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New general decay result for a system of two singular nonlocal viscoelastic equations with general source terms and a wide class of relaxation functions

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Abstract

This work is concerned with a system of two singular viscoelastic equations with general source terms and nonlocal boundary conditions. We discuss the stabilization of this system under a very general assumption on the behavior of the relaxation function k_i, namely,

$$k_i'(t) \leq -\xi_i(t)\Psi_i(k_i(t)), \quad i = 1, 2.$$

We establish a new general decay result that improves most of the existing results in the literature related to this system. Our result allows for a wider class of relaxation functions, from which we can recover the exponential and polynomial rates when $k_i(s) = s^p$ and p covers the full admissible range $[1, 2)$.

Keywords: Viscoelasticity; Stability; Nonlocal boundary conditions; Relaxation function; Convex functions

1 Introduction

In this paper, we consider the following system:

$$\begin{cases} u_{tt}(x,t) - \frac{1}{x}(xu_x(x,t))_x + \int_0^t k_1(t-s)\frac{1}{x}(xu_x(x,s))_x\,ds = f_1(u,v), \\ \quad x \in \Omega, t > 0, \\ v_{tt}(x,t) - \frac{1}{x}(xv_x(x,t))_x + \int_0^t k_2(t-s)\frac{1}{x}(xv_x(x,s))_x\,ds = f_1(u,v), \\ \quad x \in \Omega, t > 0, \\ u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad v(x,0) = v_0(x), \\ v_t(x,0) = v_1(x), \quad x \in \Omega, \\ u(L,t) = v(L,t) = 0, \qquad \int_0^L xu(x,t)\,dx = \int_0^L xv(x,t)\,dx = 0, \end{cases} \tag{1}$$

where $\Omega = (0,L)$, $k_i : [0,+\infty) \longrightarrow (0,+\infty)$, $(i = 1,2)$, are non-increasing differentiable functions satisfying more general conditions to be mentioned later and

$$\begin{cases} f_1(u,v) = a|u+v|^{2(r+1)}(u+v) + b|u|^r u|v|^{r+2}, \\ f_2(u,v) = a|u+v|^{2(r+1)}(u+v) + b|v|^r v|u|^{r+2}, \end{cases} \tag{2}$$

where $r > -1$ and $a, b > 0$.

Mixed nonlocal problems for parabolic and hyperbolic partial differential equations have received a great attention during the last few decades. These problems are especially inspired by modern physics and technology. They aim to describe many physical and biological phenomena. For instance, physical phenomena are modeled by initial boundary value problems with nonlocal constraints such as integral boundary conditions, when the data cannot be measured directly on the boundary, but the average value of the solution on the domain is known. Initial boundary value problems for second-order evolution partial differential equations and systems having nonlocal boundary conditions can be encountered in many scientific domains and many engineering models and are widely applied in heat transmission theory, underground water flow, medical science, biological processes, thermoelasticity, chemical reaction diffusion, plasma physics, chemical engineering, heat conduction processes, population dynamics, and control theory. See in this regard the work by Cannon [1], Shi [2], Capasso and Kunisch [3], Cahlon and Shi [4], Ionkin and Moiseev [5], Shi and Shilor [6], Choi and Chan [7], and Ewing and Lin [8]. In early work, most of the research on nonlocal mixed problems was devoted to the classical solutions. Later, mixed problems with integral conditions for both parabolic and hyperbolic equations were studied by Pulkina [9, 10], Yurchuk [11], Kartynnik [12], Mesloub and Bouziani [13], Mesloub and Messaoudi [14, 15], Mesloub [16], and Kamynin [17]. For instance, Said Mesloub and Fatiha Mesloub [18] obtained existence and uniqueness of the solution to the following problem:

$$u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t k(t-s)\frac{1}{x}(xu_x)_x\,ds + au_t = f(t,x,u,u_x), \quad x \in (0,1), t > 0, \tag{3}$$

and proved that the solution blows up for large initial data and decays for sufficiently small initial data. Mesloub and Messaoud [14] considered the following nonlocal singular problem:

$$u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g(t-s)\frac{1}{x}(xu_x)_x\,ds = |u|^p u, \quad x \in (0,a), t > 0, \tag{4}$$

and proved blow-up result for large initial data and decay results of sufficiently small initial data enough for $p > 2$. In [19], Draifia et al. proved a general decay result for the following singular one-dimensional viscoelastic system:

$$\begin{cases} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g_1(t-s)\frac{1}{x}(xu_x(x,s))_x\,ds = |v|^{q+1}|u|^{p-1}u, & \text{in } Q, \\ v_{tt} - \frac{1}{x}(xv_x)_x + \int_0^t g_2(t-s)\frac{1}{x}(xv_x(x,s))_x\,ds = |u|^{p+1}|v|^{q-1}v, & \text{in } Q, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in (0,\alpha), \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad x \in (0,\alpha), \\ u(\alpha,t) = v(\alpha,t) = 0, \qquad \int_0^\alpha xu(x,t)\,dx = \int_0^\alpha xv(x,t)\,dx = 0, \end{cases} \tag{5}$$

where $Q = (0,\alpha) \times (0,t)$ and $p,q > 1$. Piskin and Ekinci [20] studied problem (1) when the Bessel operator has been replaced by a Kirchhoff operator with a degenerate damping terms. They proved the global existence and established a decay rate of solution and also a finite time blow up. Recently, Boulaaras et al. [21] treated problem (1) and proved the existence of a global solution to the problem using the potential-well theory. Moreover, they established a general decay result in which the relaxation functions k_1 and k_2 satisfy

$$k_i'(t) \leq -\xi(t)k_i^p(t), \quad 1 \leq p < \frac{3}{2}. \tag{6}$$

Motivated by the above work, we prove a general stability result of system (1) replacing the condition (6) used in [21] by a more general assumption of the form:

$$k_i'(t) \leq -\xi_i(t)\Psi_i\bigl(k_i(t)\bigr), \quad i = 1,2.$$

Our decay result improves all the existing results in the literature related to this system.

This paper is divided into four sections. In Sect. 2, we state some assumptions needed in our work. Some technical lemmas will be given in Sect. 3. The statement with proof of the main result and some examples will be given in Sect. 4.

2 Preliminaries

In this section, we present some materials needed in the proof of our results. We also state, without proof, the global existence result for problem (1). Let $L_x^p = L_x^p(0,L)$ be the weighted Banach space equipped with the norm

$$\|u\|_{L_x^p} = \left(\int_0^L xu^p\,dx\right)^{\frac{1}{p}}.$$

$L_x^2(0,L)$ is the Hilbert space of square integral functions having the finite norm

$$\|u\|_{L_x^2} = \left(\int_0^L xu^2\,dx\right)^{\frac{1}{2}},$$

$V = V_x^1(0,L)$ is the Hilbert space equipped with the norm

$$\|u\|_V = \bigl(\|u\|_{L_x^2}^2 + \|u_x\|_{L_x^2}^2\bigr)^{\frac{1}{2}}$$

and

$$V_0 = \bigl\{u \in V \text{ such that } u(L) = 0\bigr\}.$$

Lemma 2.1 ([14]) $\forall w \in V_0$, *a Poincaré-type inequality is*

$$\|w\|_{L_x^2}^2 \leq C_p\|w_x\|_{L_x^2}^2. \tag{7}$$

Remark 2.1 Notice that $\|u\|_{V_0} = \|u_x\|_{L_x^2}$ defines an equivalent norm on V_0.

2.1 Assumptions

$(A1)$ $k_i : \mathbb{R}_+ \to \mathbb{R}_+$ (for $i = 1, 2$) are C^1 non-increasing functions satisfying

$$k_i(0) > 0, \quad 1 - \int_0^{+\infty} k_i(s)\,ds =: \ell_i > 0. \tag{8}$$

$(A2)$ There exist non-increasing differentiable functions $\xi_i : [0, +\infty) \longrightarrow (0, +\infty)$ and $\boldsymbol{C}^1$ functions $\Psi_i : [0, +\infty) \longrightarrow [0, +\infty)$ which are linear or strictly increasing and strictly convex $\boldsymbol{C}^2$ functions on $(0, \varepsilon]$, $\varepsilon \le k_i(0)$, with $\Psi_i(0) = \Psi_i'(0) = 0$ such that

$$k_i'(t) \le -\xi_i(t)\Psi_i\bigl(k_i(t)\bigr), \quad \forall t \ge 0 \text{ and for } i = 1, 2. \tag{9}$$

Remark 2.2 The given functions f_1 and f_2 satisfy

$$uf_1(u,v) + vf_2(u,v) = 2(r+2)F(u,v), \quad \forall (u,v) \in \mathbb{R}^2,$$

where

$$2(r+2)F(u,v) = \bigl[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\bigr].$$

Lemma 2.2 (Jensen's inequality) *Let $G : [a,b] \longrightarrow \mathbb{R}$ be a convex function. Assume that the functions $f : (0,L) \longrightarrow [a,b]$ and $h : (0,L) \longrightarrow \mathbb{R}$ are integrable such that $h(x) \ge 0$, for any $x \in (0,L)$ and $\int_0^L h(x)\,dx = k > 0$. Then*

$$G\left(\frac{1}{k}\int_0^L f(x)h(x)\,dx\right) \le \frac{1}{k}\int_0^L G\bigl(f(x)\bigr)h(x)\,dx.$$

Remark 2.3 If Ψ is a strictly increasing, strictly convex C^2 function over $(0, \varepsilon]$ and satisfying $\Psi(0) = \Psi'(0) = 0$, then it has an extension, $\overline{\Psi}$, that is also strictly increasing and strictly convex C^2 over $(0, \infty)$. For example, if $\Psi(\varepsilon) = a$, $\Psi'(\varepsilon) = b$, $\Psi''(\varepsilon) = c$, and for $t > \varepsilon$, $\overline{\Psi}$ can be defined by

$$\overline{\Psi}(t) = \frac{c}{2}t^2 + (b - c\varepsilon)t + \left(a + \frac{c}{2}\varepsilon^2 - b\varepsilon\right). \tag{10}$$

Remark 2.4 Since Ψ_i is strictly convex on $(0, \varepsilon]$ and $\Psi_i(0) = 0$,

$$\Psi_i(\theta z) \le \theta \Psi_i(z), \quad 0 \le \theta \le 1, \forall z \in (0, \varepsilon] \text{ and } i = 1, 2. \tag{11}$$

The modified energy functional E associated to problem (1) is

$$\begin{aligned} E(t) = {} & \frac{1}{2}\|u_t\|_{L_x^2}^2 + \|v_t\|_{L_x^2}^2 + \frac{1}{2}\left(1 - \int_0^t k_1(s)\,ds\right)\|u_x\|_{L_x^2}^2 \\ & + \frac{1}{2}\left(1 - \int_0^t k_2(s)\,ds\right)\|v_x\|_{L_x^2}^2 \\ & + \frac{1}{2}\bigl[(k_1 o u_x)(t) + (k_2 o v_x)(t)\bigr] - \int_0^L xF(u,v)\,dx, \end{aligned} \tag{12}$$

where, for any $w \in L^2_{loc}([0,+\infty); L^2_x(0,L))$ and $i = 1,2$,

$$(k_i \circ w)(t) := \int_0^t k_i(t-s)\big\|w(t)-w(s)\big\|^2_{L^2_x}\,ds.$$

Using (1) with direct differentiation gives

$$\begin{aligned}\frac{dE(t)}{dt} &= -\frac{1}{2}\big(k_1' o u_x\big)(t) - \frac{1}{2}k_1(t)\|u_x\|^2_{L^2_x} - \frac{1}{2}\big(k_2' o v_x\big)(t) - \frac{1}{2}k_2(t)\|v_x\|^2_{L^2_x}\\ &\le \frac{1}{2}\big(k_1' o u_x\big)(t) + \frac{1}{2}\big(k_2' o v_x\big)(t) \le 0.\end{aligned} \tag{13}$$

2.2 Local and global existence

In this subsection, we state, without proof, the local and global existence results for system (1), which can be proved similarly to the ones in [14, 18] and [21].

Theorem 2.1 *Assume that* (*A*1) *and* (*A*2) *hold. If* $(u_0,v_0) \in V_0^2$ *and* $(u_1,v_1) \in (L^2_x)^2$. *Then problem* (1) *has a unique local solution.*

For the global existence, we introduce the following functionals:

$$\begin{aligned}J(t) &= \frac{1}{2}\left(1-\int_0^t k_1(s)\,ds\right)\|u_x\|^2_{L^2_x} + \frac{1}{2}\left(1-\int_0^t k_2(s)\,ds\right)\|v_x\|^2_{L^2_x}\\ &\quad + \frac{1}{2}\big[(k_1 o u_x)(t) + (k_2 o v_x)(t)\big] - \int_0^L x\big[a|u+v|^{2(r+2)} + 2b|uv|^{(r+2)}\big]\,dx\end{aligned} \tag{14}$$

and

$$\begin{aligned}I(t) &= \left(1-\int_0^t k_1(s)\,ds\right)\|u_x\|^2_{L^2_x} + \left(1-\int_0^t k_2(s)\,ds\right)\|v_x\|^2_{L^2_x} + (k_1 o u_x)(t) + (k_2 o v_x)(t)\\ &\quad - 2(r+2)\int_0^L x\big[a|u+v|^{2(r+2)} + 2b|uv|^{(r+2)}\big]\,dx.\end{aligned} \tag{15}$$

We notice that $E(t) = J(t) + \frac{1}{2}\|u_t\|^2_{L^2_x} + \frac{1}{2}\|v_t\|^2_{L^2_x}$.

Lemma 2.3 *Suppose that* (*A*1) *and* (*A*2) *hold. Then, for any* $(u_0,v_0) \in V_0^2$ *and* $(u_1,v_1) \in (L^2_x)^2$ *satisfying*

$$\begin{cases}\beta = \eta\big[\frac{2(r+2)}{r+1}E(0)\big]^{r+1} < 1,\\ I(0) = I(u_0,v_0) > 0,\end{cases} \tag{16}$$

there exists $t_* > 0$ *such that*

$$I(t) = I\big(u(t),v(t)\big) > 0, \quad \forall t \in [0,t_*). \tag{17}$$

Remark 2.5 We can easily deduce from Lemma 2.3 that

$$\ell_1\|u_x\|^2_{L^2_x} + \ell_2\|v_x\|^2_{L^2_x} \le \frac{2(\rho+2)}{\rho+1}E(0), \quad \forall t \ge 0. \tag{18}$$

Theorem 2.2 *Assume that* $(A1)$ *and* $(A2)$ *hold. If* $(u_0, v_0) \in V_0^2$ *and* $(u_1, v_1) \in (L_x^2)^2$ *and satisfies* (16), *then the solution of* (1) *is global and bounded.*

3 Technical lemmas

In this section, we establish several lemmas needed for the proof of our main result.

Lemma 3.1 *There exist two positive constants* c_1 *and* c_2 *such that*

$$\int_0^L x\left|f_i(u,v)\right|^2 dx \le c_i\left(\ell_1\|u_x\|_{L_x^2}^2 + \ell_2\|v_x\|_{L_x^2}^2\right)^{2r+3}, \quad i = 1,2. \tag{19}$$

Proof We prove inequality (19) for f_1 and the same result holds for f_2. It is clear that

$$\begin{aligned}\left|f_1(u,v)\right| &\le C\left(|u+v|^{2r+3} + |u|^{r+1}|v|^{r+2}\right)\\ &\le C\left(|u|^{2r+3} + |v|^{2r+3} + |u|^{r+1}|v|^{r+2}\right).\end{aligned} \tag{20}$$

From (20) and Young's inequality, with

$$q = \frac{2r+3}{r+1}, \qquad q' = \frac{2r+3}{r+2},$$

we get

$$|u|^{r+1}|v|^{r+2} \le c_1|u|^{2r+3} + c_2|v|^{2r+3},$$

hence

$$\left|f_1(u,v)\right| \le C\left[|u|^{2r+3} + |v|^{2r+3}\right].$$

Consequently, by using (7), (12), (13) and the embedding $V_0 \hookrightarrow L^{2(2r+3)}$, we obtain

$$\begin{aligned}\int_0^L x\left|f_1(u,v)\right|^2 dx &\le C\left(\|u\|_{L_x^{2(2r+3)}}^{2(2r+3)} + \|v\|_{L_x^{2(2r+3)}}^{2(2r+3)}\right)\\ &\le c_1\left(\ell_1\|u_x\|_{L_x^2}^2 + \ell_2\|v_x\|_{L_x^2}^2\right)^{2r+3}.\end{aligned}$$

This completes the proof of Lemma 3.1. □

Lemma 3.2 ([22]) *There exist positive constants* d *and* t_0 *such that, for any* $t \in [0, t_0]$, *we have*

$$k_i'(t) \le -dk_i(t), \quad i = 1,2. \tag{21}$$

Lemma 3.3 *If* $(A1)$ *holds. Then, for any* $w \in V_0$, $0 < \alpha < 1$ *and* $i = 1,2$, *we have*

$$\int_0^L x\left(\int_0^t k_i(t-s)\left(w(t)-w(s)\right) ds\right)^2 dx \le C_{\alpha,i}(h_i \circ w)(t), \tag{22}$$

where $C_{\alpha,i} := \int_0^\infty \frac{k_i^2(s)}{\alpha k_i(s) - k_i'(s)}\,ds$ *and* $h_i(t) := \alpha k_i(t) - k_i'(t)$.

Proof The proof of this lemma goes similar to the one in [22]. □

Lemma 3.4 *Under the assumptions* $(A1)$ *and* $(A2)$, *the functional*

$$\Phi(t) := \int_0^L xuu_t\,dx + \int_0^L xvv_t\,dx,$$

satisfies, along with the solution of system (1), *the estimate*

$$\begin{aligned}\Phi'(t) \le{}& \|u_t\|^2_{L^2_x} + \|v_t\|^2_{L^2_x} - \frac{\ell_1}{2}\|u_x\|^2_{L^2_x} - \frac{\ell_2}{2}\|v_x\|^2_{L^2_x} \\ &+ C_{\alpha,1}(h_1 \circ u_x)(t) + C_{\alpha,2}(h_2 \circ v_x)(t) + \int_0^L xF(u,v)\,dx. \end{aligned} \tag{23}$$

Proof Direct differentiation, using (1), yields

$$\begin{aligned}\Phi'(t) ={}& \int_0^L xu_t^2\,dx + \left(1 - \int_0^t k_1(s)\,ds\right)\int_0^L xu_x^2\,dx \\ &+ \int_0^L xu_x \int_0^t k_1(t-s)\big(u_x(s) - u_x(t)\big)\,ds\,dx \\ &+ \int_0^L xv_t^2\,dx + \left(1 - \int_0^t k_1(s)\,ds\right)\int_0^L xv_t^2\,dx \\ &+ \int_0^L xv_x \int_0^t k_2(t-s)\big(v_x(s) - v_x(t)\big)\,ds\,dx \\ &+ \int_0^L x\big(uf_1(u,v) + vf_2(u,v)\big)\,dx. \end{aligned} \tag{24}$$

Using Young's inequality, we obtain, for any $\delta_1, \delta_2 \in (0,1)$,

$$\begin{aligned}\Phi'(t) \le{}& \int_0^L xu_t^2\,dx - \ell_1\int_0^L xu_x^2\,dx + \frac{\delta_1}{2}\int_0^L xu_x^2\,dx \\ &+ \frac{1}{2\delta_1}\int_0^L x\left(\int_0^t k_1(t-s)\big(u_x(s) - u_x(t)\big)\,ds\right)^2 dx \\ &+ \int_0^L xv_t^2\,dx - \ell_2\int_0^L xv_x^2\,dx + \frac{\delta_2}{2}\int_0^L xv_x^2\,dx \\ &+ \frac{1}{2\delta_2}\int_0^L x\left(\int_0^t k_2(t-s)\big(v_x(s) - v_x(t)\big)\,ds\right)^2 dx \\ &+ \int_0^L xF(u,v)\,dx. \end{aligned} \tag{25}$$

Taking $\delta_1 = \ell_1$ and $\delta_2 = \ell_2$ and using Lemma 3.3, we have

$$\begin{aligned}\Phi'(t) \le{}& \int_0^L xu_t^2\,dx - \frac{\ell_1}{2}\int_0^L xu_x^2\,dx + cC_{\alpha,1}(h_1 \circ u_x)(t) \\ &+ \int_0^L xv_t^2\,dx - \frac{\ell_1}{2}\int_0^L xv_x^2\,dx + cC_{\alpha,2}(h_2 \circ v_x)(t) + \int_0^L xF(u,v)\,dx. \end{aligned} \tag{26}$$

□

Let us introduce the functionals

$$\chi_1(t) := -\int_0^L xu_t \int_0^t k_1(t-s)\big(u(t)-u(s)\big)\,ds\,dx$$

and

$$\chi_2(t) := -\int_0^L xv_t \int_0^t k_2(t-s)\big(v(t)-v(s)\big)\,ds\,dx.$$

Lemma 3.5 *Assume that* (*A*1) *and* (*A*2) *hold. Then the functional*

$$\chi(t) := \chi_1(t) + \chi_2(t)$$

satisfies, along with the solution of (1), *the following estimate*:

$$\begin{aligned}\chi'(t) \le{}& -\left(\int_0^t k_1(s)\,ds - \delta\right)\|u_t\|^2_{L^2_x} + c\delta\|u_x\|^2_{L^2_x} + \frac{c}{\delta}(C_{\alpha,1}+1)(h_1\circ u_x)(t)\\ &-\left(\int_0^t k_2(s)\,ds - \delta\right)\|v_t\|^2_{L^2_x} + c\delta\|v_x\|^2_{L^2_x} + \frac{c}{\delta}(C_{\alpha,2}+1)(h_2\circ v_x)(t),\end{aligned} \tag{27}$$

where $0<\delta<1$.

Proof Direct differentiation, using (1), gives

$$\begin{aligned}\chi_1'(t) ={}& -\left(\int_0^t k_1(s)\,ds\right)\int_0^L xu_t^2\\ &+\left(1-\int_0^t k_1(s)\,ds\right)\int_0^L xu_x(t)\int_0^t k_1(t-s)\big(u_x(t)-u_x(s)\big)\,ds\,dx\\ &+\int_0^L x\left(\int_0^t k_1(t-s)\big(u_x(t)-u_x(s)\big)\,ds\right)^2 dx\\ &-\int_0^L xf_1(u,v)\int_0^t k_1(t-s)\big(u(t)-u(s)\big)\,ds\,dx\\ &-\int_0^L xu_t\int_0^t k_1'(t-s)\big(u(t)-u(s)\big)\,ds\,dx.\end{aligned} \tag{28}$$

Using Young's inequality and Lemma 3.3, we get, for any $0<\delta<1$, the following:

$$\begin{aligned}&\left(1-\int_0^t k_1(s)\,ds\right)\int_0^L xu_x(t)\int_0^t k_1(t-s)\big(u_x(t)-u_x(s)\big)\,ds\,dx\\ &\quad+\int_0^L x\left(\int_0^t k_1(t-s)\big|u_x(t)-u_x(s)\big|\,ds\right)^2 dx\\ &\le \delta\int_0^L xu_x^2 + \frac{c}{\delta}\int_0^L x\left(\int_0^t k_1(t-s)\big|u_x(t)-u_x(s)\big|\,ds\right)^2 dx\\ &\le \delta\int_0^L xu_x^2 + \frac{c}{\delta}C_{\alpha,1}(h_1\circ u_x)(t).\end{aligned} \tag{29}$$

Using Young's inequality, (18), (19) and (22), we have

$$\begin{aligned}
&\int_0^L xf_1(u,v)\int_0^t k_1(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad\le \delta\left(\int_0^L x\big|f_1(u,v)\big|^2\,dx\right)+\frac{1}{4\delta}\int_0^L x\left(\int_0^t k_1(t-s)\big(u(t)-u(s)\big)\,ds\right)^2 dx \\
&\quad\le c_1\delta\big(\ell_1\|u_x\|_{L_x^2}^2+\ell_2\|v_x\|_{L_x^2}^2\big)^{2r+3}+\frac{c}{\delta}C_{\alpha,1}(h_1\circ u_x)(t) \\
&\quad\le c_1\delta\left(\frac{2(r+2)}{r+1}E(0)\right)^{2r+1}\big(\ell_1\|u_x\|_{L_x^2}^2+\ell_2\|v_x\|_{L_x^2}^2\big)+\frac{c}{\delta}C_{\alpha,1}(h_1\circ u_x)(t) \\
&\quad\le c\delta\|u_x\|_{L_x^2}^2+c\delta\|v_x\|_{L_x^2}^2+\frac{c}{\delta}C_{\alpha,1}(h_1\circ u_x)(t).
\end{aligned}\tag{30}$$

Also, by applying Young's inequality and Lemma 3.3, we obtain, for any $0<\delta<1$,

$$\begin{aligned}
&-\int_0^L xu_t\int_0^t k_1'(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad=\int_0^L xu_t\int_0^t h_1(t-s)\big(u(t)-u(s)\big)\,ds\,dx-\int_0^L xu_t\int_0^t \alpha k_1(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad\le \delta\|u_t\|_{L_x^2}^2+\frac{1}{2\delta}\left(\int_0^t h_1(s)\,ds\right)(h_1\circ u)(t)+\frac{c}{\delta}C_{\alpha,1}(h_1\circ u)(t) \\
&\quad\le \delta\|u_t\|_{L_x^2}^2+\frac{c}{\delta}(C_{\alpha,1}+1)(h_1\circ u_x)(t).
\end{aligned}\tag{31}$$

Similarly, we have

$$-\int_0^L xv_t\int_0^t k_2'(t-s)\big(v(t)-v(s)\big)\,ds\,dx\le \delta\|v_t\|_{L_x^2}^2+\frac{c}{\delta}(C_{\alpha,2}+1)(h_2\circ v_x)(t).\tag{32}$$

A combination of all the above estimates gives

$$\chi_1'(t)\le -\left(\int_0^t k_1(s)\,ds-\delta\right)\|u_t\|_{L_x^2}^2+c\delta\|u_x\|_{L_x^2}^2+\frac{c}{\delta}(C_{\alpha,1}+1)(h_1\circ u_x)(t).\tag{33}$$

Repeating the same calculations with χ_2, we obtain

$$\chi_2'(t)\le -\left(\int_0^t k_2(s)\,ds-\delta\right)\|v_t\|_{L_x^2}^2+c\delta\|v_x\|_{L_x^2}^2+\frac{c}{\delta}(C_{\alpha,2}+1)(h_2\circ v_x)(t).\tag{34}$$

Therefore, (33) and (34) imply (27), which completes the proof of Lemma 3.5. □

Lemma 3.6 *Assume that* (A1) *and* (A2) *hold. Then the functionals* J_1 *and* J_2 *defined by*

$$J_1(t):=\int_0^L x\int_0^t K_1(t-s)\big|u_x(s)\big|^2\,ds\,dx$$

and

$$J_2(t):=\int_0^L x\int_0^t K_2(t-s)\big|v_x(s)\big|^2\,ds\,dx$$

satisfy, along with the solution of (1), *the estimates*

$$J_1'(t) \le 3(1-\ell)\|u_x\|^2_{L^2_x} - \frac{1}{2}(k_1 \circ u_x)(t), \tag{35}$$

$$J_2'(t) \le 3(1-\ell)\|v_x\|^2_{L^2_x} - \frac{1}{2}(k_2 \circ v_x)(t), \tag{36}$$

where $K_i(t) := \int_t^\infty k_i(s)\,ds$ (*for* $i = 1,2$) *and* $\ell = \min\{\ell_1, \ell_2\}$.

Proof We will prove inequality (35) and the same proof also holds for (36). By Young's inequality and the fact that $K_1'(t) = -k_1(t)$, we see that

$$\begin{aligned} J_1'(t) &= K_1(0)\int_0^L x\big|u_x(t)\big|^2\,dx - \int_0^L x\int_0^t k_1(t-s)\big|u_x(s)\big|^2\,dx \\ &= -\int_0^L x\int_0^t k_1(t-s)\big|u_x(s)-u_x(t)\big|^2\,ds\,dx \\ &\quad - 2\int_0^L xu_x(t).\int_0^t k_1(t-s)\big(u_x(s)-u_x(t)\big)\,ds\,dx + K_1(t)\int_0^L x\big|u_x(t)\big|^2\,dx. \end{aligned}$$

Now,

$$\begin{aligned} &-2\int_0^L xu_x(t).\int_0^t k_1(t-s)\big(u_x(s)-u_x(t)\big)\,ds\,dx \\ &\quad \le 2(1-\ell_1)\int_0^L x\big|u_x(t)\big|^2\,dx + \frac{\int_0^t k_1(s)\,ds}{2(1-\ell_1)}\int_0^L x\int_0^t k_1(t-s)\big|u_x(s)-u_x(t)\big|^2\,ds\,dx. \end{aligned}$$

Using the facts that $K_1(0) = 1-\ell_1$ and $\int_0^t k_1(s)\,ds \le 1-\ell_1$, (35) is established. □

Lemma 3.7 *The functional L defined by*

$$L(t) := NE(t) + N_1\phi(t) + N_2\chi(t)$$

satisfies, for a suitable choice of $N, N_1, N_2 \ge 1$,

$$L(t) \sim E(t) \tag{37}$$

and the estimate

$$\begin{aligned} L'(t) &\le -4(1-\ell)\big(\|u_x\|^2_{L^2_x} + \|v_x\|^2_{L^2_x}\big) - \big(\|u_t\|^2_{L^2_x} + \|v_t\|^2_{L^2_x}\big) \\ &\quad + c\int_0^L xF(u,v)\,dx + \frac{1}{4}\big[(k_1\circ u_x)(t) + (k_2\circ v_x)(t)\big], \quad \forall t \ge t_0, \end{aligned} \tag{38}$$

where t_0 *is introduced in Lemma* 3.2 *and* $\ell = \min\{\ell_1, \ell_2\}$.

Proof It is not difficult to prove that $L(t) \sim E(t)$. To establish (38), we choose $\delta = \frac{\ell}{4cN_2}$ where $\ell = \min\{\ell_1, \ell_2\}$. We set $C_\alpha = \max\{C_{\alpha,1}, C_{\alpha,2}\}$ and $k_0 = \min\{\int_0^{t_0} k_1(s)\,ds, \int_0^{t_0} k_2(s)\,ds\} > 0$. Now using (23) and (28) and recalling the fact that $k_i' = \alpha k_i - h_i$, we obtain, for any $t \ge t_0$,

$$L'(t) \le -\frac{\ell}{4}(2N_1 - 1)\big(\|u_x\|^2_{L^2_x} + \|v_x\|^2_{L^2_x}\big) - \bigg(k_0N_2 - \frac{\ell}{4c} - N_1\bigg)\big(\|u_t\|^2_{L^2_x} + \|v_t\|^2_{L^2_x}\big)$$

$$
\begin{aligned}
& - N_1 \int_0^L xF(u,v)\,dx + \frac{\alpha}{2} N\big[(k_1 \circ u_x)(t) + (k_2 \circ v_x)(t)\big] \\
& - \left[\frac{1}{2}N - \frac{4c^2}{\ell}N_2^2 - C_\alpha\left(\frac{4c^2}{\ell}N_2^2 + cN_1\right)\right]\big[(h_1 \circ u_x)(t) + (h_2 \circ v_x)(t)\big].
\end{aligned}
$$

First, we choose N_1 so large such that $\frac{\ell}{4}(2N_1 - 1) > 4(1-\ell)$.

Then we select N_2 large enough so that $k_0 N_2 - \frac{\ell}{4c} - N_1 > 1$. Now, one can use the Lebesgue dominated convergence theorem with the fact that $\frac{\alpha k_i^2(s)}{\alpha k_i(s) - k_i'(s)} < k_i(s)$, for $i = 1, 2$, to prove that

$$
\lim_{\alpha \to 0^+} \alpha C_\alpha = 0.
$$

Therefore, there exists $\alpha_0 \in (0,1)$ such that if $\alpha < \alpha_0$, then, we get $\alpha C_\alpha < \frac{1}{8[\frac{4c^2}{\ell}N_2^2 + cN_1]}$. Then, by letting $\alpha = \frac{1}{2N} < \alpha_0$, we get $\frac{1}{4}N - \frac{4c^2}{\ell}N_2^2 > 0$. This leads to

$$
\frac{1}{2}N - \frac{4c^2}{\ell}N_2^2 - C_\alpha\left[\frac{4c^2}{\ell}N_2^2 + cN_1\right] > \frac{1}{4}N - \frac{4c^2}{\ell}N_2^2 > 0.
$$

Then, (38) is established. □

4 General decay result

In this section, we state and prove our main result.

Theorem 4.1 *Let $(u_0, v_0) \in V_0^2$ and $(u_1, v_1) \in (L_x^2)^2$ be given and satisfying* (16). *Assume that* $(A1)$ *and* $(A2)$ *hold. If Ψ_1 and Ψ_2* are linear, *then there exist two positive constants λ_1 and λ_2 such that the solution to problem* (1) *satisfies the estimate*

$$
E(t) \le \lambda_2 \exp\left(-\lambda_1 \int_{t_0}^t \xi(s)\,ds\right), \quad \forall t \ge t_0, \tag{39}
$$

where t_0 is introduced in Lemma 3.2 *and* $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$.

Proof Using (21) and (13) we have, for any $t \ge t_0$,

$$
\int_0^{t_0} k_1(s)\big\|u_x(t) - u_x(t-s)\big\|_{L_x^2}^2\,ds + \int_0^{t_0} k_2(s)\big\|v_x(t) - v_x(t-s)\big\|_{L_x^2}^2\,ds \le -cE'(t).
$$

Using this inequality, the estimate (38) becomes, for some $m > 0$ and for any $t \ge t_0$,

$$
\begin{aligned}
L'(t) \le{}& -mE(t) - cE'(t) + c\int_{t_0}^t k_1(s)\big\|u_x(t) - u_x(t-s)\big\|_{L_x^2}^2\,ds \\
& + c\int_{t_0}^t k_2(s)\big\|v_x(t) - v_x(t-s)\big\|_{L_x^2}^2\,ds.
\end{aligned}
$$

Let $\mathcal{L} := L + cE \sim E$, we obtain

$$
\mathcal{L}'(t) \le -mE(t) + c\int_{t_0}^t k_1(s)\big\|u_x(t) - u_x(t-s)\big\|_{L_x^2}^2\,ds
$$

$$+ c\int_{t_0}^{t} k_2(s)\left\|v_x(t)-v_x(t-s)\right\|_{L_x^2}^2\,ds, \quad \forall t \geq t_0. \tag{40}$$

Multiply both sides of (40) by $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$ where ξ is non-increasing function and using $(A2)$ and (13) we get, for any $t \geq t_0$ and $m > 0$, the following:

$$\begin{aligned}
\xi(t)\mathcal{L}'(t) &\leq -m\xi(t)E(t) + c\int_0^t \xi_1(s)k_1(s)\left\|u_x(t)-u_x(t-s)\right\|_{L_x^2}^2\,ds \\
&\quad + c\int_0^t \xi_2(s)k_2(s)\left\|v_x(t)-v_x(t-s)\right\|_{L_x^2}^2\,ds \\
&\leq -m\xi(t)E(t) - c\int_0^t k_1'(s)\left\|u_x(t)-u_x(t-s)\right\|_{L_x^2}^2\,ds \\
&\quad \times c\int_0^t k_2'(s)\left\|v_x(t)-v_x(t-s)\right\|_{L_x^2}^2\,ds \\
&\leq -m\xi(t)E(t) - cE'(t).
\end{aligned}$$

Since ξ is non-increasing, we have

$$(\xi\mathcal{L} + cE)'(t) \leq -m\xi(t)E(t), \quad \forall t \geq t_0.$$

Integrating over (t_0, t) and using the fact that $\xi\mathcal{L} + cE \sim E$, then, for any $\lambda_1, \lambda_2 > 0$, we obtain

$$E(t) \leq \lambda_2 \exp\left(-\lambda_1\int_{t_0}^t \xi(s)\,ds\right), \quad \forall t \geq t_0.$$

□

Theorem 4.2 *Let $(u_0, v_0) \in V_0^2$ and $(u_1, v_1) \in (L_x^2)^2$ be given and satisfying* (16). *Assume that* $(A1)$ *and* $(A2)$ *hold. If Ψ_1 or Ψ_2* is nonlinear, *then there exist two positive constants λ_1 and λ_2 such that the solution to problem* (1) *satisfies the estimate*

$$E(t) \leq \lambda_2\Psi_*^{-1}\left(\lambda_1\int_{t_0}^t \xi(s)\,ds\right), \quad \forall t > t_0, \tag{41}$$

where

$$\Psi_*(t) = \int_t^r \frac{1}{sH(s)}\,ds \quad \text{with } H(t) = \min\left\{\Psi_1'(t), \Psi_2'(t)\right\}.$$

Proof Using Lemmas 3.6 and 3.7, we easily see that

$$\mathcal{L}_1(t) := L(t) + J_1(t) + J_2(t)$$

is nonnegative and, for any $t \geq t_0$, and, for some $C > 0$,

$$\mathcal{L}_1'(t) \leq -cE(t).$$

Therefore, we arrive at

$$\int_0^\infty E(s)\,ds < +\infty. \tag{42}$$

Now, we define the following functionals:

$$I_1(t) := \gamma \int_{t_0}^{t} \|u_x(t) - u_x(t-s)\|_{L_x^2}^2 \, ds, \qquad I_2(t) := \gamma \int_{t_0}^{t} \|v_x(t) - v_x(t-s)\|_{L_x^2}^2 \, ds.$$

Thanks to (42), one can choose $0 < \gamma < 1$ so that

$$I_i(t) < 1, \quad \forall t \geq t_0 \text{ and } i = 1, 2. \tag{43}$$

Without loss of the generality, we assume that $I_i(t) > 0$, for any $t > t_0$; otherwise, we get an exponential decay from (38). We also define the following functionals:

$$\eta_1(t) := -\int_{t_0}^{t} k_1'(s) \|u_x(t) - u_x(t-s)\|_{L_x^2}^2 \, ds, \ \eta_2(t) := -\int_{t_0}^{t} k_2'(s) \|v_x(t) - v_x(t-s)\|_{L_x^2}^2 \, ds$$

and observe that

$$\eta_1(t) + \eta_2(t) \leq -cE'(t), \quad \forall t \geq t_0. \tag{44}$$

Using (2.4), Assumption $(A2)$, inequality (43) and Jensen's inequality, we obtain

$$\begin{aligned}
\eta_1(t) &\leq \frac{1}{\gamma I_1(t)} \int_{t_0}^{t} \gamma I_1(t) \xi_1(s) \Psi_1\big(k_1(s)\big) \|u_x(t) - u_x(t-s)\|_{L_x^2}^2 \, ds \\
&\leq \frac{\xi_1(t)}{\gamma I_1(t)} \int_{t_0}^{t} \gamma \Psi_1\big(I_1(t) k_1(s)\big) \|u_x(t) - u_x(t-s)\|_{L_x^2}^2 \, ds \\
&\leq \frac{\xi_1(t)}{\gamma} \Psi_1\left(\frac{1}{I_1(t)} \int_{t_0}^{t} \gamma I_1(t) k_1(s) \|u_x(t) - u_x(t-s)\|_{L_x^2}^2 \, ds \right) \\
&= \frac{\xi_1(t)}{\gamma} \bar{\Psi}_1\left(\gamma \int_{t_0}^{t} k_1(s) \|u_x(t) - u_x(t-s)\|_{L_x^2}^2 \, ds \right), \quad \forall t \geq t_0,
\end{aligned}$$

where $\bar{\Psi}_1$ is defined in Remark (2.3). Then, we have

$$\int_{t_0}^{t} k_1(s) \|u(t) - u(t-s)\|_{L_x^2}^2 \, ds \leq \frac{1}{\gamma} \bar{\Psi}_1^{-1}\left(\frac{\gamma \eta_1(t)}{\xi_1(t)} \right), \quad t \geq t_0.$$

Similarly, we can have

$$\int_{t_0}^{t} k_2(s) \|v(t) - v(t-s)\|_{L_x^2}^2 \, ds \leq \frac{1}{\gamma} \bar{\Psi}_2^{-1}\left(\frac{\gamma \eta_2(t)}{\xi_2(t)} \right), \quad t \geq t_0.$$

Thus, the estimate (40) becomes

$$F'(t) \leq -mE(t) + c\bar{\Psi}_1^{-1}\left(\frac{\gamma \eta_1(t)}{\xi_1(t)} \right) + c\bar{\Psi}_2^{-1}\left(\frac{\gamma \eta_2(t)}{\xi_2(t)} \right), \quad t \geq t_0. \tag{45}$$

Set $H = \min\{\bar{\Psi}_1', \bar{\Psi}_2'\}$ and define the functional

$$F_1(t) := H\left(\varepsilon_0 \frac{E(t)}{E(0)} \right) F(t) + E(t), \quad \text{for } \varepsilon_0 \in (0, \varepsilon) \text{ and } t \geq t_0.$$

Using the fact that $\bar{\Psi}_i' > 0$, $\bar{\Psi}_i'' > 0$ and $E' \le 0$, we also deduce that $F_1 \sim E$. Further, we get

$$F_1'(t) = \varepsilon_0 \frac{E'(t)}{E(0)} H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) F(t) + H\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) F'(t) + E'(t), \quad \text{for a.e } t \ge t_0.$$

Recalling that $E' \le 0$, then we drop the first and last terms of the above identity. Therefore, by using the estimate (45), we have

$$\begin{aligned} F_1'(t) \le{}& -mE(t)H\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cH\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \bar{\Psi}_1^{-1}\left(\frac{\gamma \eta_1(t)}{\xi_1(t)}\right) \\ &+ cH\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \bar{\Psi}_2^{-1}\left(\frac{\gamma \eta_2(t)}{\xi_2(t)}\right), \quad \text{for a.e } t \ge t_0. \end{aligned} \tag{46}$$

In the sense of Young [23], we let $\bar{\Psi}_i^*$ be the convex conjugate of $\bar{\Psi}_i$ such that

$$\bar{\Psi}_i^*(s) = s\left(\bar{\Psi}_i'\right)^{-1}(s) - \bar{\Psi}_i\left[\left(\bar{\Psi}_i'\right)^{-1}(s)\right], \quad \text{for } i = 1, 2, \tag{47}$$

and it satisfies the following generalized Young inequality:

$$AB_i \le \bar{\Psi}_i^*(A) + \bar{\Psi}_i(B_i), \quad \text{for } i = 1, 2. \tag{48}$$

By letting $A = H(\varepsilon_0 \frac{E(t)}{E(0)})$, $B_i = \bar{\Psi}_i^{-1}(\frac{\gamma \eta_i(t)}{\xi_i(t)})$, for $i = 1, 2$, and combining (46)–(48), we have, for almost every $t \ge t_0$,

$$\begin{aligned} F_1'(t) \le{}& -mE(t)H\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\bar{\Psi}_1^*\left[H\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right] + c\frac{\gamma \eta_1(t)}{\xi_1(t)} \\ &+ c\bar{\Psi}_2^*\left[H\left(\varepsilon \frac{E(t)}{E(0)}\right)\right] + c\frac{\gamma \eta_2(t)}{\xi_2(t)} \\ \le{}& -mE(t)H\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cH\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\left(\bar{\Psi}_1'\right)^{-1}\left[\bar{\Psi}_1'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right] + c\frac{\gamma \eta_1(t)}{\xi_1(t)} \\ &+ cH\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\left(\bar{\Psi}_2'\right)^{-1}\left[\bar{\Psi}_2'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right] + c\frac{\gamma \eta_2(t)}{\xi_2(t)} \\ \le{}& -\left(mE(0) - c\varepsilon_0\right)\frac{E(t)}{E(0)} H\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\left(\frac{\gamma \eta_1(t)}{\xi_1(t)} + \frac{\gamma \eta_2(t)}{\xi_2(t)}\right). \end{aligned}$$

Multiplying the above estimate by $\xi(t) = \min\{\xi_1(t), \xi_2(t)\} > 0$ and using the fact in (44), we get

$$\xi(t)F_1'(t) \le -\left(mE(0) - c\varepsilon_0\right)\xi(t)\frac{E(t)}{E(0)} H\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t), \quad \text{for a.e } t \ge t_0.$$

Select ε_0 small enough so that $k_0 := mE(0) - c\varepsilon_0 > 0$, and we obtain

$$\xi(t)F_1'(t) \le -k_0\xi(t)\frac{E(t)}{E(0)} H\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t), \quad \text{for a.e } t \ge t_0.$$

Let $F_2 = \xi F_1 + cE \sim E$, we have, for some $\alpha_1, \alpha_2 > 0$, the following equivalent inequality:

$$\alpha_1 F_2(t) \le E(t) \le \alpha_2 F_2(t), \quad \forall t \ge t_0. \tag{49}$$

Hence, we have

$$F_2'(t) \le -k_0 \xi(t) \frac{E(t)}{E(0)} H\left(\varepsilon_0 \frac{E(t)}{E(0)}\right), \quad \text{for a.e } t \ge t_0. \tag{50}$$

Now, we set

$$H_0(t) = tH(\varepsilon_0 t), \quad \forall t \in [0,1].$$

Using the fact that $\Psi_i' > 0$ and $\Psi_i'' > 0$ on $(0,r]$ (for $i = 1,2$), we deduce that $H_0, H_0' > 0$ *a.e.* on $(0,1]$. Now, we define the following functional:

$$R(t) := \frac{\alpha_1 F_2(t)}{E(0)}$$

and use (49) and (50) to show that $R \sim E$ and, for some $\beta_1 > 0$,

$$R'(t) \le -\beta_1 \xi(t) H_0\big(R(t)\big), \quad \text{for a.e } t \ge t_0.$$

Integrating over the interval (t_0, t) and using a change of variables, we get

$$\int_{\varepsilon_0 R(t)}^{\varepsilon_0 R(0)} \frac{1}{sH(s)}\, ds \ge \beta_1 \int_{t_0}^{t} \xi(s)\, ds;$$

which gives

$$R(t) \le \frac{1}{\varepsilon_0} \Psi_*^{-1}\left(\beta_1 \int_{t_0}^{t} \xi(s)\, ds\right) \quad \forall t \ge t_0,$$

where $\Psi_*(t) := \int_t^r \frac{1}{sH(s)}\, ds$. Since $R \sim E$, we have, for $\beta_2 > 0$,

$$E(t) \le \beta_2 \Psi_*^{-1}\left(\beta_1 \int_{0}^{t} \xi(s)\, ds\right) \quad \forall t \ge t_0.$$

This completes the proof. □

Example 4.3

(1) Let $k_1(t) = ae^{-\alpha t}$ and $k_2(t) = \frac{b}{(1+t)^q}$, $q > 1$. The constants a and b are chosen so that $(A1)$ is satisfied. Then there exists $C > 0$ such that

$$E(t) \le \frac{C}{(1+t)^q}, \quad \forall t > 0.$$

(2) Let $k_1(t) = \frac{a}{(1+t)^m}$ and $k_2(t) = \frac{b}{(1+t)^n}$ with $m, n > 1$. The constants a and b are chosen so that $(A1)$ is satisfied. Then there exists $C > 0$ such that, for any $t > 0$,

$$E(t) \le \frac{C}{(1+t)^\nu}, \quad \text{with } \nu = \min\{m, n\}.$$

(3) Let $k_1(t) = ae^{-\beta t}$ and $k_2(t) = be^{-(1+t)^q}$ with $0 < q < 1$. The constants a and b are chosen so that $(A1)$ is satisfied. Then there exist positive constants C and α_1 such that

$$E(t) \leq Ce^{-\alpha_1(1+t)^\nu}, \quad \text{for } t \text{ large}.$$

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