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On a system of fractional q-differential inclusions via sum of two multi-term functions on a time scale

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Abstract

Nowadays most researchers have been focused on fractional calculus because it has been proved that fractional derivatives could describe most phenomena better than usual derivations. Numerical parts of fractional calculus such as q-derivations are considered by researchers. In this work, our aim is to review the existence of solution for an m-dimensional system of fractional q-differential inclusions via sum of two multi-term functions under some boundary conditions on the time scale $\mathbb{T}_{t_0} = \{t : t = t_0 q^n\} \cup \{0\}$, where $n \geq 1$, $t_0 \in \mathbb{R}$, and $q \in (0, 1)$. By using the Banach contraction principle and some sufficient conditions, we guarantee the existence of solutions for the system of q-differential inclusions. Also, we provide an example, some algorithms, and a figure to illustrate our main result.

MSC: Primary 34A08; 34B16; secondary 39A13

Keywords: Multi-term function; Sum boundary value condition; Caputo q-derivative; Riemann–Liouville q-derivative; Time scale

1 Introduction

The fractional calculus provides a meaningful generalization for the classical integration and differentiation to any order. Also, the quantum calculus is equivalent to traditional infinitesimal calculus without the notion of limits. It defines q-calculus, where q stands for quantum. Despite the old history of the theories, the investigation of their properties has remained untouched until recent time. After its introduction by Jackson in 1910 [1], many researchers have extended this field (see, for example, [2–8]). It is important that we increase our abilities by investigating complicate fractional differential equations and applications (see [9–24]). One of the methods in this way is working on fractional differential inclusions which are an appropriate extension for fractional differential equations [25–30]. Finally, it is known that fractional difference equations need time scales as a discrete system [31, 32].

In 2013, Ahmad *et al.* studied the fractional inclusion problem ${}^{c}\mathcal{D}^{\beta} k(t) \in T(t, k(t))$ with the integral boundary conditions $k^j(0) - c_i k^j(\delta) = a_i \int_0^1 f_j(r, k(r))\,\mathrm{d}r$ for $j = 0, 1, 2$, where T is a multifunction [26]. In 2014, Ghorbanian *et al.* investigated the existence of solution for the fractional differential inclusion problems

$$ {}^{c}\mathcal{D}^{\sigma_1}[z](t) \in T_1\big(t, z(t), z'(t), z''(t)\big) $$

and

$$ {}^{c}\mathcal{D}^{\sigma_2}[z](t) \in T_2\big(t, z(t), \qquad {}^{c}\mathcal{D}^{\beta_1}[z](t), \dots, {}^{c}\mathcal{D}^{\beta_n}[z](t)\big), $$

with integral boundary value conditions

$$ \begin{cases} z(0) + z(\eta) + z(1) = \int_0^1 f_0(r, z(r))\,\mathrm{d}r, \\ {}^{c}\mathcal{D}^{\zeta}[z](0) + {}^{c}\mathcal{D}^{\zeta}[z](\eta) + {}^{c}\mathcal{D}^{\zeta}[z](1) = \int_0^1 f_1(r, z(r))\,\mathrm{d}r, \\ {}^{c}\mathcal{D}^{\beta}[z](0) + {}^{c}\mathcal{D}^{\beta}[z](\eta) + {}^{c}\mathcal{D}^{\beta}[z](1) = \int_0^1 f_2(r, z(r))\,\mathrm{d}r, \\ z(0) + az(1) = \sum_{i=1}^{n} \mathcal{I}^{\beta_i}[z](\eta), \\ Z'(0) + bz'(1) = \sum_{i=1}^{n} {}^{c}\mathcal{D}^{\beta_i}[z](\eta), \end{cases} $$

where $t \in J$, $2 < \sigma_1 \leq 3$, $1 < \sigma_2 \leq 2$, $0 < \eta, \zeta, \beta_i < 1$, $1 < \beta < 2$, $\sigma_2 - \beta_i \geq 1$, for $1 \leq i \leq n$,

$$ a > \sum_{i=1}^{n} \frac{\eta^{\beta_i+1}}{\Gamma(\beta_i + 2)}, \qquad b > \sum_{i=1}^{n} \frac{\eta^{1-\beta_i}}{\Gamma(2 - \beta_i)}, $$

$n \in \mathbb{N}$, $T_1 : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to P_{\mathrm{cp}}(\mathbb{R})$, $T_2 : J \times \mathbb{R}^{n+1} \to P_{\mathrm{cp}}(\mathbb{R})$ are multifunctions, $f_i : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions for $i = 0, 1, 2$, and $P_{\mathrm{cp}}(\mathbb{R})$ is the set of all compact subsets of $\mathbb{R}$ [28]. In 2015, Agarwal *et al.* reviewed the fractional derivative inclusions ${}^{c}\mathcal{D}^{\beta}[x](t) \in F_1(t, x(t))$ and ${}^{c}\mathcal{D}^{\beta}[x](t) \in F_2(t, x(t), {}^{c}\mathcal{D}^{\zeta}x(t))$ with the boundary value conditions $x(0) = a\int_0^{\nu} x(s)ds$, $x(1) = b\int_0^{\eta} x(s)\,\mathrm{d}s$ and $x(1) + x'(1) = \int_0^{\eta} x(s)\,\mathrm{d}s$, $x(0) = 0$, respectively, where $t \in J$, $\zeta, \eta, \nu \in (0,1)$, $\beta \in (1,2]$ with $\beta - \zeta > 1$, $a, b \in \mathbb{R}$, ${}^{c}\mathcal{D}^{\beta}$ is the Caputo differentiation and $F_1 : J \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$, $F_2 : J \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ are compact-valued multifunctions [25]. In 2019, Samei *et al.* studied the existence of solutions for the hybrid inclusion

$$ \begin{cases} {}^{C}_{H}D^{\alpha}\Big[\frac{x(t) - f(t, x(t), I^{\beta_1}h_1(t, x(t)), I^{\beta_2}h_2(t, x(t)), \dots, I^{\beta_n}h_n(t, x(t)))}{g(t, x(t), I^{\gamma_1}[x](t), I^{\gamma_2}[x](t), \dots, I^{\gamma_m}[x](t))}\Big] \in K(t, x(t)), \\ x(1) = \mu(x), \quad x(e) = \eta(x), \end{cases} $$

where ${}^{C}_{H}\mathcal{D}^{\alpha}$ and ${}^{H}\mathcal{I}^{\alpha}$ denote the Caputo–Hadamard fractional derivative and Hadamard integral of order α, respectively, $t \in J = [1, e]$, $n, m \in \mathbb{N}$, $1 < \alpha \leq 2$, $\beta_i > 0$ for $i = 1, 2, \dots, n$, $\gamma_i > 0$ for $i = 1, 2, \dots, m$, the functions $f : J \times \mathbb{R}^{n+1} \to \mathbb{R}$, $g : J \times \mathbb{R}^{m+1} \to \mathbb{R} - \{0\}$, $h_i : J \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2, \dots, n$, functions μ, η map $C(J, \mathbb{R})$ into $\mathbb{R}$ and the multifunction $K : J \times \mathbb{R} \to P(\mathbb{R})$ satisfies certain conditions [30]. Also, Ntouyas *et al.* studied the boundary value problem of first-order fractional differential equations given by

$$ {}^{c}D^{\beta_1}_{0^+}[f_1](x) = w_1\big(x, f_1(x), f_2(x)\big), \qquad {}^{c}D^{\beta_2}_{0^+}[f_2](x) = w_2\big(x, f_1(x), f_2(x)\big), \quad \big(t \in [0,1]\big) $$

with Riemann–Liouville integral boundary conditions of different order $f_1(0) = c_1 I^{\alpha_1}[f_1](a_1)$ and $f_2(0) = c_2 I^{\alpha_2}[f_2](a_2)$ for $0 < a_1, a_2 < 1$, $\beta_i \in (0,1]$, $\alpha_i, c_i \in \mathbb{R}$ where $i = 1, 2$

[33]. On the other hand, Samei and Ntouyas investigated a multi-term nonlinear fractional q-integro-differential equation

$$ {}^{c}D_q^{\alpha}[x](t) = w\big(t, x(t), (\varphi_1 x)(t), (\varphi_2 x)(t), \qquad {}^{c}D_q^{\beta_1}[x](t), {}^{c}D_q^{\beta_2}[x](t), \dots, {}^{c}D_q^{\beta_n}[x](t)\big) $$

under some boundary conditions [7]. In 2019, Samei *et al.* discussed the fractional hybrid q-differential inclusions

$$ {}^{c}\mathcal{D}_q^{\alpha}\left(\frac{k}{f(t,k,\mathcal{I}_q^{\alpha_1}[k],\dots,\mathcal{I}_q^{\alpha_n}[k])}\right) \in F\big(t,k,\mathcal{I}_q^{\beta_1}[k],\dots,\mathcal{I}_q^{\beta_k}[k]\big) $$

with the boundary conditions $k(0) = k_0$ and $k(1) = k_1$, where $1 < \alpha \le 2$, $q \in (0,1)$, $k_0, k_1 \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1,2,\dots,n$, $\beta_j > 0$ for $j = 1,2,\dots,m$, $n,m \in \mathbb{N}$, ${}^{c}\mathcal{D}_q^{\alpha}$ denotes Caputo type q-derivative of order α, $\mathcal{I}_q^{\beta}$ denotes Riemann–Liouville type q-integral of order β, $f : J \times \mathbb{R}^n \to (0,\infty)$ is continuous, and $F : J \times \mathbb{R}^m \to P(\mathbb{R})$ is a multifunction [34].

Now by mixing the main ideas of the works, we investigate the existence of solutions for a system of fractional q-differential inclusions via sum of two multi-term functions:

$$ \begin{cases} {}^{c}\mathcal{D}_q^{\sigma_1}[k_1](t) \in \mathcal{T}_{1i}(t,k_1(t),\dots,k_m(t),\mathcal{I}_q^{\zeta_1}[k_1](t),\dots,\mathcal{I}_q^{\zeta_m}[k_m(t)]) \\ \qquad\qquad + \mathcal{T}_{21}(t,k_1(t),\dots,k_m(t),{}^{c}\mathcal{D}_q^{\zeta_1}[k_1](t),\dots,{}^{c}\mathcal{D}_q^{\zeta_m}[k_m](t)), \\ {}^{c}\mathcal{D}_q^{\sigma_2}[k_2](t) \in \mathcal{T}_{12}(t,k_1(t),\dots,k_m(t),\mathcal{I}_q^{\zeta_1}[k_1](t),\dots,\mathcal{I}_q^{\zeta_m}[k_m](t)) \\ \qquad\qquad + \mathcal{T}_{22}(t,k_1(t),\dots,k_m(t),{}^{c}\mathcal{D}_q^{\zeta_1}[k_1](t),\dots,{}^{c}\mathcal{D}_q^{\zeta_m}[k_m](t)), \\ \vdots \\ {}^{c}\mathcal{D}_q^{\sigma_m}[k_m](t) \in \mathcal{T}_{1m}(t,[k_1](t),\dots,k_m(t),\mathcal{I}_q^{\zeta_1}[k_1](t),\dots,\mathcal{I}_q^{\zeta_m}[k_m](t)) \\ \qquad\qquad + \mathcal{T}_{2m}(t,k_1(t),\dots,k_m(t),{}^{c}\mathcal{D}_q^{\zeta_1}[k_1](t),\dots,{}^{c}\mathcal{D}_q^{\zeta_m}[k_m](t)), \end{cases} \tag{1} $$

with the boundary conditions

$$ \begin{aligned} &k_i(0) + \mathcal{I}_q^{\zeta_i}[k_i](0) + {}^{c}\mathcal{D}_q^{\zeta_i}[k_i](0) = -k_i(1), \\ &k_i(1) + \mathcal{I}_q^{\zeta_i}[k_i](1) + {}^{c}\mathcal{D}_q^{\zeta_i}[k_i](1) = -k_i(0), \end{aligned} \tag{2} $$

where $1 < \sigma_i \le 2$, $0 < \zeta_i < 1$, $t \in \bar{J} = [0,1]$, and $\mathcal{T}_{1i}, \mathcal{T}_{2i} : \bar{J} \times \mathbb{R}^{2m} \to 2^{\mathbb{R}}$ are some set-valued maps for $i = 1,\dots,m$.

2 Essential preliminaries

In this work, we apply the time scales calculus notation of the book [32]. In fact, we consider the fractional q-calculus on the specific time scale $\mathbb{T}_{t_0} = \{0\} \cup \{t : t = t_0 q^n\}$, where $n \ge 0$, $t_0 \in \mathbb{R}$, and $q \in (0,1)$. Let $a \in \mathbb{R}$. Define $[a]_q = \frac{1-q^a}{1-q}$ [1]. The power function $(x-y)_q^n$ with $n \in \mathbb{N}_0$ is defined by $(x-y)_q^{(0)} = 1$ and $(x-y)_q^{(n)} = \prod_{k=0}^{n-1}(x - yq^k)$ for $n \ge 1$, where x and y are real numbers [31]. Also, for $\alpha \in \mathbb{R}$ and $a \ne 0$, we have $(x-y)_q^{(\alpha)} = x^{\alpha} \prod_{k=0}^{\infty} \frac{x-yq^k}{x-yq^{\alpha+k}}$. If $y = 0$, then it is clear that $x^{(\alpha)} = x^{\alpha}$ [31] (Algorithm 1). The q-gamma function is given by $\Gamma_q(z) = (1-q)^{(z-1)}/(1-q)^{z-1}$, where $z \in \mathbb{R} \backslash \{0,-1,-2,\dots\}$ [1]. Note that $\Gamma_q(z+1) = [z]_q \Gamma_q(z)$. Algorithm 2 shows a pseudo-code description of the technique for estimating q-gamma function of order n. The q-derivative of function f is defined by $(\mathcal{D}_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and

Algorithm 1 The proposed method for calculated $(a-b)_q^{(\alpha)}$

```
function p = powerfunction(a, b, n, q)
%Power Gamma (a-b)^(n)
s=1;
if n==0
 p=1
else
for k=1:n-1
 s=s*(a-b*q^k)/(a- b*q^(alpha+k));
end;
p=a^alpha * s;
end;
end
```

Algorithm 2 The proposed method for calculated $\Gamma_q(x)$

```
function g = qGamma(q, x, n)
%q-Gamma Function
p=1;
for k=0:n
 p=p*(1-q^(k+1))/(1- q^(x+k));
end;
g=p/(1-q)^(x-1);
end
```

Algorithm 3 The proposed method for calculated $(D_q f)(x)$

```
function g = Dq(q, x, n, fun)
if x==0
 g=limit ((fun(x)-fun(q*x))/((1-q)*x),x,0);
else
 g=(fun(x)-fun(q*x))/((1-q)*x);
end;
end
```

$(\mathcal{D}_q f)(0) = \lim_{x\to 0}(\mathcal{D}_q f)(x)$, which is shown in Algorithm 3 [2, 3]. Furthermore, the higher-order q-derivative of a function f is defined by $D_q^n[f](x) = D_q[D_q^{n-1}[f]](x)$ for $n \geq 1$, where $(D_q^0 f)(x) = f(x)$ [2, 3]. The q-integral of a function f is defined by

$$I_q f(x) = \int_0^x f(s)\, \mathrm{d}_q s = x(1-q)\sum_{k=0}^{\infty} q^k f\left(xq^k\right)$$

for $0 \leq x \leq b$, provided the series is absolutely convergent [2, 3]. If x in $[0, T]$, then

$$\begin{aligned}\int_x^T f(r)\, \mathrm{d}_q r &= I_q f(T) - I_q f(x) \\ &= (1-q)\sum_{k=0}^{\infty} q^k\left[Tf\left(Tq^k\right) - xf\left(xq^k\right)\right],\end{aligned}$$

whenever the series exists. The operator I_q^n is given by $(I_q^0 h)(x) = h(x)$ and $(I_q^n h)(x) = (I_q(I_q^{n-1}h))(x)$ for $n \geq 1$ and $h \in C([0,T])$ [2, 3]. It has been proved that $(D_q(I_q h))(x) = h(x)$ and $(I_q(D_q h))(x) = h(x) - h(0)$ whenever h is continuous at $x = 0$ [2, 3].

Algorithm 4 The proposed method for calculated $I_q^{\alpha}[x]$

```
function g = Iq_alpha(q, alpha,  x, n, fun)
p=0;
for k=0:n
    s1=1;
    for i=0:k-1
        s1=s1*(1-q^(alpha+i));
    end
    s2=1;
    for i=0:k-1
        s2=s2*(1-q^(i+1));
    end
    p=p + q^k*s1*eval(subs(fun, t*q^k))/s2;
end;
g=round((t^alpha)* ((1-q)^alpha)* p, 6);
end
```

The fractional Riemann–Liouville type q-integral of the function h on $J = (0,1)$ for $\sigma \geq 0$ is defined by

$$\begin{aligned}\mathcal{I}_q^{\sigma}[h](t) &= \frac{1}{\Gamma_q(\sigma)}\int_0^t (t-qs)^{(\sigma-1)}h(s)\,\mathrm{d}_q s\\ &= t^{\sigma}(1-q)^{\sigma}\sum_{k=0}^{\infty} q^k \frac{\prod_{i=1}^{k-1}(1-q^{\sigma+i})}{\prod_{i=1}^{k-1}(1-q^{i+1})} h\bigl(tq^k\bigr)\end{aligned} \tag{3}$$

and $\mathcal{I}_q^0[h](t) = h(t)$ for $t \in J$ [5]. Also, the Caputo fractional q-derivative of a function w is defined by

$$\begin{aligned}{}^c\mathcal{D}_q^{\sigma}[w](t) &= \mathcal{I}_q^{[\sigma]-\sigma}\bigl[{}^c\mathcal{D}_q^{[\sigma]}[w]\bigr](t)\\ &= \frac{1}{\Gamma_q([\sigma]-\alpha)}\int_0^t (t-qs)^{([\sigma]-\sigma-1)}{}^c\mathcal{D}_q^{[\sigma]}[w](s)\,\mathrm{d}_q s\\ &= \frac{1}{t^{\sigma}(1-q)^{\sigma}}\sum_{k=0}^{\infty} q^k \frac{\prod_{i=1}^{k-1}(1-q^{i-\sigma})}{\prod_{i=1}^{k-1}(1-q^{i+1})} w\bigl(tq^k\bigr),\end{aligned} \tag{4}$$

where $t \in J$ and $\sigma > 0$ [5, 35]. It has been proved that $\mathcal{I}_q^{\beta}(\mathcal{I}_q^{\alpha}[h])(x) = \mathcal{I}_q^{\alpha+\beta}[h](x)$ and $\mathcal{D}_q^{\alpha}(\mathcal{I}_q^{\alpha}[h])(x) = h(x)$, where $\alpha, \beta \geq 0$ [5]. Algorithm 4 shows pseudo-code $\mathcal{I}_q^{\alpha}[h](x)$.

Let $(\mathcal{E}, \rho)$ be a metric space. Denote by $\mathcal{P}(\mathcal{E})$ and $2^{\mathcal{E}}$ the class of all subsets and the class of all nonempty subsets of $\mathcal{E}$, respectively. Thus, $\mathcal{P}_{\mathrm{cl}}(\mathcal{E})$, $\mathcal{P}_{\mathrm{bd}}(\mathcal{E})$, $\mathcal{P}_{\mathrm{cv}}(\mathcal{E})$, and $\mathcal{P}_{\mathrm{cp}}(\mathcal{E})$ denote the class of all closed, bounded, convex, and compact subsets of $\mathcal{E}$, respectively. A mapping $\mathcal{T} : \mathcal{E} \to 2^{\mathcal{E}}$ is called a multifunction on $\mathcal{E}$ and $e \in \mathcal{E}$ is called a fixed point of $\mathcal{T}$ whenever $e \in \mathcal{T}(e)$. A multifunction $\mathcal{T} : \mathcal{E} \to \mathcal{P}_{\mathrm{cl}}(\mathcal{E})$ is lower semi-continuous if, for any open set $\mathcal{O}$ of $\mathcal{E}$, the set

$$\mathcal{T}^{-1}(\mathcal{O}) := \bigl\{z \in \mathcal{E} : \mathcal{T}(z) \cap \mathcal{O} \neq \emptyset\bigr\}$$

is open [29]. If the set $\{z \in \mathcal{X} : \mathcal{T}(z) \subset \mathcal{O}\}$ is open for every open set $\mathcal{O}$ of $\mathcal{E}$, then we say that $\mathcal{T}$ is upper semi-continuous [29]. Also, $\mathcal{T} : \mathcal{E} \to \mathcal{P}_{\mathrm{cp}}(\mathcal{E})$ is called compact if $\overline{\mathcal{T}(\mathcal{B})}$ is compact for each bounded subset $\mathcal{B}$ of $\mathcal{E}$ [29]. A multifunction $\mathcal{T} = [0,1] : \bar{J} \to \mathcal{P}_{\mathrm{cl}}(\mathbb{R})$ is said to be measurable whenever, for each $y \in \mathbb{R}$, the function $t \mapsto \rho(y, \mathcal{T}(t)) = \inf\{|y-z| :$

$z \in \mathcal{T}(t)\}$ is measurable [36]. The Pompeiu–Hausdorff metric $P_\rho : 2^{\mathcal{E}} \times 2^{\mathcal{E}} \to [0,\infty)$ is defined by

$$P_\rho(S,T) = \max\Big\{\sup_{s\in S} \rho(s,T), \sup_{t\in T} \rho(S,t)\Big\}, \tag{5}$$

where $\rho(S,t) = \inf_{s\in S} \rho(s;t)$ [29]. Then $(\mathcal{P}_{b,\mathrm{cl}}(\mathcal{E}), P_\rho)$ is a metric space and $(\mathcal{P}_{\mathrm{cl}}(\mathcal{E}), P_\rho)$ is a generalized metric space [29]. A multifunction $\mathcal{T} : \mathcal{E} \to \mathcal{P}_{\mathrm{cl}}(\mathcal{E})$ is called λ-contraction whenever there exists $\lambda \in (0,1)$ such that $P_\rho(\mathcal{T}(e_1), \Theta(e_2)) \le \lambda\rho(e_1,e_2)$ for all $e_1, e_2 \in \mathcal{E}$.

In 1970, Covitz and Nadler proved that each closed-valued contractive multifunction on a complete metric space has a fixed point [37]. We say that $\mathcal{T} : \overline{J} \times \mathbb{R}^{2m} \to 2^{\mathbb{R}}$ is a Carathéodory multifunction whenever $t \mapsto \mathcal{T}(t, r_1, \dots, r_{2m})$ is measurable for all $r_1, \dots, r_{2m} \in \mathbb{R}$ and $(r_1, \dots, r_{2m}) \mapsto \mathcal{T}(t, r_1, \dots, r_{2m})$ is an upper semi-continuous map for almost all $t \in \overline{J}$ [27, 29, 38]. Also, a Carathéodory multifunction $\mathcal{T} : \overline{J} \times \mathbb{R}^{2m} \to 2^{\mathbb{R}}$ is called L^1-Carathéodory whenever, for each $\eta > 0$, there exists $\Upsilon_\eta \in L^1(\overline{J}, \mathbb{R}^+)$ such that

$$\big\|\mathcal{T}(t, r_1, \dots, r_{2m})\big\| = \sup\big\{|k| : k \in \mathcal{T}(t, r_1, \dots, r_{2m})\big\} \le \Upsilon_\eta(t)$$

for all $|r_1|, \dots, |r_{2m}| \le \eta$ and for almost all $t \in \overline{J}$ [27, 29, 38]. For each i, define the space $E_i = \{k(t) : k(t), {}^c\mathcal{D}_q^{\zeta_i}[k](t) \in \mathcal{A}\}$ endowed with the norm

$$\|k\|_i = \max_{t\in\overline{J}} \big|k(t)\big| + \max_{t\in\overline{J}} \big|{}^c\mathcal{D}_q^{\zeta_i}[k](t)\big|,$$

where $\mathcal{A} = C(\overline{J}, \mathbb{R})$. Also, consider the product space $\mathcal{E} = E_1 \times \cdots \times E_m$ endowed with the norm $\|(k_1, \dots, k_m)\| = \sum_{i=1}^m \|k_i\|_i$. Then $(\mathcal{E}, \|.\|)$ is a Banach space [39]. By using the idea of some works such as [26], define the set of the selections of $\mathcal{S}_{1i}, \mathcal{S}_{2i}$ at k by

$$S_{\mathcal{T}_{1i},k} = \big\{p \in L^1(\overline{J}) : p(t) \in \mathcal{T}_{1i}\big(t, k_1(t), \dots, k_m(t), \mathcal{I}_q^{\zeta_1}[k_1](t), \dots, \mathcal{I}_q^{\zeta_m}[k_m](t)\big)\big\},$$
$$S_{\mathcal{T}_{2i},k} = \big\{p \in L^1(\overline{J}) : p(t) \in \mathcal{T}_{2i}\big(t, k_1(t), \dots, k_m(t), {}^c\mathcal{D}_q^{\zeta_1}[k_1](t), \dots, {}^c\mathcal{D}_q^{\zeta_m}[k_m](t)\big)\big\}$$

for all $t \in \overline{J}$, $k = (k_1, \dots, k_m) \in \mathcal{E}$ and $1 \le i \le m$. One can check that $S_{\mathcal{T}_{1i},k} \ne \emptyset$ for all $k \in \mathcal{E}$ whenever $\dim \mathcal{E} < \infty$ [40]. We need the following results.

Lemma 1 ([38]) *If $\mathcal{T} : \mathcal{E} \to \mathcal{P}_{\mathrm{cl}}(\mathcal{F})$ is upper semicontinuous, then* $\mathrm{Gr}(\mathcal{T})$ *is a closed subset of $\mathcal{E} \times \mathcal{F}$. Conversely, if $\mathcal{T}$ is completely continuous and has a closed graph, then it is upper semicontinuous.*

Lemma 2 ([40]) *Let $\mathcal{E}$ be a Banach space, $\mathcal{T} : \overline{J} \times \mathcal{E} \to \mathcal{P}_{\mathrm{cp,cv}}(\mathcal{E})$ be an L^1-Carathéodory multifunction, and $\mathscr{L}$ be a linear continuous mapping from $L^1(\overline{J}, \mathcal{E})$ to $C(\overline{J}, \mathcal{E})$. Then the operator $\mathscr{L} o S_{\mathcal{T}} : C(\overline{J}, \mathcal{E}) \to \overline{P}_{\mathrm{cp,cv}}(C(\overline{J}), \mathcal{E})$ defined by $(\mathscr{L} o S_{\mathcal{T}})(k) = \mathscr{L}(S_{\mathcal{T},k})$ is a closed graph operator in $C(\overline{J}, \mathcal{E}) \times C(\overline{J}, \mathcal{E})$.*

Lemma 3 ([41]) *Let $\mathcal{E}$ be a Banach space, $\mathcal{F} \in \mathcal{P}_{\mathrm{bd,cl,cv}}(\mathcal{E})$ and $\mathcal{M}, \mathcal{N} : \mathcal{F} \to \mathcal{P}_{\mathrm{cp,cv}}(\mathcal{E})$ be two multi-valued operators. If $\mathcal{M}(k) + \mathcal{N}(k) \subset \mathcal{F}$ for all $k \in \mathcal{F}m$, $\mathcal{M}$ is a contraction and $\mathcal{N}$ is upper semicontinuous and compact, then there exists $k \in \mathcal{F}$ such that $k \in \mathcal{M}(k) + \mathcal{N}(k)$.*

3 Main results

Now, we are ready to provide our main results.

Lemma 4 *Let $z \in \mathcal{A}$, $\sigma \in (1,2]$ and $\zeta \in (0,1)$ with $\sigma - \zeta > 1$. Then the unique solution of the fractional problem ${}^{c}\mathcal{D}_q^{\sigma}[k](t) = z(t)$ with the boundary value conditions*

$$\begin{aligned} &k(0) + \mathcal{I}_q^{\zeta}[k](0) + {}^{c}\mathcal{D}_q^{\zeta}[k](0) = -k(1), \\ &k(1) + \mathcal{I}_q^{\zeta}[k](1) + {}^{c}\mathcal{D}_q^{\zeta}[k](1) = -k(0), \end{aligned}$$

is given by

$$\begin{aligned} k(t) &= \mathcal{I}_q^{\sigma}[z](t) + A_1(t,q,\zeta)\mathcal{I}_q^{\sigma}[z](1) \\ &\quad + A_2(t,q,\zeta)\mathcal{I}_q^{\sigma+\zeta}[z](1) + A_3(t,q,\zeta)\mathcal{I}_q^{\sigma-\zeta}[z](1) \\ &= \int_0^1 G_q(t,r,\sigma,\zeta)z(r)\,\mathrm{d}_q r, \end{aligned}$$

where

$$\begin{aligned} A_1(t,q,\zeta) &= \Sigma\bigl[t\Gamma_q(\zeta+2)\Gamma_q(2-\zeta) + \Gamma_q(\zeta+1)\Gamma_q(\zeta+2) + \Gamma_q(\zeta+1)\Gamma_q(2-\zeta)\bigr], \\ A_2(t,q,\zeta) &= \Sigma\bigl[(2t-1)\Gamma_q(\zeta+1)\Gamma_q(\zeta+2)\Gamma_q(2-\zeta)\bigr], \\ A_3(t,q,\zeta) &= \Sigma\bigl[(2t-1)\Gamma_q(\zeta+1)\Gamma_q(\zeta+2)\Gamma_q(2-\zeta)\bigr], \\ \Sigma &= \bigl[\Gamma_q(\zeta+2)\Gamma_q(2-\zeta) - 2\Gamma_q(\zeta+1)\Gamma_q(2+\zeta) - 2\Gamma_q(\zeta+1)\Gamma_q(2-\zeta)\bigr]^{-1} \end{aligned} \tag{6}$$

and

$$G_q(t,r,\sigma,\zeta) = \begin{cases} \frac{(1-qr)^{(\sigma-1)}}{\Gamma_q(\sigma)}[1 + A_1(t,q,\zeta)] \\ \quad + \frac{(1-qr)^{(\sigma+\zeta-1)}}{\Gamma_q(\sigma+\zeta)}A_2(t,q,\zeta) \\ \quad + \frac{(1-qr)^{(\sigma-\zeta-1)}}{\Gamma_q(\sigma-\zeta)}A_3(t,q,\zeta), & 0 < r < t < 1, \\ \frac{(1-qr)^{(\sigma-1)}}{\Gamma_q(\sigma)}A_1(t,q,\zeta) \\ \quad + \frac{(1-qr)^{(\sigma+\zeta-1)}}{\Gamma_q(\sigma+\zeta)}A_2(t,q,\zeta) \\ \quad + \frac{(1-qr)^{(\sigma-\zeta-1)}}{\Gamma_q(\sigma-\zeta)}A_3(t,q,\zeta), & 0 < t < r < 1. \end{cases}$$

Proof It is known that the general solution of the equation ${}^{c}\mathcal{D}_q^{\sigma}[k](t) = z(t)$ is given by

$$k(t) = \mathcal{I}_q^{\sigma}[z](t) + d_0 + d_1 t = \frac{1}{\Gamma_q(\sigma)}\int_0^t (t-qs)^{(\sigma-1)}z(r)\,\mathrm{d}_q r + d_0 + d_1 t,$$

where d_0, d_1 are real constants and $t \in \overline{J}$ (see [42]). Thus,

$$\begin{aligned} {}^{c}\mathcal{D}_q^{\zeta}[k](t) &= \mathcal{I}_q^{\sigma-\zeta}[z](t) + \frac{t^{1-\zeta}d_1}{\Gamma_q(2-\zeta)} \\ &= \frac{1}{\Gamma_q(\sigma-\zeta)}\int_0^t (t-qr)^{(\sigma-\zeta-1)}z(r)\,\mathrm{d}_q r + \frac{t^{1-\zeta}d_1}{\Gamma_q(2-\zeta)} \end{aligned}$$

and

$$\mathcal{I}_q^{\zeta}[k](t) = \frac{1}{\Gamma_q(\sigma+\zeta)}\int_0^t (t-qr)^{(\sigma+\zeta-1)} z(r)\,\mathrm{d}_q r + \frac{d_0 t^{\zeta}}{\Gamma_q(\zeta+1)} + \frac{d_1 t^{\zeta+1}}{\Gamma_q(\zeta+2)}.$$

Hence, we get $k(0) = -d_0 - d_1 - \frac{1}{\Gamma_q(\sigma)}\int_0^1 (1-qr)^{\sigma-1} z(r)\,\mathrm{d}_q r$ and

$$\begin{aligned} &k(1) + {}^c\mathcal{D}_q^{\zeta}[k](1) + \mathcal{I}_q^{\zeta}[k](1) \\ &\quad= \frac{1}{\Gamma_q(\sigma)}\int_0^1 (1-qr)^{(\sigma-1)} z(r)\,\mathrm{d}_q r \\ &\qquad+ \frac{1}{\Gamma_q(\sigma+\zeta)}\int_0^1 (1-qr)^{(\sigma+\zeta-1)} z(r)\,\mathrm{d}_q r \\ &\qquad+ \frac{1}{\Gamma_q(\sigma-\zeta)}\int_0^1 (1-qr)^{(\sigma-\zeta-1)} z(r)\,\mathrm{d}_q r \\ &\qquad+ d_0\left[\frac{1+\Gamma_q(\zeta+1)}{\Gamma_q(\zeta+1)}\right] \\ &\qquad+ d_1\left[\frac{\Gamma_q(\zeta+2)\Gamma_q(2-\zeta)+\Gamma_q(\zeta+2)+\Gamma_q(2-\zeta)}{\Gamma_q(\zeta+2)\Gamma_q(2-\zeta)}\right]. \end{aligned}$$

By using the boundary conditions, we obtain

$$\begin{aligned} 2d_0 + d_1 &= -\frac{1}{\Gamma_q(\sigma)}\int_0^1 (1-qr)^{(\sigma-1)} z(r)\,\mathrm{d}_q r \\ &\quad+ d_0\left[\frac{1+2\Gamma_q(\zeta+1)}{\Gamma_q(\zeta+1)}\right] \\ &\quad+ d_1\left[\frac{\Gamma_q(\zeta+2)\Gamma_q(2-\zeta)+\Gamma_q(\zeta+2)+\Gamma_q(2-\zeta)}{\Gamma_q(\zeta+2)\Gamma_q(2-\zeta)}\right] \\ &= -\frac{1}{\Gamma_q(\sigma)}\int_0^1 (1-qr)^{(\sigma-1)} z(r)\,\mathrm{d}_q r \\ &\quad- \frac{1}{\Gamma_q(\sigma-\zeta)}\int_0^1 (1-qr)^{(\sigma-\zeta-1)} z(r)\,\mathrm{d}_q r \\ &\quad- \frac{1}{\Gamma_q(\sigma+\zeta)}\int_0^1 (1-qr)^{(\sigma+\zeta-1)} z(r)\,\mathrm{d}_q r. \end{aligned}$$

Thus,

$$\begin{aligned} d_0 &= \Sigma\big[\Gamma_q(\zeta+1)\big(\Gamma_q(\zeta+2)+\Gamma_q(2-\zeta)\big)\big]\mathcal{I}_q^{\alpha}[z](1) \\ &\quad- \Sigma\Gamma_q(\zeta+1)\Gamma_q(\zeta+2)\Gamma_q(2-\zeta)\mathcal{I}_q^{\sigma+\zeta}[z](1) \\ &\quad- \Sigma\Gamma_q(\zeta+1)\Gamma_q(\zeta+2)\Gamma_q(2-\zeta)\mathcal{I}_q^{\sigma-\zeta}[z](1) \end{aligned}$$

and

$$d_1 = \Sigma\Gamma_q(\zeta+2)\Gamma_q(2-\zeta))\mathcal{I}_q^{\sigma}[z](1)$$

$$+ \Sigma 2\Gamma_q(\zeta+1)\Gamma_q(\zeta+2)\Gamma_q(2-\zeta)\mathcal{I}_q^{\sigma+\zeta}[z](1)$$
$$+ \Sigma 2\Gamma_q(\zeta+1)\Gamma_q(\zeta+2)\Gamma_q(2-\zeta)\mathcal{I}_q^{\sigma-\zeta}[z](1).$$

Hence,

$$\begin{aligned} k(t) &= \frac{1}{\Gamma_q(\sigma)}\int_0^t (t-qr)^{(\sigma-1)}z(r)\,\mathrm{d}_q r + A_1(t,q,\zeta)\mathcal{I}_q^{\sigma}[z](1) \\ &\quad + A_2(t,q,\zeta)\mathcal{I}_q^{\sigma+\zeta}[z](1) + A_3(t,q,\zeta)\mathcal{I}_q^{\sigma-\zeta}[z](1) \\ &= \int_0^1 G_q(t,r,\sigma,\zeta)z(r)\,\mathrm{d}_q r. \end{aligned}$$

The converse part concludes with some straight calculation. This completes the proof. □

Definition 5 A function $(k_1,k_2,\dots,k_m)\in\prod_{i=1}^m \mathrm{AC}^1(\overline{J})$ is a solution for the system of fractional inclusions whenever it satisfies the boundary conditions and there exists a function $(z_1,z_2,\dots,z_m)$, $(z_1',z_2',\dots,z_m')\in\prod_{i=1}^m L^1(\overline{J})$ such that

$$\begin{aligned} z_i(t) &\in \mathcal{T}_{1i}\bigl(t,k_1(t),\dots,k_m(t),\mathcal{I}_q^{\zeta_1}[k_1](t),\dots,\mathcal{I}_q^{\zeta_m}[k_m](t)\bigr), \\ z_i'(t) &\in \mathcal{T}_{2i}\bigl(t,k_1(t),\dots,k_m(t),{}^c\mathcal{D}_q^{\zeta_1}[k_1](t),\dots,{}^c\mathcal{D}_q^{\zeta_m}[k_m](t)\bigr) \end{aligned}$$

and

$$\begin{aligned} k_i(t) &= \frac{1}{\Gamma_q(\sigma_i)}\int_0^t (t-qr)^{(\sigma_i-1)}z_i(r)\,\mathrm{d}_q r + A_1(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i}[z_i](1) \\ &\quad + A_2(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i+\zeta_i}[z_i](1) + A_3(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i-\zeta_i}[z_i](1) \\ &\quad + \frac{1}{\Gamma_q(\sigma_i)}\int_0^t (t-qr)^{(\sigma_i-1)}z_i'(r)\,\mathrm{d}_q r + A_1(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i}\bigl[z_i'\bigr](1) \\ &\quad + A_2(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i+\zeta_i}\bigl[z_i'\bigr](1) + A_3(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i-\zeta_i}\bigl[z_i'\bigr](1) \\ &= \int_0^1 G_q(t,r,\sigma_i,\zeta_i)z_i(r)\,\mathrm{d}_q r + \int_0^1 G_q(t,r,\sigma_i,\zeta_i)z_i'(r)\,\mathrm{d}_q r \end{aligned}$$

for $t\in\overline{J}$ and $1\le i\le m$.

Theorem 6 *Let $\mathcal{T}_{1i}:\overline{J}\times\mathbb{R}^{2m}\to\mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R})$ be a set-valued map and, for each $1\le i\le m$, $\mathcal{T}_{2i}:\overline{J}\times\mathbb{R}^{2m}\to\mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R})$ be a Caratheodory multifunction. Assume that there exist continuous functions $h_{1i},h_{2i},\gamma_i:\overline{J}\to(0,\infty)$ $(i=1,\dots,m)$ such that $t\vdash\mathcal{T}_{1i}(t,u_1,\dots,u_m,v_1,\dots,v_m)$ is measurable,*

$$\begin{aligned} &\bigl\|\mathcal{T}_{1i}\bigl(t,k_1(t),\dots,k_m(t),\mathcal{I}_q^{\zeta_1}[k_1](t),\dots,\mathcal{I}_q^{\zeta_m}[k_m](t)\bigr)\bigr\| \\ &\quad = \sup\bigl\{|z| : v\in\mathcal{T}_{1i}\bigl(t,k_1(t),\dots,k_m(t),\mathcal{I}_q^{\zeta_1}[k_1](t),\dots,\mathcal{I}_q^{\zeta_m}[k_m](t)\bigr)\bigr\} \\ &\quad \le h_{1i}(t), \\ &\bigl\|\mathcal{T}_{2i}\bigl(t,k_1(t),\dots,k_m(t),{}^c\mathcal{D}_q^{\zeta_1}[k_1](t),\dots,{}^c\mathcal{D}_q^{\zeta_m}[k_m](t)\bigr)\bigr\| \end{aligned}$$

$$= \sup\bigl\{|z| : z \in \mathcal{T}_{2i}\bigl(t, k_1(t), \ldots, k_m(t), {}^c\mathcal{D}_q^{\zeta_1}[k_1](t), \ldots, {}^c\mathcal{D}_q^{\zeta_m}[k_m](t)\bigr)\bigr\} \\ \le h_{2i}(t),$$

and

$$P_\rho\bigl(\mathcal{T}_{1i}(t, u_1, \ldots, u_k, v_1, \ldots, v_m), \mathcal{T}_{1i}\bigl(t, u'_1, \ldots, u'_k, v'_1, \ldots, v'_m\bigr)\bigr) \\ \le \sum_{i=1}^m \gamma_i(t)\bigl(\bigl|u_i - u'_i\bigr| + \bigl|v_i - v'_i\bigr|\bigr)$$

for all $t \in \overline{J}$, $(k_1, \ldots, k_m) \in \mathcal{E}$, u_i, v_i, u'_i *and* $v'_i \in \mathbb{R}$ *and* $1 \le i \le m$. *If*

$$\Delta = \sum_{i=1}^m \|\gamma_i\|_\infty \biggl(\frac{1 + \Gamma_q(\zeta_i + 1)}{\Gamma_q(\zeta_i + 1)}\biggr)(\Lambda_{1i} + \Lambda_{2i}) < 1,$$

then the system of fractional inclusions has a solution, where

$$\|\gamma_i\|_\infty = \max_{t \in \overline{J}}\bigl|\gamma_i(t)\bigr|$$

and

$$\begin{aligned} \Lambda_{1i} = {}& \frac{1}{\Gamma_q(\sigma_i + 1)}\bigl(1 + |\Sigma_i|\bigl[\Gamma_q(\zeta_i + 2)\Gamma_q(2 - \zeta_i) \\ &+ \Gamma_q(\zeta_i + 1)\Gamma_q(\zeta_i + 2) + \Gamma_q(\zeta_i + 1)\Gamma_q(2 - \zeta_i)\bigr]\bigr) \\ &+ \frac{|\Sigma_i|\Gamma_q(\zeta_i + 1)\Gamma_q(\zeta_i + 2)\Gamma_q(2 - \zeta_i)}{\Gamma_q(\sigma_i + \zeta_i + 1)} \\ &+ \frac{|\Sigma_i|\Gamma_q(\zeta_i + 1)\Gamma_q(\zeta_i + 2)\Gamma_q(2 - \zeta_i)}{\Gamma_q(\sigma_i - \zeta_i + 1)}, \end{aligned} \tag{7}$$

$$\begin{aligned} \Lambda_{2i} = {}& \frac{1}{\Gamma_q(\sigma_i - \zeta_i + 1)} + \frac{|\Sigma_i|\Gamma_q(\zeta_i + 2)}{\Gamma_q(\sigma_i + 1)} \\ &+ \frac{2|\Sigma_i|(\Gamma_q(\zeta_i + 1) + \Gamma_q(\zeta_i + 2))}{\Gamma_q(\sigma_i + \zeta_i + 1)} + \frac{|\Sigma_i|(2\Gamma_q(\zeta_i + 1) + \Gamma_q(\zeta_i + 2))}{\Gamma_q(\sigma_i - \zeta_i + 1)} \end{aligned} \tag{8}$$

for all $1 \le i \le m$.

Proof Consider the subset $\mathcal{F} = \{(k_1, \ldots, k_m) \in \mathcal{E} : \|(k_1, \ldots, k_m)\| \le M\}$ of $\mathcal{E}$, where

$$M = \sum_{i=1}^m \bigl(\|h_{1i}\|_\infty + \|h_{2i}\|_\infty\bigr)(\Lambda_{1i} + \Lambda_{2i}).$$

It is easy to see that $\mathcal{F}$ is a closed, bounded, and convex subset of the Banach space $\mathcal{E}$. Now, define the multi-valued operators $\mathcal{M}, \mathcal{N} : \mathcal{F} \to \mathcal{P}(\mathcal{E})$ by

$$\mathcal{M}(k_1, \ldots, k_m) = \begin{pmatrix} M_1(k_1, \ldots, k_m) \\ M_2(k_1, \ldots, k_m) \\ \vdots \\ M_m(k_1, \ldots, k_m) \end{pmatrix},$$

$$\mathcal{N}(k_1,\dots,k_m) = \begin{pmatrix} N_1(k_1,\dots,k_m) \\ N_2(k_1,\dots,k_m) \\ \vdots \\ N_m(k_1,\dots,k_m) \end{pmatrix},$$

where the multifunctions $M_i(k_1,\dots,k_m)$ and $N_i(k_1,\dots,k_m)$ are the set of all $\theta \in \mathcal{E}_i$ with the feature that there exist $z \in S_{\mathcal{T}_{1i},(k_1,\dots,k_m)}$ and $\theta \in S_{\mathcal{T}_{2i},(k_1,\dots,k_m)}$, respectively, such that

$$\theta(t) = \int_0^1 G_q(t,r,\sigma_i,\zeta_i)z(r)\,\mathrm{d}_qs$$

for all $t \in \bar{J}$ and $1 \le i \le m$. Thus, the system of fractional q-differential inclusions is equivalent to the inclusion problem $k \in \mathcal{M}(k) + \mathcal{N}(k)$. We show that the set-valued maps $\mathcal{M}$ and $\mathcal{N}$ satisfy the conditions of Lemma 3 on $\mathcal{F}$. First, we show that $\mathcal{M}$ is compact-valued on $\mathcal{F}$. Note that $M_i = \mathscr{L}_i \circ S_{\mathcal{T}_{1i}}$, where $\mathscr{L}_i$ is the continuous linear operator on $L^1(\bar{J},\mathbb{R})$ into E_i defined by $\mathscr{L}_i[z](t) = \int_0^1 G_q(t,r,\sigma_i,\zeta_i)z(r)\,\mathrm{d}_qs$. Let $(k_1,\dots,k_m) \in \mathcal{F}$ and $\{z_n\}$ be a sequence in $S_{\mathcal{T}_{1i},(k_1,\dots,k_m)}$. Then, by the definition of $S_{\mathcal{T}_{1i},(k_1,\dots,k_m)}$, we have

$$z_n(t) \in \mathcal{T}_{1i}\bigl(t,k_1(t),\dots,k_m(t),\mathcal{I}_q^{\zeta_1}[k_1](t),\dots,\mathcal{I}_q^{\zeta_m}[k_m](t)\bigr)$$

for almost $t \in \bar{J}$. Since $\mathcal{T}_{1i}(t,k_1(t),\dots,k_m(t),\mathcal{I}_q^{\zeta_1}[k_1](t),\dots,\mathcal{I}_q^{\zeta_m}[k_m](t))$ is compact for all $t \in \bar{J}$, there is a convergent subsequence of $\{z_n(t)\}$, call it again $\{z_n(t)\}$), that converges in measure to some $z(t) \in S_{\mathcal{T}_{1i},(k_1,\dots,k_m)}$ for almost all $t \in \bar{J}$. Since $\mathscr{L}_i$ is continuous, we conclude that $\mathscr{L}_i[z_n](t) \to \mathscr{L}_i[z](t)$ pointwise on $\bar{J}$. In order to show that the convergence is uniform, we have to show that $\{\mathscr{L}_i[z_n]\}$ is an equicontinuous sequence. Let $t_1 < t \in \bar{J}$. Then we have

$$\begin{aligned}
&\bigl|\mathscr{L}_i[z_n](t) - \mathscr{L}_i[z_n](t_1)\bigr| \\
&\quad\le \int_0^1 \bigl|G_q(t,r,\sigma_i,\zeta_i) - G_q(t_1,r,\sigma_i,\zeta_i)\bigr|\bigl|z_n(r)\bigr|\,\mathrm{d}_qr \\
&\quad\le \|h_{1i}\|_\infty\biggl[\frac{1}{\Gamma_q(\sigma_i+1)}\bigl(t^{\sigma_i} - t_1^{\sigma_i} + |\Sigma_i|(t-t_1)\Gamma_q(\zeta_i+2)\Gamma_q(\zeta_i+1)\bigr) \\
&\qquad + \frac{2|\Sigma_i|(t-t_1)\Gamma_q(\zeta_i+2)\Gamma_q(\zeta_i+1)\Gamma_q(2-\zeta_i)}{\Gamma_q(\sigma_i+\zeta_i+1)} \\
&\qquad + \frac{2|\Sigma_i|(t-t_1)\Gamma_q(\zeta_i+2)\Gamma_q(\zeta_i+1)\Gamma_q(2-\zeta_i)}{\Gamma_q(\sigma_i-\zeta_i+1)}\biggr]
\end{aligned}$$

and

$$\begin{aligned}
&\bigl|{}^c\mathcal{D}_q^{\zeta_i}\bigl(\mathscr{L}_i[z_n](t)\bigr) - {}^c\mathcal{D}_q^{\zeta_i}\bigl(\mathscr{L}_i[z_n](t_1)\bigr)\bigr| \\
&\quad\le \|h_{1i}\|_\infty\biggl[\frac{t^{\sigma_i-\zeta_i} - t_1^{\sigma_i-\zeta_i}}{\Gamma_q(\sigma_i-\zeta_i+1)} + \frac{|\Sigma_i|(t^{1-\zeta_i} - t_1^{1-\zeta_i})\Gamma_q(2+\zeta_i)}{\Gamma_q(\sigma_i+1)} \\
&\qquad + \frac{2|\Sigma_i|(t^{1-\zeta_i} - t_1^{1-\zeta_i})\Gamma_q(2+\zeta_i)\Gamma_q(1+\zeta_i)}{\Gamma_q(\sigma_i+\zeta_i+1)} \\
&\qquad + \frac{2|\Sigma_i|(t^{1-\zeta_i} - t_1^{1-\zeta_i})\Gamma_q(2+\zeta_i)\Gamma_q(1+\zeta_i)}{\Gamma_q(\sigma_i-\zeta_i+1)}\biggr].
\end{aligned}$$

Hence, the right-hand side of the inequalities tends to 0 as $t \to t_1$, and so the sequence $\{\mathscr{L}_i[z_n]\}$ is equicontinuous. Now, by using the Arzela–Ascoli theorem we deduce that there is a uniformly convergent subsequence. Thus, there is a subsequence of $\{z_n\}$, we show it again by $\{z_n\}$, such that $\mathscr{L}_i[z_n](t) \to \mathscr{L}_i[z](t)$ for each $t \in \overline{J}$. Note that $\mathscr{L}_i[z] \in \ell_i(S_{\mathcal{T}_{1i},(k_1,\ldots,k_m)})$. Hence,

$$M_i(k_1,\ldots,k_m) = \mathscr{L}_i(S_{\mathcal{T}_{1i},(k_1,\ldots,k_m)})$$

is compact for all $(k_1,\ldots,k_m) \in \mathcal{F}$ and $i = 1,\ldots,m$, and so $\mathcal{M}(k_1,\ldots,k_m)$ is compact. Now, we show that $\mathcal{M}(k_1,\ldots,k_m)$ is convex for all $(k_1,\ldots,k_m) \in \mathcal{F}$. Let $(x_1,\ldots,x_m)$, $(x'_1,\ldots,x'_m) \in \mathcal{M}(k)$. Choose z_i, $z'_i \in S_{\mathcal{T}_{1i},(k_1,\ldots,k_m)}$ such that

$$\begin{aligned} x_i(t) &= \int_0^1 G_q(t,r,\sigma_i,\zeta_i)z_i(r)\,\mathrm{d}_q r, \\ x'_i(t) &= \int_0^1 G_q(t,r,\sigma_i,\zeta_i)z'_i(r)\,\mathrm{d}_q r \end{aligned}$$

for almost all $t \in \overline{J}$ and $1 \le i \le m$. Let $0 \le \lambda \le 1$. Then we have

$$\bigl[\lambda x_i + (1-\lambda)x'_i\bigr](t) = \int_0^1 G_q(t,r,\sigma_i,\zeta_i)\bigl[\lambda z_i(r) + (1-\lambda)z'_i(r)\bigr]\,\mathrm{d}_q r.$$

Since $\mathcal{T}_{1i}$ is convex-valued for all $1 \le i \le m$,

$$\bigl[\lambda x_i + (1-\lambda)x'_i\bigr] \in M_i(k_1,\ldots,k_m).$$

Thus,

$$\lambda(x_1,\ldots,x_m) + (1-\lambda)\bigl(x'_1,\ldots,x'_m\bigr) = \bigl(\lambda x_1 + (1-\lambda)x'_1,\ldots,\lambda x_m + (1-\lambda)x'_m\bigr) \in \mathcal{M}(k).$$

Similarly, $\mathcal{N}$ is compact and convex-valued. Here, we show that $\mathcal{M}(f) + \mathcal{N}(f) \subset \mathcal{F}$ for all $f \in \mathcal{F}$. Let $f \in \mathcal{F}$ and $(x_1,\ldots,x_m) \in \mathcal{M}(f)$ and $(x'_1,\ldots,x'_m) \in \mathcal{N}(f)$. Then we can choose $(z_1,\ldots,z_m) \in S_{\mathcal{T}_{11},f} \times \cdots \times S_{\mathcal{T}_{1m},f}$ and $(z'_1,\ldots,z'_m) \in S_{\mathcal{T}_{21},f} \times \cdots \times S_{\mathcal{T}_{2m},f}$ such that $x_i(t) = \int_0^1 G_q(t,r,\sigma_i,\zeta_i)z_i(r)\,\mathrm{d}_q r$ and $x'_i(t) = \int_0^1 G_q(t,r,\sigma_i,\zeta_i)z'_i(r)\,\mathrm{d}_q r$ for almost all $t \in \overline{J}$ and $1 \le i \le m$. Hence, we get

$$\begin{aligned} \bigl|x_i(t) + x'_i(t)\bigr| \le{}& \mathcal{I}_q^{\sigma_i}\bigl[|z_i| + \bigl|z'_i\bigr|\bigr](t) + A_1(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i}\bigl[|z_i| + \bigl|z'_i\bigr|\bigr](1) \\ &+ A_2(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i+\zeta_i}\bigl[|z_i| + \bigl|z'_i\bigr|\bigr](1) \\ &+ A_3(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i-\zeta_i}\bigl[|z_i| + \bigl|z'_i\bigr|\bigr](1) \end{aligned}$$

and

$$\begin{aligned} \bigl|{}^c\mathcal{D}_q^{\zeta_i}[x_i](t) + {}^c\mathcal{D}_q^{\zeta_i}\bigl[x'_i\bigr](t)\bigr| \le{}& \mathcal{I}_q^{\sigma_i-\zeta_i}\bigl[|z_i| + \bigl|z'_i\bigr|\bigr](t) \\ &+ \frac{1}{t^{\zeta_i}}A_1(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i}\bigl[|z_i| + \bigl|z'_i\bigr|\bigr](1) \\ &+ \frac{t}{(t-1)t^{\zeta_i}}A_2(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i+\zeta_i}\bigl[|z_i| + \bigl|z'_i\bigr|\bigr](1) \end{aligned}$$

$$+ \frac{t}{(t-1)t^{\zeta_i}} A_3(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i-\zeta_i}\big[|z_i| + |z_i'|\big](1).$$

Hence, $\max_{t\in\overline{J}} |x_i(t) + x_i'(t)| \le (\|h_{1i}\|_\infty + \|h_{2i}\|_\infty)\Lambda_{1i}$ and

$$\max_{t\in\overline{J}} \big|{}^c\mathcal{D}_q^{\zeta_i}[z_i](t) + {}^c\mathcal{D}_q^{\zeta_i}\big[x_i'\big](t)\big| \le \big(\|h_{1i}\|_\infty + \|h_{2i}\|_\infty\big)\Lambda_{2i}$$

for $1 \le i \le m$, and so

$$\begin{aligned}\big\|(x_1,\dots,x_m) + \big(x_1',\dots,x_m'\big)\big\| &= \sum_{i=1}^m \big\|x_i + x_i'\big\|_i \\ &\le \sum_{i=1}^m \big(\|h_{1i}\|_\infty + \|h_{2i}\|_\infty\big)(\Lambda_{1i} + \Lambda_{2i}) = M.\end{aligned}$$

In this step, we show that the operator $\mathcal{N}$ is compact on $\mathcal{F}$. To do this, it is enough to prove that $\mathcal{N}(\mathcal{F})$ is uniformly bounded and equicontinuous. Let $(x_1,\dots,x_m) \in \mathcal{N}(\mathcal{F})$. Choose $(z_1,\dots,z_m) \in S_{\mathcal{T}_{21},k} \times \cdots \times S_{\mathcal{T}_{2m},k}$ such that $x_i(t) = \int_0^1 G_q(t,r,\sigma_i,\zeta_i)z_i(r)\,\mathrm{d}_qr$ for some $k \in \mathcal{F}$ and all $1 \le i \le m$. Hence,

$$\begin{aligned}\big|x_i(t)\big| \le{}& \mathcal{I}_q^{\sigma_i}\big[|z_i|\big](t) + A_1(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i}\big[|z_i|\big](1) + A_2(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i+\zeta_i}\big[|z_i|\big](1) \\ &+ A_3(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i-\zeta_i}\big[|z_i|\big](1)\end{aligned}$$

and

$$\begin{aligned}\big|{}^c\mathcal{D}_q^{\zeta_i}[x_i](t)\big| \le{}& \mathcal{I}_q^{\sigma_i-\zeta_i}\big[|z_i|\big](t) + \frac{1}{t^{\zeta_i}}A_1(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i}\big[|z_i|\big](1) \\ &+ \frac{t}{(t-1)t^{\zeta_i}}A_2(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i+\zeta_i}\big[|z_i|\big](1) \\ &+ \frac{t}{(t-1)t^{\zeta_i}}A_3(t,q,\zeta_i)\mathcal{I}_q^{\sigma_i-\zeta_i}\big[|z_i|\big](1).\end{aligned}$$

Thus, $\max_{t\in\overline{J}} |x_i(t)| \le \|h_{1i}\|_\infty\Lambda_{2i}$ and $\max_{t\in\overline{J}} |{}^c\mathcal{D}_q^{\zeta_i}[x_i](t)| \le \|h_{2i}\|_\infty\Lambda_{2i}$ for $1 \le i \le m$, and so

$$\big\|(x_1,\dots,x_m)\big\| = \sum_{i=1}^m \|x_i\|_\infty \le \sum_{i=1}^m \|h_{2i}\|_\infty(\Lambda_{1i} + \Lambda_{2i}).$$

Now, we show that $\mathcal{N}$ maps $\mathcal{F}$ to equicontinuous subsets of $\mathcal{E}$. Let $t, t_1 \in \overline{J}$ with $t_1 < t$, $k \in \mathcal{F}$, and $(x_1,\dots,x_m) \in \mathcal{N}(k)$. Choose $(z_1,\dots,z_m) \in S_{\mathcal{T}_{21},k} \times \cdots \times S_{\mathcal{T}_{2k},k}$ such that $x_i(t) = \int_0^1 G_q(t,r,\sigma_i,\zeta_i)z_i(r)\,\mathrm{d}_qr$ for all $1 \le i \le m$. Then we have

$$\begin{aligned}\big|x_i(t) - x_i(t_1)\big| \le{}& \|h_{2i}\|_\infty \\ &\times \Bigg[\frac{1}{\Gamma_q(\sigma_i+1)}\big(t^{\sigma_i} - t_1^{\sigma_i} + |\Sigma_i|(t-t_1)\Gamma_q(\zeta_i+2)\Gamma_q(\zeta_i+1)\big) \\ &+ \frac{2|\Sigma_i|(t-t_1)\Gamma_q(\zeta_i+2)\Gamma_q(\zeta_i+1)\Gamma_q(2-\zeta_i)}{\Gamma_q(\sigma_i+\zeta_i+1)}\end{aligned}$$

$$+\frac{2|\Sigma_i|(t-t_1)\Gamma_q(\zeta_i+2)\Gamma_q(\zeta_i+1)\Gamma_q(2-\zeta_i)}{\Gamma_q(\sigma_i-\zeta_i+1)}\Bigg]$$

and

$$\begin{aligned}\left|{}^c\mathcal{D}_q^{\zeta_i}[x_i](t)-{}^c\mathcal{D}_q^{\zeta_i}[x_i](t_1)\right| \le \|h_{2i}\|_\infty \Bigg[&\frac{t^{\sigma_i-\zeta_i}-t_1^{\sigma_i-\zeta_i}}{\Gamma_q(\sigma_i-\zeta_i+1)}\\ &+\frac{|\Sigma_i|(t^{1-\zeta_i}-t_1^{1-\zeta_i})\Gamma_q(2+\zeta_i)}{\Gamma_q(\sigma_i+1)}\\ &+\frac{2|\Sigma_i|(t^{1-\zeta_i}-t_1^{1-\zeta_i})\Gamma_q(2+\zeta_i)\Gamma_q(1+\zeta_i)}{\Gamma_q(\sigma_i+\zeta_i+1)}\\ &+\frac{2|\Sigma_i|(t^{1-\zeta_i}-t_1^{1-\zeta_i})\Gamma_q(2+\zeta_i)\Gamma_q(1+\zeta_i)}{\Gamma_q(\sigma_i-\zeta_i+1)}\Bigg].\end{aligned}$$

Note that the right-hand side of these inequalities tends to 0 as $t \to t_1$. By using the Arzela–Ascoli theorem, $\mathcal{N}$ is compact. Here, we show that $\mathcal{N}$ has a closed graph. Let $(k_1^n,\dots,k_m^n) \in \mathcal{F}$ and $(x_1^n,\dots,x_m^n) \in \mathcal{N}(k_1^n,\dots,k_m^n)$ be such that $(k_1^n,\dots,k_m^n) \to (k_1^0,\dots,k_m^0)$ and also $(x_1^n,\dots,x_m^n) \to (x_1^0,\dots,x_m^0)$ for all n. We show that $(x_1^0,\dots,x_m^0) \in \mathcal{N}(k_1^0,\dots,k_m^0)$. For each natural number n, choose $(z_1^n,\dots,z_m^n) \in S_{\mathcal{T}_{21},(k_1^n,\dots,k_m^n)} \times \cdots \times S_{\mathcal{T}_{2m},(k_1^n,\dots,k_m^n)}$ such that

$$x_i^n(t) = \int_0^1 G_q(t,r,\sigma_i,\zeta_i) z_i^n(r)\,\mathrm{d}_q r$$

for all $t \in \overline{J}$ and $1 \le i \le m$. Again, consider the continuous linear operator $\mathscr{L}_i : L^1(\overline{J},\mathbb{R}) \to E_i$ by $\mathscr{L}_i[z](t) = \int_0^1 G_q(t,r,\sigma_i,\zeta_i)z(r)\,\mathrm{d}_q r$. By using Lemma 2, $\mathscr{L}_i \circ S_{\mathcal{T}_{2i}}$ is a closed graph operator. Since $x_i^n \in \mathscr{L}_i(S_{\mathcal{T}_{2i},(k_1^n,\dots,k_m^n)})$ for all n, $1 \le i \le m$ and $(k_1^n,\dots,k_m^n) \to (k_1^0,\dots,k_m^0)$, there exists $z_i^0 \in S_{\mathcal{T}_{2i},(k_1^0,\dots,k_m^0)}$ such that $x_i^0(t) = \int_0^1 G_q(t,r,\sigma_i,\zeta_i)z_i^0(r)\,\mathrm{d}_q r$. This implies that $x_i^0 \in N_i(k_1^0,\dots,k_m^0)$ for all $1 \le i \le m$. Thus, N_i has a closed graph for all $1 \le i \le m$, and so $\mathcal{N}$ has a closed graph. This shows that the operator $\mathcal{N}$ is upper semi-continuous. Now, we show that $\mathcal{M}$ is a contractive multifunction. Let $k = (k_1,\dots,k_m), f = (f_1,\dots,f_m) \in \mathcal{E}$, and $(x_1,\dots,x_m) \in \mathcal{M}(f)$. Then we can choose $(z_1,\dots,z_m) \in S_{\mathcal{T}_{11},f} \times S_{\mathcal{T}_{12},f} \times \cdots \times S_{\mathcal{T}_{1m},f}$ such that $x_i(t) = \int_0^1 G_q(t,r,\sigma_i,\zeta_i)z_i(r)\,\mathrm{d}_q r$ for all $t \in \overline{J}$ and $i = 1,\dots,m$. Since

$$\begin{aligned}&P_\rho\big(\mathcal{T}_{1i}\big(t,k_1(t),\dots,k_m(t),\mathcal{I}_q^{\zeta_1}[k_1](t),\dots,\mathcal{I}_q^{\zeta_m}[k_m](t)\big),\\ &\quad \mathcal{T}_{1i}\big(t,f_1(t),\dots,f_m(t),\mathcal{I}_q^{\zeta_1}[f_1](t),\dots,\mathcal{I}_q^{\zeta_1}[f_m](t)\big)\big)\\ &\quad \le \gamma_i(t)\sum_{i=1}^m \big(\big|k_i(t)-f_i(t)\big| + \big|\mathcal{I}_q^{\zeta_i}[k_i](t)-\mathcal{I}_q^{\zeta_i}[k_i](t)\big|,\end{aligned}$$

for almost all $t \in \overline{J}$ and $i = 1,\dots,m$, by using (5), there exists

$$u_i \in \mathcal{T}_{1i}\big(t,k_1(t),\dots,k_m(t),\mathcal{I}_q^{\zeta_1}[k_1](t),\dots,\mathcal{I}_q^{\zeta_k}[k_m](t)\big)$$

such that

$$\big|z_i(t)-u_i\big| \le \gamma_i(t)\sum_{i=1}^m \big(\big|k_i(t)-f_i(t)\big| + \big|\mathcal{I}_q^{\zeta_i}[k_i](t)-\mathcal{I}_q^{\zeta_i}[f_i](t)\big|\big)$$

for almost all $t \in \overline{J}$ and $i = 1, \ldots, m$. Consider the set-valued mapping $\Omega_i : \overline{J} \to 2^{\mathbb{R}}$ defined by

$$\Omega_i(t) = \{ y \in \mathbb{R} : |z_i(t) - y| \leq \gamma_i(t) g(t) \text{ for almost all } t \in \overline{J} \},$$

where $g(t) = \sum_{i=1}^m (|k_i(t) - f_i(t)| + |\mathcal{I}_q^{\zeta_i}[k_i](t) - \mathcal{I}_q^{\zeta_i}[f_i](t)|$. Put

$$\varrho_i = \gamma_i \sum_{i=1}^m \big(|k_i - f_i| + |\mathcal{I}_q^{\zeta_i}[k_i] - \mathcal{I}_q^{\zeta_i}[f_i]| \big).$$

Since z_i and ϱ_i are measurable for all i,

$$\Omega_i(\cdot) \cap \mathcal{T}_{1i}\big(t, k_1(\cdot), \ldots, k_m(\cdot), \mathcal{I}_q^{\zeta_1}[k_1](\cdot), \ldots, \mathcal{I}_q^{\zeta_m}[k_m](\cdot)\big)$$

is a measurable multifunction. Thus, we can choose

$$z_i'(t) \in \mathcal{T}_{1i}\big(t, k_1(t), \ldots, k_m(t), \mathcal{I}_q^{\zeta_1}[k_1](t), \ldots, \mathcal{I}_q^{\zeta_m}[k_m](t)\big)$$

such that

$$\begin{aligned} |z_i(t) - z_i'(t)| &\leq \gamma_i(t) \sum_{i=1}^m \big(|k_i(t) - f_i(t)| + |\mathcal{I}_q^{\zeta_i}[k_i](t) - \mathcal{I}_q^{\zeta_i}[f_i](t)| \big) \\ &\leq \gamma_i(t) \sum_{i=1}^m \left(|k_i(t) - f_i(t)| + \frac{\|k_i - f_i\|_i}{\Gamma_q(\zeta_i + 1)} \right) \end{aligned}$$

and

$$x_i'(t) = \int_0^1 G_q(t, r, \sigma_i, \zeta_i) z_i'(r) \, \mathrm{d}_q r$$

for all $t \in \overline{J}$ and $i = 1, \ldots, m$. Since

$$\begin{aligned} |x_i(t) - x_i'(t)| \leq{}& \mathcal{I}_q^{\sigma_i}\big[|z_i - z_i'|\big](t) + A_1(t, q, \zeta_i) \mathcal{I}_q^{\sigma_i}\big[|z_i - z_i'|\big](1) \\ &+ A_2(t, q, \zeta_i) \mathcal{I}_q^{\sigma_i + \zeta_i}\big[|z_i - z_i'|\big](1) \\ &+ A_3(t, q, \zeta_i) \mathcal{I}_q^{\sigma_i - \zeta_i}\big[|z_i - z_i'|\big](1) \end{aligned}$$

and

$$\begin{aligned} \big|{}^c\mathcal{D}^{\zeta_i}[x_i](t) + {}^c\mathcal{D}^{\zeta_i}[x_i'](t)\big| \leq{}& \mathcal{I}_q^{\sigma_i - \zeta_i}\big[|z_i - z_i'|\big](t) \\ &+ \frac{1}{t^{\zeta_i}} A_1(t, q, \zeta_i) \mathcal{I}_q^{\sigma_i}\big[|z_i - z_i'|\big](1) \\ &+ \frac{t}{(t-1)t^{\zeta_i}} A_2(t, q, \zeta_i) \mathcal{I}_q^{\sigma_i + \zeta_i}\big[|z_i - z_i'|\big](1) \\ &+ \frac{t}{(t-1)t^{\zeta_i}} A_3(t, q, \zeta_i) \mathcal{I}_q^{\sigma_i - \zeta_i}\big[|z_i - z_i'|\big](1), \end{aligned}$$

we get $\max_{t\in\bar{J}} |x_i(t) - x_i'(t)| \leq \|\gamma_i\|_\infty (\frac{1+\Gamma_q(\zeta_i+1)}{\Gamma_q(\zeta_i+1)}) \Lambda_{1i}\|k - f\|$ and

$$\max_{t\in\bar{J}} \left| {}^c\mathcal{D}_q^{\zeta_i} x_i(t) - {}^c\mathcal{D}_q^{\zeta_i} z_i'(t) \right| \leq \|\gamma_i\|_\infty \left(\frac{1 + \Gamma_q(\zeta_i + 1)}{\Gamma_q(\zeta_i + 1)} \right) \Lambda_{2i}\|k - f\|$$

for each $1 \leq i \leq m$. Thus,

$$\begin{aligned} &\left\| (x_1, \dots, x_m) - \left(x_1', \dots, x_m'\right) \right\| \\ &\quad = \sum_{i=1}^{m} \left\| x_i - x_i' \right\|_i \\ &\quad \leq \sum_{i=1}^{m} \|\gamma_i\|_\infty \left(\frac{1 + \Gamma_q(\zeta_i + 1)}{\Gamma_q(\zeta_i + 1)} \right) (\Lambda_{1i} + \Lambda_{2i})\|k - f\|. \end{aligned}$$

This implies that $P_\rho(\mathcal{M}(k), \mathcal{M}(f)) \leq \lambda\|k - f\|$. Now, by using Lemma 3, the operator inclusion $k \in \mathcal{M}(k) + \mathcal{N}(k)$ has a solution which is a solution for the system of q-fractional inclusions. This completes the proof. □

Now, we give an example to illustrate our main result. In this way, we give a computational technique for checking the system. We need to present a simplified analysis that is able to execute the values of the q-gamma function. For this purpose, we provide a pseudo–code description of the method for calculation of the q-gamma function of order n in Algorithms 2, 3, 4, 5, and 6.

Example 1 Consider the three-dimensional system of fractional q-differential inclusions

$$\begin{cases} {}^c\mathcal{D}_q^{\frac{3}{2}}[k_1](t) \in \mathcal{T}_{1i}(t, k_1(t), k_2(t), k_3(t), \mathcal{I}_q^{\frac{1}{4}}[k_1](t), \mathcal{I}_q^{\frac{1}{2}}[k_2](t), \mathcal{I}_q^{\frac{3}{5}}[k_3](t)) \\ \qquad\qquad + \mathcal{T}_{21}(t, k_1(t), k_2(t), k_3(t), {}^c\mathcal{D}_q^{\frac{1}{4}}[k_1](t), {}^c\mathcal{D}_q^{\frac{1}{2}}[k_2](t), {}^c\mathcal{D}_q^{\frac{3}{5}}[k_3](t)), \\ {}^c\mathcal{D}_q^{\frac{7}{4}}[k_2](t) \in \mathcal{T}_{12}(t, k_1(t), k_2(t), k_3(t), \mathcal{I}_q^{\frac{1}{4}}[k_1](t), \mathcal{I}_q^{\frac{1}{2}}[k_2](t), \mathcal{I}_q^{\frac{3}{5}}[k_3](t)) \\ \qquad\qquad + \mathcal{T}_{22}(t, k_1(t), k_2(t), k_3(t), {}^c\mathcal{D}_q^{\frac{1}{4}}[k_1](t), {}^c\mathcal{D}_q^{\frac{1}{2}}[k_2](t), {}^c\mathcal{D}_q^{\frac{3}{5}}[k_3](t)), \\ {}^c\mathcal{D}_q^{\frac{9}{5}}[k_3](t) \in \mathcal{T}_{13}(t, k_1(t), k_2(t), k_3(t), \mathcal{I}_q^{\frac{1}{4}}[k_1](t), \mathcal{I}_q^{\frac{1}{2}}[k_2](t), \mathcal{I}_q^{\frac{3}{5}}[k_3](t)) \\ \qquad\qquad + \mathcal{T}_{23}(t, k_1(t), k_2(t), k_3(t), {}^c\mathcal{D}_q^{\frac{1}{4}}[k_1](t), {}^c\mathcal{D}_q^{\frac{1}{2}}[k_2](t), {}^c\mathcal{D}_q^{\frac{3}{5}}[k_3](t)) \end{cases} \tag{9}$$

with boundary conditions $k(0) + \mathcal{I}_q^{\frac{1}{4}}[k_1](0) + {}^c\mathcal{D}_q^{\frac{1}{4}}[k_1](0) = -k_1(1)$, $k_1(1) + \mathcal{I}_q^{\frac{1}{4}}[k_1](1) + {}^c\mathcal{D}_q^{\frac{1}{4}}[k_1](1) = -k_1(0)$, $k_2(0) + \mathcal{I}_q^{\frac{1}{2}}[k_2](0) + {}^c\mathcal{D}_q^{\frac{1}{2}}[k_2](0) = -k_2(1)$, $k_2(1) + \mathcal{I}_q^{\frac{1}{2}}[k_2](1) + {}^c\mathcal{D}_q^{\frac{1}{2}}[k_2](1) = -k_2(0)$, $k_3(0) + \mathcal{I}_q^{\frac{3}{5}}[k_3](0) + {}^c\mathcal{D}_q^{\frac{3}{5}}[k_3](0) = -k_3(1)$, and $k_3(1) + \mathcal{I}_q^{\frac{3}{5}}[k_3](1) +$

Algorithm 5 The proposed method for calculated $\int_a^b f(r)\, d_q r$

```
function g = Iq(q, x, n, fun)
p=1;
for k=0:n
 p=p+ q^k*fun(x*q^k);
end;
g=x* (1-q) * p;
end
```

Algorithm 6 The proposed method for calculated Λ_{1i}, Λ_{2i}, and Δ

```
function [Sigma gamzeta Lambda1 Lambda2 sumlumbda Delta]= ...
     funclambda(q, sigma, zeta, m, k, normgamma)
[xp yp]=size(q);

for n=1:k
    gamzeta(n,1)=n;
    Sigma(n,1)=n;
    Lambda1(n,1)=n;
    Lambda2(n,1)=n;
    sumlumbda(n,1)=n;
    Delta(n,1)=n;
end;
column=2;
for s=1:yp
    for t=1:m
        for n=1:k
            s1=qGamma(q(s), sigma(t)+1, n);
            s2=qGamma(q(s), sigma(t)+zeta(t)+1, n);
            s3=qGamma(q(s), sigma(t)-zeta(t)+1, n);
            d1=qGamma(q(s), zeta(t)+2, n);
            d2=qGamma(q(s), 2-zeta(t), n);
            gamzeta(n, column)=qGamma(q(s), zeta(t)+1, n);
            Sigma(n,column)=1/(d1*d2-2*d1*gamzeta(n, ...
                 column)-2*d2*gamzeta(n, column));
            Lambda1(n,column)=(1+(d1*d2+d1*gamzeta(n, ...
                 column)+d2*gamzeta(n, ...
                 column))/(abs(d1*d2-2*d1*gamzeta(n, ...
                 column)-2*d2*gamzeta(n, column))))/s1+ ...
                 d1*d2*gamzeta(n, column)/(s2*abs(d1*d2-2*d1*gamzeta(n, ...
                 column)-2*d2*gamzeta(n, column))) + d1*d2*gamzeta(n, ...
                 column)/(s3*abs(d1*d2-2*d1*gamzeta(n, ...
                 column)-2*d2*gamzeta(n, column)));
            Lambda2(n,column)=1/s3 + d1/(s1 * abs(d1*d2-2*d1*gamzeta(n, ...
                 column)-2*d2*gamzeta(n, column))) + 2* (d1+gamzeta(n, ...
                 column))/(s2*abs(d1*d2-2*d1*gamzeta(n, ...
                 column)-2*d2*gamzeta(n, column)))+ 2* (d1+gamzeta(n, ...
                 column))/(s3*abs(d1*d2-2*d1*gamzeta(n, ...
                 column)-2*d2*gamzeta(n, column)));
            sumlumbda(n,column)=Lambda1(n,column)+ Lambda2(n,column);
        end;
        column=column+1;
    end;
end;

for n=1:k
    D=0;
    column=1;
    for t=1:m
        D=D+normgamma(t)*(1+gamzeta(n, ...
             column+t))*sumlumbda(n,column+t)/gamzeta(n, column+t);
    end;
    Delta(n,2)=D;
end;

for n=1:k
    D=0;
    column=4;
    for t=1:m
        D=D+normgamma(t)*(1+gamzeta(n, ...
             column+t))*sumlumbda(n,column+t)/gamzeta(n, column+t);
    end;
    Delta(n,3)=D;
end;

for n=1:k
    D=0;
    column=7;
    for t=1:m
        D=D+normgamma(t)*(1+gamzeta(n, ...
             column+t))*sumlumbda(n,column+t)/gamzeta(n, column+t);
    end;
    Delta(n,4)=D;
end;


end
```

${}^{c}\mathcal{D}_q^{\frac{3}{5}}[k_3](1) = -k_3(0)$, where $\mathcal{T}_{ij} : \bar{J} \times \mathbb{R}^6 \to \mathcal{P}_{\text{cp,cv}}(\mathbb{R})$ is such that

$$\begin{aligned}
\mathcal{T}_{11}(t,k_1,k_2,k_3,k_4,k_5,k_6) = \bigg[0, &\frac{\sin k_1}{270(1+t^2)} + \frac{t|k_2|}{135(1+|k_2|)} \\
&+ \frac{1}{270}\cos k_3 + \frac{|k_4|}{270(1+|k_4|)} \\
&+ \frac{tk_5^2}{270(1+k_5^2)} + \frac{t}{270(1+|k_6|)} + t^2 + t\bigg], \\
\mathcal{T}_{12}(t,k_1,k_2,k_3,k_4,k_5,k_6) = \bigg[0, &\frac{t^2|k_1|}{330(1+|k_1|)} + \frac{|k_2|}{330(1+|k_2|)} \\
&+ \frac{1}{330}\sin k_3 + \frac{1}{330}\cos k_4 + e^t \\
&+ \frac{t|k_5|}{330(2+|k_5|)} + \frac{|k_6|e^t}{330(1+e^t|k_6|)} + 2\bigg], \\
\mathcal{T}_{13}(t,k_1,k_2,k_3,k_4,k_5,k_6) = \bigg[0, &\frac{t}{190(1+|k_1|)} + \frac{t|k_2|}{190(1+|k_2|)} \\
&+ \frac{1}{190}\cos k_3 + \frac{1}{190}\sin k_4 + e^t \\
&+ \frac{e^t|k_5|}{190(1+e^t|k_5|)} + \frac{|k_6|}{190(1+k_6|)} + \frac{1}{95}\bigg], \\
\mathcal{T}_{21}(t,k_1,k_2,k_3,k_4,k_5,k_6) = \bigg[&e^{-|k_1|} - \frac{|k_2|}{1+|k_2|} + e^{-|k_3|} + \sin k_4 \\
&- \frac{\cos(k_5k_6)}{1+\cos(k_5k_6)} + t^2, \cos k_1 + \cos k_2 \\
&+ \cos(k_3k_4) + \frac{k_5^2k_6^2}{1+k_5^2k_6^2} + t + \sin t\bigg], \\
\mathcal{T}_{22}(t,k_1,k_2,k_3,k_4,k_5,k_6) = \bigg[&-1, \frac{\sin^2 k_1}{1+t} + t\cos^2 k_2 + e^t\sin(k_3) \\
&+ e^t\cos(k_4) + \frac{e^{k_5k_6}}{1+e^{k_5k_6}}\bigg], \\
\mathcal{T}_{23}(t,k_1,k_2,k_3,k_4,k_5,k_6) = \bigg[&\frac{k_1}{4(1+k_1)} + \frac{k_3}{1+k_3} + \frac{e^{-|k_6|}}{4(1+k_6)}, t\cos^2 k_1 \\
&+ \sin k_3 + \frac{\cos^3 k_6}{1+t^3}\bigg].
\end{aligned}$$

Define $h_{11}(t) = t^2 + \frac{46}{45}t + \frac{1}{270(1+t^2)} + \frac{1}{135}$, $h_{12}(t) = \frac{1}{330}t^2 + \frac{1}{330}t + e^t + \frac{332}{165}$, $h_{13} = \frac{1}{95}t + \frac{3}{95} + e^t$, $h_{21}(t) = \sin t + t + 4$, $h_{22}(t) = 2e^t + t + \frac{1}{1+t} + 1$, $h_{23} = \frac{1}{1+t^3} + t + \frac{5}{2}$, $\gamma_1(t) = \frac{1}{270(1+t^2)} + \frac{2}{135}t + \frac{1}{135}$, $\gamma_2(t) = \frac{1}{330}t^2 + \frac{1}{330}t + \frac{4}{330}$, and $\gamma_3(t) = \frac{1}{95}t + \frac{2}{95}$. Put $\sigma_1 = \frac{3}{2}$, $\sigma_2 = \frac{7}{4}$, $\sigma_3 = \frac{9}{5}$, $\zeta_1 = \frac{1}{4}$, $\zeta_2 = \frac{1}{2}$, and $\zeta_2 = \frac{3}{5}$. It is easy to check that $\|\mathcal{T}_{ij}(t,k_1,k_2,k_3,k_4,k_5,k_6)\| \le h_{ij}(t)$ and

$$P_\rho\big(\mathcal{T}_{1j}(t,k_1,k_2,k_3,k_4,k_5,k_6), \mathcal{T}_{1j}\big(t,k_1',k_2',k_3',k_4',k_5',k_6'\big)\big) \le \gamma_j(t)\left(\sum_{i=1}^{6}\big|k_i - k_i'\big|\right)$$

for $i = 1, 2$ and $j = 1, 2, 3$. By using (6) and (7), (8), we obtain

$$\begin{aligned}
\Sigma_1 &= \left[\Gamma_q(\zeta_1+2)\Gamma_q(2-\zeta_1) - 2\Gamma_q(\zeta_1+1)\Gamma_q(2+\zeta_1) - 2\Gamma_q(\zeta_1+1)\Gamma_q(2-\zeta_1)\right]^{-1} \\
&= \left[\Gamma_q\left(\frac{1}{4}+2\right)\Gamma_q\left(2-\frac{1}{4}\right) - 2\Gamma_q\left(\frac{1}{4}+1\right)\Gamma_q\left(2+\frac{1}{4}\right)\right. \\
&\quad \left. - 2\Gamma_q\left(\frac{1}{4}+1\right)\Gamma_q\left(2-\frac{1}{4}\right)\right]^{-1} \\
&= \left[\Gamma_q\left(\frac{9}{4}\right)\Gamma_q\left(\frac{7}{4}\right) - 2\Gamma_q\left(\frac{5}{4}\right)\Gamma_q\left(\frac{9}{4}\right) - 2\Gamma_q\left(\frac{5}{4}\right)\Gamma_q\left(\frac{7}{4}\right)\right]^{-1},
\end{aligned} \tag{10}$$

$$\begin{aligned}
\Lambda_{11} &= \frac{1}{\Gamma_q(\sigma_1+1)}\bigl(1 + |\Sigma_1|\bigl[\Gamma_q(\zeta_1+2)\Gamma_q(2-\zeta_1) \\
&\quad + \Gamma_q(\zeta_1+1)\Gamma_q(\zeta_1+2) + \Gamma_q(\zeta_1+1)\Gamma_q(2-\zeta_1)\bigr]\bigr) \\
&\quad + \frac{|\Sigma_1|\Gamma_q(\zeta_1+1)\Gamma_q(\zeta_1+2)\Gamma_q(2-\zeta_1)}{\Gamma_q(\sigma_1+\zeta_1+1)} \\
&\quad + \frac{|\Sigma_1|\Gamma_q(\zeta_1+1)\Gamma_q(\zeta_1+2)\Gamma_q(2-\zeta_1)}{\Gamma_q(\sigma_1-\zeta_1+1)} \\
&= \frac{1}{\Gamma_q(\frac{3}{2}+1)}\left(1 + |\Sigma_1|\left[\Gamma_q\left(\frac{1}{4}+2\right)\Gamma_q\left(2-\frac{1}{4}\right)\right.\right. \\
&\quad \left.\left. + \Gamma_q\left(\frac{1}{4}+1\right)\Gamma_q\left(\frac{1}{4}+2\right) + \Gamma_q\left(\frac{1}{4}+1\right)\Gamma_q\left(2-\frac{1}{4}\right)\right]\right) \\
&\quad + \frac{|\Sigma_1|\Gamma_q(\frac{1}{4}+1)\Gamma_q(\frac{1}{4}+2)\Gamma_q(2-\frac{1}{4})}{\Gamma_q(\frac{3}{2}+\frac{1}{4}+1)} \\
&\quad + \frac{|\Sigma_1|\Gamma_q(\frac{1}{4}+1)\Gamma_q(\frac{1}{4}+2)\Gamma_q(2-\frac{1}{4})}{\Gamma_q(\frac{3}{2}-\frac{1}{4}+1)} \\
&= \frac{1}{\Gamma_q(\frac{5}{2})}\left(1 + |\Sigma_1|\left[\Gamma_q\left(\frac{9}{4}\right)\Gamma_q\left(\frac{7}{4}\right)\right.\right. \\
&\quad \left.\left. + \Gamma_q\left(\frac{5}{4}\right)\Gamma_q\left(\frac{9}{4}\right) + \Gamma_q\left(\frac{5}{4}\right)\Gamma_q\left(\frac{7}{4}\right)\right]\right) \\
&\quad + \frac{|\Sigma_1|\Gamma_q(\frac{5}{4})\Gamma_q(\frac{9}{4})\Gamma_q(\frac{7}{4})}{\Gamma_q(\frac{11}{4})} + \frac{|\Sigma_1|\Gamma_q(\frac{5}{4})\Gamma_q(\frac{9}{4})\Gamma_q(\frac{7}{4})}{\Gamma_q(\frac{9}{4})},
\end{aligned} \tag{11}$$

$$\begin{aligned}
\Lambda_{21} &= \frac{1}{\Gamma_q(\sigma_1-\zeta_1+1)} + \frac{|\Sigma_1|\Gamma_q(\zeta_1+2)}{\Gamma_q(\sigma_1+1)} + \frac{2|\Sigma_1|(\Gamma_q(\zeta_1+1)+\Gamma_q(\zeta_1+2))}{\Gamma_q(\sigma_1+\zeta_1+1)} \\
&\quad + \frac{2|\Sigma_1|(2\Gamma_q(\zeta_1+1)+\Gamma_q(\zeta_1+2))}{\Gamma_q(\sigma_1-\zeta_1+1)} \\
&= \frac{1}{\Gamma_q(\frac{3}{2}-\frac{1}{4}+1)} + \frac{|\Sigma_1|\Gamma_q(\frac{1}{4}+2)}{\Gamma_q(\frac{3}{2}+1)} + \frac{2|\Sigma_1|(\Gamma_q(\frac{1}{4}+1)+\Gamma_q(\frac{1}{4}+2))}{\Gamma_q(\frac{3}{2}+\frac{1}{4}+1)} \\
&\quad + \frac{2|\Sigma_1|(2\Gamma_q(\frac{1}{4}+1)+\Gamma_q(\frac{1}{4}+2))}{\Gamma_q(\frac{3}{2}-\frac{1}{4}+1)} \\
&= \frac{1}{\Gamma_q(\frac{9}{4})} + \frac{|\Sigma_1|\Gamma_q(\frac{9}{4})}{\Gamma_q(\frac{5}{2})} + \frac{2|\Sigma_1|(\Gamma_q(\frac{5}{4})+\Gamma_q(\frac{9}{4}))}{\Gamma_q(\frac{11}{4})}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2|\Sigma_1|(2\Gamma_q(\frac{5}{4})+\Gamma_q(\frac{9}{4}))}{\Gamma_q(\frac{9}{4})}, \qquad (12)\\
\Sigma_2 &= \left[\Gamma_q(\zeta_2+2)\Gamma_q(2-\zeta_2)-2\Gamma_q(\zeta_2+1)\Gamma_q(2+\zeta_2)-2\Gamma_q(\zeta_2+1)\Gamma_q(2-\zeta_2)\right]^{-1}\\
&= \left[\Gamma_q\left(\frac{1}{2}+2\right)\Gamma_q\left(2-\frac{1}{2}\right)-2\Gamma_q\left(\frac{1}{2}+1\right)\Gamma_q\left(2+\frac{1}{2}\right)\right.\\
&\quad\left.-2\Gamma_q\left(\frac{1}{2}+1\right)\Gamma_q\left(2-\frac{1}{2}\right)\right]^{-1}\\
&= \left[\Gamma_q\left(\frac{5}{2}\right)\Gamma_q\left(\frac{3}{2}\right)-2\Gamma_q\left(\frac{3}{2}\right)\Gamma_q\left(\frac{5}{2}\right)-2\Gamma_q\left(\frac{3}{2}\right)\Gamma_q\left(\frac{3}{2}\right)\right]^{-1}, \qquad (13)\\
\Lambda_{12} &= \frac{1}{\Gamma_q(\sigma_2+1)}\big(1+|\Sigma_2|\big[\Gamma_q(\zeta_2+2)\Gamma_q(2-\zeta_2)\\
&\quad+\Gamma_q(\zeta_2+1)\Gamma_q(\zeta_2+2)+\Gamma_q(\zeta_2+1)\Gamma_q(2-\zeta_2)\big]\big)\\
&\quad+\frac{|\Sigma_2|\Gamma_q(\zeta_2+1)\Gamma_q(\zeta_2+2)\Gamma_q(2-\zeta_2)}{\Gamma_q(\sigma_2+\zeta_2+1)}\\
&\quad+\frac{|\Sigma_2|\Gamma_q(\zeta_2+1)\Gamma_q(\zeta_2+2)\Gamma_q(2-\zeta_2)}{\Gamma_q(\sigma_2-\zeta_2+1)}\\
&= \frac{1}{\Gamma_q(\frac{7}{4}+1)}\left(1+|\Sigma_2|\left[\Gamma_q\left(\frac{1}{2}+2\right)\Gamma_q\left(2-\frac{1}{2}\right)\right.\right.\\
&\quad\left.\left.+\Gamma_q\left(\frac{1}{2}+1\right)\Gamma_q\left(\frac{1}{2}+2\right)+\Gamma_q\left(\frac{1}{2}+1\right)\Gamma_q\left(2-\frac{1}{2}\right)\right]\right)\\
&\quad+\frac{|\Sigma_2|\Gamma_q(\frac{1}{2}+1)\Gamma_q(\frac{1}{2}+2)\Gamma_q(2-\frac{1}{2})}{\Gamma_q(\frac{7}{4}+\frac{1}{2}+1)}\\
&\quad+\frac{|\Sigma_2|\Gamma_q(\frac{1}{2}+1)\Gamma_q(\frac{1}{2}+2)\Gamma_q(2-\frac{1}{2})}{\Gamma_q(\frac{7}{4}-\frac{1}{2}+1)}\\
&= \frac{1}{\Gamma_q(\frac{11}{4})}\left(1+|\Sigma_2|\left[\Gamma_q\left(\frac{5}{2}\right)\Gamma_q\left(\frac{3}{2}\right)\right.\right.\\
&\quad\left.\left.+\Gamma_q\left(\frac{3}{2}\right)\Gamma_q\left(\frac{5}{2}\right)+\Gamma_q\left(\frac{3}{2}\right)\Gamma_q\left(\frac{3}{2}\right)\right]\right)\\
&\quad+\frac{|\Sigma_2|\Gamma_q(\frac{3}{2})\Gamma_q(\frac{5}{2})\Gamma_q(\frac{3}{2})}{\Gamma_q(\frac{13}{4})}+\frac{|\Sigma_2|\Gamma_q(\frac{3}{2})\Gamma_q(\frac{5}{2})\Gamma_q(\frac{3}{2})}{\Gamma_q(\frac{9}{4})}, \qquad (14)\\
\Lambda_{22} &= \frac{1}{\Gamma_q(\sigma_2-\zeta_2+1)}+\frac{|\Sigma_2|\Gamma_q(\zeta_2+2)}{\Gamma_q(\sigma_2+1)}\\
&\quad+\frac{2|\Sigma_2|(\Gamma_q(\zeta_2+1)+\Gamma_q(\zeta_2+2))}{\Gamma_q(\sigma_2+\zeta_2+1)}\\
&\quad+\frac{2|\Sigma_2|(2\Gamma_q(\zeta_2+1)+\Gamma_q(\zeta_2+2))}{\Gamma_q(\sigma_2-\zeta_2+1)}\\
&= \frac{1}{\Gamma_q(\frac{7}{4}-\frac{1}{2}+1)}+\frac{|\Sigma_2|\Gamma_q(\frac{1}{2}+2)}{\Gamma_q(\frac{1}{2}+1)}\\
&\quad+\frac{2|\Sigma_2|(\Gamma_q(\frac{1}{2}+1)+\Gamma_q(\frac{1}{2}+2))}{\Gamma_q(\frac{7}{4}+\frac{1}{2}+1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2|\Sigma_2|(\Gamma_q(\frac{1}{2}+1)+\Gamma_q(\frac{1}{2}+2))}{\Gamma_q(\frac{7}{4}-\frac{1}{2}+1)} \\
&=\frac{1}{\Gamma_q(\frac{9}{4})}+\frac{|\Sigma_2|\Gamma_q(\frac{5}{2})}{\Gamma_q(\frac{3}{2})}+\frac{2|\Sigma_2|(\Gamma_q(\frac{3}{2})+\Gamma_q(\frac{5}{2}))}{\Gamma_q(\frac{13}{4})} \\
&+\frac{2|\Sigma_2|(\Gamma_q(\frac{3}{2})+\Gamma_q(\frac{5}{2}))}{\Gamma_q(\frac{9}{4})},
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\Sigma_3 &= \left[\Gamma_q(\zeta_3+2)\Gamma_q(2-\zeta_3)-2\Gamma_q(\zeta_3+1)\Gamma_q(2+\zeta_3)-2\Gamma_q(\zeta_3+1)\Gamma_q(2-\zeta_3)\right]^{-1} \\
&= \left[\Gamma_q\left(\frac{3}{5}+2\right)\Gamma_q\left(2-\frac{3}{5}\right)-2\Gamma_q\left(\frac{3}{5}+1\right)\Gamma_q\left(2+\frac{3}{5}\right)\right. \\
&\quad \left.-2\Gamma_q\left(\frac{3}{5}+1\right)\Gamma_q\left(2-\frac{3}{5}\right)\right]^{-1} \\
&= \left[\Gamma_q\left(\frac{13}{5}\right)\Gamma_q\left(\frac{7}{5}\right)-2\Gamma_q\left(\frac{8}{5}\right)\Gamma_q\left(\frac{13}{5}\right)-2\Gamma_q\left(\frac{8}{5}\right)\Gamma_q\left(\frac{7}{5}\right)\right]^{-1},
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\Lambda_{13} &= \frac{1}{\Gamma_q(\sigma_3+1)}\big(1+|\Sigma_3|\big[\Gamma_q(\zeta_3+2)\Gamma_q(2-\zeta_3) \\
&\quad +\Gamma_q(\zeta_3+1)\Gamma_q(\zeta_3+2)+\Gamma_q(\zeta_3+1)\Gamma_q(2-\zeta_3)\big]\big) \\
&\quad +\frac{|\Sigma_3|\Gamma_q(\zeta_3+1)\Gamma_q(\zeta_3+2)\Gamma_q(2-\zeta_3)}{\Gamma_q(\sigma_3+\zeta_3+1)} \\
&\quad +\frac{|\Sigma_3|\Gamma_q(\zeta_3+1)\Gamma_q(\zeta_3+2)\Gamma_q(2-\zeta_3)}{\Gamma_q(\sigma_3-\zeta_3+1)} \\
&= \frac{1}{\Gamma_q(\frac{9}{5}+1)}\left(1+|\Sigma_3|\left[\Gamma_q\left(\frac{3}{5}+2\right)\Gamma_q\left(2-\frac{3}{5}\right)\right.\right. \\
&\quad \left.\left.+\Gamma_q\left(\frac{3}{5}+1\right)\Gamma_q\left(\frac{3}{5}+2\right)+\Gamma_q\left(\frac{3}{5}+1\right)\Gamma_q\left(2-\frac{3}{5}\right)\right]\right) \\
&\quad +\frac{|\Sigma_3|\Gamma_q(\frac{3}{5}+1)\Gamma_q(\frac{3}{5}+2)\Gamma_q(2-\frac{3}{5})}{\Gamma_q(\frac{9}{5}+\frac{3}{5}+1)} \\
&\quad +\frac{|\Sigma_3|\Gamma_q(\frac{3}{5}+1)\Gamma_q(\frac{3}{5}+2)\Gamma_q(2-\frac{3}{5})}{\Gamma_q(\frac{9}{5}-\frac{3}{5}+1)} \\
&= \frac{1}{\Gamma_q(\frac{14}{5})}\left(1+|\Sigma_3|\left[\Gamma_q\left(\frac{13}{5}\right)\Gamma_q\left(\frac{7}{5}\right)\right.\right. \\
&\quad \left.\left.+\Gamma_q\left(\frac{8}{5}\right)\Gamma_q\left(\frac{13}{5}\right)+\Gamma_q\left(\frac{8}{5}\right)\Gamma_q\left(\frac{7}{5}\right)\right]\right) \\
&\quad +\frac{|\Sigma_3|\Gamma_q(\frac{8}{5})\Gamma_q(\frac{13}{5})\Gamma_q(\frac{7}{5})}{\Gamma_q(\frac{17}{5})}+\frac{|\Sigma_3|\Gamma_q(\frac{8}{5})\Gamma_q(\frac{13}{5})\Gamma_q(\frac{7}{5})}{\Gamma_q(\frac{11}{5})},
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\Lambda_{23} &= \frac{1}{\Gamma_q(\sigma_3-\zeta_3+1)}+\frac{|\Sigma_3|\Gamma_q(\zeta_3+2)}{\Gamma_q(\sigma_3+1)} \\
&\quad +\frac{2|\Sigma_3|(\Gamma_q(\zeta_3+1)+\Gamma_q(\zeta_3+2))}{\Gamma_q(\sigma_3+\zeta_3+1)} \\
&\quad +\frac{2|\Sigma_3|(2\Gamma_q(\zeta_3+1)+\Gamma_q(\zeta_3+2))}{\Gamma_q(\sigma_3-\zeta_3+1)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\Gamma_q(\frac{9}{5} - \frac{3}{5} + 1)} + \frac{|\Sigma_3|\Gamma_q(\frac{3}{5} + 2)}{\Gamma_q(\frac{9}{5} + 1)} \\
&\quad + \frac{2|\Sigma_3|(\Gamma_q(\frac{3}{5} + 1) + \Gamma_q(\frac{3}{5} + 2))}{\Gamma_q(\frac{9}{5} + \frac{3}{5} + 1)} \\
&\quad + \frac{2|\Sigma_3|(\Gamma_q(\frac{3}{5} + 1) + \Gamma_q(\frac{3}{5} + 2))}{\Gamma_q(\frac{9}{5} - \frac{3}{5} + 1)} \\
&= \frac{1}{\Gamma_q(\frac{11}{5})} + \frac{|\Sigma|\Gamma_q(\frac{13}{5})}{\Gamma_q(\frac{14}{5})} + \frac{2|\Sigma_3|(\Gamma_q(\frac{8}{5}) + \Gamma_q(\frac{13}{5}))}{\Gamma_q(\frac{17}{5})} \\
&\quad + \frac{2|\Sigma_3|(\Gamma_q(\frac{8}{5}) + \Gamma_q(\frac{13}{5}))}{\Gamma_q(\frac{11}{5})}.
\end{aligned} \tag{18}
$$

Note that Tables 1 and 2 show that $\Lambda_{11} \approx 2.5743$, $\Lambda_{12} \approx 2.5222$, $\Lambda_{13} \approx 2.5131$, $\Lambda_{21} \approx 3.9450$, $\Lambda_{22} \approx 3.9032$, $\Lambda_{23} \approx 3.8920$. Table 3 shows that $\Lambda_{11} \approx 2.3064$, $\Lambda_{12} \approx 2.1178$, $\Lambda_{13} \approx 2.0901$, $\Lambda_{21} \approx 3.7015$, $\Lambda_{22} \approx 3.5288$, $\Lambda_{23} \approx 3.4997$, and Table 4 leads us to $\Lambda_{11} \approx 2.1482$, $\Lambda_{12} \approx 1.8883$, $\Lambda_{13} \approx 1.8536$, $\Lambda_{21} \approx 3.5434$, $\Lambda_{22} \approx 3.3065$, and $\Lambda_{23} \approx 3.2801$ for $q = \frac{1}{10}$, $q = \frac{1}{2}$, and $q = \frac{6}{7}$ respectively. By the definition of γ_i, for $j = 1, 2, 3$, we get $\|\gamma_1\|_\infty = \frac{7}{270}$, $\|\gamma_2\|_\infty = \frac{1}{55}$, and $\|\gamma_3\|_\infty = \frac{3}{95}$. Now, by using Eqs. (7), (8), (10), (11), (12), (13),

Table 1 Some numerical results for Σ_1, Σ_2, and Σ_3 in Example 1 for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$

n	$q = \frac{1}{10}$			$q = \frac{1}{2}$			$q = \frac{6}{7}$		
	Σ_1	Σ_2	Σ_3	Σ_1	Σ_2	Σ_3	Σ_1	Σ_2	Σ_3
1	−0.3442	−0.3434	−0.3409	−0.3142	−0.2946	−0.2815	−0.1587	−0.0793	−0.0596
2	−0.3446	−0.3439	−0.3414	−0.3381	−0.326	−0.3147	−0.1837	−0.1125	−0.09
3	−0.3446	−0.344	−0.3414	−0.3501	−0.342	−0.3317	−0.2076	−0.1435	−0.1197
4	−0.3446	−0.344	−0.3414	−0.3562	−0.35	−0.3403	−0.2292	−0.1716	−0.1476
5	−0.3446	−0.344	−0.3414	−0.3592	−0.3541	−0.3446	−0.2484	−0.1968	−0.1731
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	−0.3446	−0.344	−0.3414	−0.3622	−0.358	−0.3488	−0.3128	−0.2832	−0.2634
11	−0.3446	−0.344	−0.3414	−0.3622	−0.3581	−0.3489	−0.3209	−0.2943	−0.2752
12	−0.3446	−0.344	−0.3414	−0.3622	−0.3581	−0.3489	−0.3279	−0.3039	−0.2855
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
56	−0.3446	−0.344	−0.3414	−0.3622	−0.3581	−0.3489	−0.3706	−0.3626	−0.3487
57	−0.3446	−0.344	−0.3414	−0.3622	−0.3581	−0.3489	−0.3706	−0.3627	−0.3488
58	−0.3446	−0.344	−0.3414	−0.3622	−0.3581	−0.3489	−0.3706	−0.3627	−0.3488
59	−0.3446	−0.344	−0.3414	−0.3622	−0.3581	−0.3489	−0.3707	−0.3627	−0.3488
60	−0.3446	−0.344	−0.3414	−0.3622	−0.3581	−0.3489	−0.3707	−0.3627	−0.3488

Table 2 Some numerical results of Λ_{1i}, Λ_{2i}, and $\Gamma_q(\zeta_i + 1)$ in Example 1 for $q = \frac{1}{10}$

n	$q = \frac{1}{10}$								
	Λ_{11}	Λ_{21}	$\Gamma_q(\zeta_1 + 1)$	Λ_{12}	Λ_{22}	$\Gamma_q(\zeta_2 + 1)$	Λ_{13}	Λ_{23}	$\Gamma_q(\zeta_3 + 1)$
1	2.5726	3.9393	0.9748	2.5202	3.897	0.9729	2.5109	3.8857	0.9761
2	2.5741	3.9444	0.9743	2.5221	3.9026	0.9723	2.5129	3.8913	0.9753
3	2.5743	3.9449	0.9743	2.5222	3.9032	0.9722	2.513	3.8919	0.9752
4	2.5743	3.945	0.9743	2.5223	3.9032	0.9722	2.5131	3.892	0.9752
5	2.5743	3.945	0.9743	2.5223	3.9032	0.9722	2.5131	3.892	0.9752
6	2.5743	3.945	0.9743	2.5223	3.9032	0.9722	2.5131	3.892	0.9752
7	2.5743	3.945	0.9743	2.5223	3.9032	0.9722	2.5131	3.892	0.9752

Table 3 Some numerical results of $\varLambda_{1i}$, $\varLambda_{2i}$, and $\varGamma_q(\zeta_i + 1)$ in Example 1 for $q = \frac{1}{2}$

n	$q = \frac{1}{2}$								
	$\varLambda_{11}$	$\varLambda_{21}$	$\varGamma_q(\zeta_1 + 1)$	$\varLambda_{12}$	$\varLambda_{22}$	$\varGamma_q(\zeta_2 + 1)$	$\varLambda_{13}$	$\varLambda_{23}$	$\varGamma_q(\zeta_3 + 1)$
1	2.082	3.0643	0.9743	1.8502	2.8525	0.9965	1.8071	2.8154	1.0157
2	2.1927	3.3761	0.9526	1.9826	3.183	0.9565	1.9472	3.1499	0.9686
3	2.2492	3.5371	0.9426	2.0498	3.354	0.9382	2.0183	3.3229	0.9472
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
13	2.3064	3.7014	0.9331	2.1177	3.5286	0.9209	2.0901	3.4995	0.9269
14	2.3064	3.7015	0.9331	2.1178	3.5287	0.9209	2.0901	3.4996	0.9269
15	2.3064	3.7015	0.9331	2.1178	3.5287	0.9209	2.0901	3.4996	0.9269
16	2.3064	3.7015	0.9331	2.1178	3.5287	0.9209	2.0901	3.4997	0.9269
17	2.3064	3.7015	0.9331	2.1178	3.5288	0.9209	2.0901	3.4997	0.9269
18	2.3064	3.7015	0.9331	2.1178	3.5288	0.9209	2.0901	3.4997	0.9269
19	2.3064	3.7015	0.9331	2.1178	3.5288	0.9209	2.0901	3.4997	0.9269
20	2.3064	3.7015	0.9331	2.1178	3.5288	0.9209	2.0901	3.4997	0.9269

Table 4 Some numerical results of $\varLambda_{1i}$, $\varLambda_{2i}$, and $\varGamma_q(\zeta_i + 1)$ in Example 1 for $q = \frac{6}{7}$

n	$q = \frac{6}{7}$								
	$\varLambda_{11}$	$\varLambda_{21}$	$\varGamma_q(\zeta_1 + 1)$	$\varLambda_{12}$	$\varLambda_{22}$	$\varGamma_q(\zeta_2 + 1)$	$\varLambda_{13}$	$\varLambda_{23}$	$\varGamma_q(\zeta_3 + 1)$
1	1.1914	1.0078	1.2002	0.6438	0.6785	1.5189	0.559	0.641	1.6878
2	1.2638	1.2688	1.1277	0.7963	0.9496	1.3487	0.7147	0.9078	1.4673
3	1.357	1.5226	1.0798	0.9337	1.2104	1.2408	0.8565	1.1673	1.3295
4	1.4502	1.7599	1.0458	1.0566	1.4544	1.1663	0.9839	1.4117	1.2356
5	1.5372	1.9772	1.0204	1.1655	1.6783	1.1121	1.0973	1.6369	1.1678
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
64	2.1482	3.5433	0.9128	1.8883	3.3064	0.8946	1.8536	3.28	0.9016
65	2.1482	3.5433	0.9128	1.8883	3.3064	0.8946	1.8536	3.28	0.9016
66	2.1482	3.5434	0.9128	1.8883	3.3064	0.8946	1.8536	3.28	0.9016
67	2.1482	3.5434	0.9128	1.8883	3.3064	0.8946	1.8536	3.28	0.9016
68	2.1482	3.5434	0.9128	1.8883	3.3065	0.8946	1.8536	3.2801	0.9016
69	2.1482	3.5434	0.9128	1.8883	3.3065	0.8946	1.8536	3.2801	0.9016
70	2.1482	3.5434	0.9128	1.8883	3.3065	0.8946	1.8536	3.2801	0.9016

(14), (15), (16), (17), and (18), we obtain

$$\begin{aligned}
\Delta &= \sum_{i=1}^{m} \|\gamma_i\|_\infty \left(\frac{1 + \varGamma_q(\zeta_i + 1)}{\varGamma_q(\zeta_i + 1)} \right) (\varLambda_{1i} + \varLambda_{2i}) \\
&= \frac{7}{270} \left(\frac{1 + \varGamma_q(\frac{1}{4} + 1)}{\varGamma_q(\frac{1}{4} + 1)} \right) (\varLambda_{11} + \varLambda_{21}) \\
&\quad + \frac{1}{55} \left(\frac{1 + \varGamma_q(\frac{1}{2} + 1)}{\varGamma_q(\frac{1}{2} + 1)} \right) (\varLambda_{12} + \varLambda_{22}) \\
&\quad + \frac{3}{95} \left(\frac{1 + \varGamma_q(\frac{3}{5} + 1)}{\varGamma_q(\frac{3}{5} + 1)} \right) (\varLambda_{13} + \varLambda_{23}).
\end{aligned}$$

By using Table 5, one finds the values of Δ for $q = \frac{1}{10}$, $q = \frac{1}{2}$, and $q = \frac{6}{7}$ (see Fig. 1). In fact, we get $\Delta \approx 0.9892 < 1$, $\Delta \approx 0.9038 < 1$, and $\Delta \approx 0.8512 < 1$ for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$, respectively. Now, by using Theorem 6, system (9) of fractional differential inclusions has a solution.

Table 5 Some numerical results of Δ, $\varLambda_{2i}$ in Example 1 for $q=\frac{1}{10},\frac{1}{2},\frac{6}{7}$

n	$q=\frac{1}{10}$	$q=\frac{1}{2}$	$q=\frac{6}{7}$
1	0.9876	0.7314	0.2047
2	0.989	0.8152	0.2653
3	0.9891	0.8589	0.3262
4	<u>0.9892</u>	0.8812	0.3845
5	0.9892	0.8925	0.4389
6	0.9892	0.8981	0.4887
⋮	⋮	⋮	⋮
11	0.9892	0.9036	0.6699
12	0.9892	0.9037	0.6943
13	0.9892	<u>0.9038</u>	0.7157
14	0.9892	0.9038	0.7343
⋮	⋮	⋮	⋮
62	0.9892	0.9038	0.8511
63	0.9892	0.9038	0.8511
64	0.9892	0.9038	0.8511
65	0.9892	0.9038	0.8511
66	0.9892	0.9038	<u>0.8512</u>
67	0.9892	0.9038	0.8512
68	0.9892	0.9038	0.8512
69	0.9892	0.9038	0.8512
70	0.9892	0.9038	0.8512

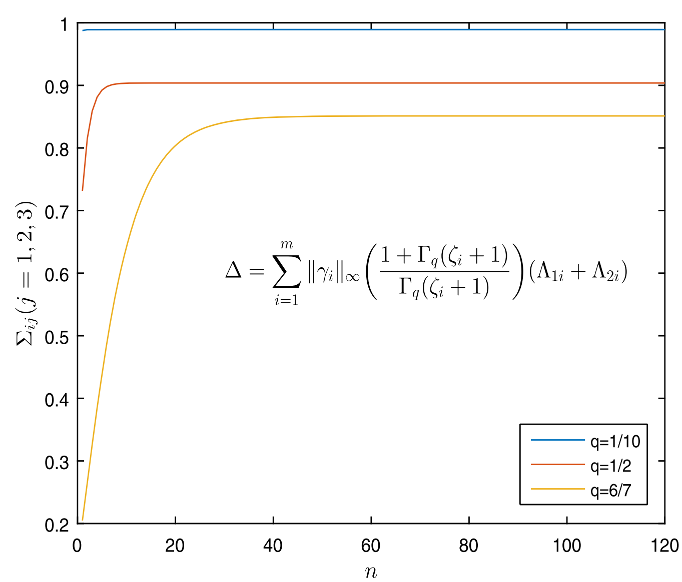

Figure 1 Numerical results of Δ where $q=\frac{1}{10},\frac{1}{2},\frac{6}{7}$ in Example 1

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