Gamma = \widetilde{\Gamma}$.*

Proof We shall prove the assertion by contradiction. Assume contrarily that $\Gamma \neq \widetilde{\Gamma}$. Without loss of generality, we can choose a point $z^* \in \Gamma \setminus \widetilde{\Gamma}$ satisfying that $f(\widetilde{z}^*) > \widetilde{f}(\widetilde{z}^*)$ with $z^* = (\widetilde{z}^*, z_3)$. Then we define the sequence

$$z_j := z^* - \frac{\delta}{j} \nu\bigl(z^*\bigr) \quad \text{for } j = 1, 2, \ldots \cdots \tag{3.8}$$

with sufficiently small $\delta > 0$ such that $z_j \in B_{\varepsilon_0}(z^*) \subseteq (\Omega_+ \cap \widetilde{\Omega}_+)$ for all $j \in \mathbb{N}_+$, where $B_{\varepsilon_0}(z^*)$ is a small ball centered at z^* with the radius $\varepsilon_0 > 0$.

Let $(\hat{v}_1(\cdot;z_j),\hat{v}_2(\cdot;z_j))$ and $(\hat{\tilde{v}}_1(\cdot;z_j),\hat{\tilde{v}}_2(\cdot;z_j))$ be the solutions to problem (3.2)–(3.6) corresponding to the same $\hat{\alpha}$-quasi-periodic incident point source $\hat{v}^i = \hat{G}(\cdot,z_j)$ with z_j defined by (3.8). Then one obtains from Lemma 3.1 that

$$u_1^s(z_j;m) = -8\pi^2 i\hat{\beta}^+_{-m}\hat{v}^+_{-m}(z_j) \quad \text{and} \quad \widetilde{u}_1^s(z_j;m) = -8\pi^2 i\hat{\beta}^+_{-m}\hat{\tilde{v}}^+_{-m}(z_j) \tag{3.9}$$

for all $m\in\mathbb{Z}^2$, where $\hat{v}^+_{-m}(z_j)$ and $\hat{\tilde{v}}^+_{-m}(z_j)$ denote the Rayleigh coefficients of the scattered fields $\hat{v}^s(\cdot;z_j)$ and $\hat{\tilde{v}}^s(\cdot;z_j)$, respectively. By the assumption that $u_1^s(\cdot;m)|_{\Gamma_+(h)} = \widetilde{u}_1^s(\cdot;m)|_{\Gamma_+(h)}$ for all incident fields $u^i(x;m)m\in\mathbb{Z}^2$, we arrive at that $\hat{v}^+_{-m}(z_j) = \hat{\tilde{v}}^+_{-m}(z_j)$, $m\in\mathbb{Z}^2$. This in combination with the Rayleigh expansions and the unique continuation principle implies that

$$\hat{v}_1(\cdot;z_j) = \hat{\tilde{v}}_1(\cdot;z_j) \quad \text{in } \Omega^+\cap\widetilde{\Omega}^+ \tag{3.10}$$

for all $j\in\mathbb{N}_+$.

Denote $D_0 := B_{\varepsilon_0}(z^*)\cap\Omega^-$ with sufficiently small $\varepsilon_0>0$ such that $D_0\subseteq(\Omega_-\cap\widetilde{\Omega}_+)$. Let $U_j := \hat{\tilde{v}}_1(\cdot;z_j)$ and $W_j := \hat{v}_2(\cdot;z_j)$, it is observed that (U_j, W_j) satisfies the following modified interior transmission problem:

$$\begin{cases} \triangle U_j - U_j = g_{1,j} & \text{in } D_0,\\ \triangle W_j - W_j = g_{2,j} & \text{in } D_0,\\ U_j - W_j = h_{1,j} & \text{on } \partial D_0,\\ \frac{\partial U_j}{\partial\nu} - \lambda\frac{\partial W_j}{\partial\nu} = h_{2,j} & \text{on } \partial D_0 \end{cases} \tag{3.11}$$

with the right terms and the boundary data

$$g_{1,j} := -\left(k_1^2+1\right)\hat{\tilde{v}}_1(\cdot;z_j), \qquad g_{2,j} := -\left(k_2^2+1\right)\hat{v}_2(\cdot;z_j),$$
$$h_{1,j} := \hat{\tilde{v}}_1(\cdot;z_j) - \hat{v}_2(\cdot;z_j), \qquad h_{2,j} := \frac{\partial\hat{\tilde{v}}_1(\cdot;z_j)}{\partial\nu} - \lambda\frac{\partial\hat{v}_2(\cdot;z_j)}{\partial\nu}.$$

Clearly, one has that $h_{1,j} = h_{2,j}$ on $\partial D_0\cap\Gamma$. Since Z^* has a positive distance from $\widetilde{\Gamma}$, we obtain that $\hat{\tilde{v}}^s(\cdot;z_j)\in H^1(D_0)$ uniformly for all $j\in\mathbb{N}_+$. In view of the fact that $\hat{G}(\cdot,z_j)\in L^2(D_0)$ uniformly for all $j\in\mathbb{N}_+$, it is deduced that $g_{1,j}\in L^2(D_0)$ uniformly for all $j\in\mathbb{N}_+$. The uniform boundedness of $g_{2,j}$ in $L^2(D_0)$ for all $j\in\mathbb{N}_+$ is a direct consequence of Corollary 2.2 in Sect. 2. Moreover, arguing similarly as in [36, Theorem 2.9], one derives from the fact that $h_{1,j} = h_{2,j}$ on $\partial D_0\cap\Gamma$ that $h_{1,j}\in H^{1/2}(\partial D_0)$ and $h_{2,j}\in H^{-1/2}(\partial D_0)$, respectively, uniformly for all $j\in\mathbb{N}_+$. Therefore, by the well-posedness of problem (3.11), we have

$$\left\|\hat{G}(\cdot,z_j)\right\|_{H^1(D_0)} - \left\|\hat{\tilde{v}}^s(\cdot;z_j)\right\|_{H^1(D_0)} \le \left\|\hat{\tilde{v}}(\cdot;z_j)\right\|_{H^1(D_0)} = \|U_j\|_{H^1(D_0)} \le C.$$

However, the above inequality is a contradiction since $\|\hat{\tilde{v}}^s(\cdot;z_j)\|_{H^1(D_0)}$ is uniformly bounded and $\|\hat{G}(\cdot,z_j)\|_{H^1(D_0)}\to\infty$ as $j\to\infty$. Therefore, one concludes that $\Gamma = \widetilde{\Gamma}$. This completes the proof of the theorem. □

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