Aphithana et al. *Boundary Value Problems* (2019) 2019:47
https://doi.org/10.1186/s13661-019-1162-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Forced oscillation of fractional differential equations via conformable derivatives with damping term

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Abstract

Based on the properties of nonlocal fractional calculus generated by conformable derivatives, we establish some sufficient conditions for oscillation of all solutions for fractional differential equations with damping term. Forced oscillation of conformable differential equations in the frame of Riemann, as well as of Caputo type, is established. Examples are provided to demonstrate the effectiveness of the main results.

MSC: 34A08; 34C10; 26A33

Keywords: Forced oscillation; Oscillation theory; Fractional differential equations; Fractional conformable integrals; Fractional conformable derivatives; Damping

1 Introduction

Fractional differential equations gained considerable importance due to their various applications in viscoelasticity, electroanalytical chemistry, control theory, many physical problems, etc. The books [1–6] summarize and organize much of fractional calculus and many of theories and applications of fractional differential equations. Many authors have studied the existence and uniqueness of solutions for different types of fractional boundary value problems; see the papers [7–18] and the references cited therein.

The oscillation theory for fractional differential and difference equations has been studied by some authors (see [19–29]). In [23] the authors studied the oscillation theory for fractional differential equations by considering fractional initial value problem of the form

$$\begin{cases} \mathcal{D}_a^q x(t) + f_1(t,x) = v(t) + f_2(t,x), & t > a, \\ \lim_{t\to a+} J_a^{1-q} x(t) = b, \end{cases} \tag{1.1}$$

where $\mathcal{D}_a^q$ denotes the Riemann–Liouville fractional derivative starting at a point a, of order q with $0 < q \le 1$, J_a^{1-q} is the Riemann–Liouville fractional integral starting at a point a, of order $1-q$, f_1, f_2 are continuous functions.

Recently, in [21] the authors studied the oscillation of a conformable initial value problem of the form

$$\begin{cases} {}_a\mathfrak{D}^{\alpha,\rho}x(t) + f_1(t,x) = r(t) + f_2(t,x), \quad t > a, \\ \lim_{t\to a^+} {}_a\mathfrak{J}^{j-\alpha,\rho}x(t) = b_j \quad (j = 1,2,\ldots,m), \end{cases} \tag{1.2}$$

where $m = \lceil\alpha\rceil = \min\{m \in \mathbb{Z} | m \geq \alpha\}$, ${}_a\mathfrak{D}^{\alpha,\rho}$ is the left conformable derivative of order $\alpha \in \mathbb{C}, \Re(\alpha) \geq 0$ in the Riemann–Liouville setting and ${}_a\mathfrak{J}^{\alpha,\rho}$ is the left conformable integral operator.

In [22] the authors studied forced oscillatory properties of solutions to the nonlinear fractional initial value problem with damping

$$\begin{cases} (\mathcal{D}^{1+\alpha}_{0^+}y)(t) + p(t)(\mathcal{D}^{\alpha}_{0^+}y)(t) + q(t)f(y(t)) = g(t), \quad t > 0, \\ (I^{1-\alpha}_{0^+}y)(0+) = b, \end{cases} \tag{1.3}$$

where b is a real number, $\alpha \in (0,1)$ is a given constant, and $\mathcal{D}^{\alpha}_{0^+}$ is the Riemann–Liouville fractional derivative of order α.

In this paper, motivated by the above papers, we study forced oscillatory properties of solutions to the conformable initial value problem with damping in the Riemann–Liouville setting as follows:

$$\begin{cases} {}_a\mathfrak{D}^{1+\alpha,\rho}x(t) + p(t){}_a\mathfrak{D}^{\alpha,\rho}x(t) + q(t)f(x(t)) = g(t), \quad t > a, \\ \lim_{t\to a^+} {}_a\mathfrak{J}^{j-\alpha,\rho}x(t) = b_j \quad (j = 1,2,\ldots,m), \end{cases} \tag{1.4}$$

where $m = \lceil\alpha\rceil$, $0 < \rho \leq 1$, $p \in \mathbb{C}(\mathbb{R}^+,\mathbb{R})$, $q \in \mathbb{C}(\mathbb{R}^+,\mathbb{R}^+)$, $g \in \mathbb{C}(\mathbb{R}^+,\mathbb{R})$, $f \in \mathbb{C}(\mathbb{R},\mathbb{R})$ are continuous functions, ${}_a\mathfrak{D}^{\alpha,\rho}$ is the left conformable derivative of order $\alpha \in \mathbb{C}, \Re(\alpha) \geq 0$ in the Riemann–Liouville setting, and ${}_a\mathfrak{J}^{\alpha,\rho}$ is the left conformable integral operator.

Moreover, we study the forced oscillation of conformable initial value problems in the Caputo setting of the form

$$\begin{cases} {}^C_a\mathfrak{D}^{1+\alpha,\rho}x(t) + p(t){}^C_a\mathfrak{D}^{\alpha,\rho}x(t) + q(t)f(x(t)) = g(t), \quad t > a, \\ {}^k_aD^{\rho}x(a) = b_k \quad (k = 0,1,\ldots,m-1), \end{cases} \tag{1.5}$$

where $m = \lceil\alpha\rceil$, and ${}^C_a\mathfrak{D}^{\alpha,\rho}x$ is the left conformable derivative of order $\alpha \in \mathbb{C}, \Re(\alpha) \geq 0$ in the Caputo setting.

Definition 1.1 The solution x of problem (1.4) (respectively (1.5)) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

This paper is organized as follows. Section 2 introduces some notations and provides the definitions of conformable fractional integral and differential operators together with some basic properties and lemmas that are needed in the proofs of the main theorems. In Sect. 3, forced oscillation of conformable fractional differential equations in the frame of Riemann is presented, while in Sect. 4 forced oscillation of conformable fractional differential equations in the frame of Caputo is established. Examples are provided in Sect. 5 to demonstrate the effectiveness of the main theorems.

2 Preliminaries

The left conformable derivative starting from a of a function $f : [a, \infty) \to \mathbb{R}$ of order $0 < \rho \leq 1$ is defined by

$$\left({}_aD^\rho f\right)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon(t-a)^{1-\rho}) - f(t)}{\epsilon}.$$

If $({}_aD^\rho f)(t)$ exists on (a,b), then $({}_aD^\rho f)(a) = \lim_{t\to a^+}({}_aD^\rho f)(t)$. If f is differentiable, then

$$\left({}_aD^\rho f\right)(t) = (t-a)^{1-\rho} f'(t). \tag{2.1}$$

The corresponding left conformable integral is defined as

$${}_aI^\rho f(x) = \int_a^x f(t) \frac{dt}{(t-a)^{1-\rho}}, \quad 0 < \rho \leq 1.$$

For the extension to the higher order $\rho > 1$, see [30].

Definition 2.1 ([31]) The left conformable integral operator is defined by

$${}_a\mathfrak{J}^{\alpha,\rho} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{(x-a)^\rho - (t-a)^\rho}{\rho} \right)^{\alpha-1} \frac{f(t)\,dt}{(t-a)^{1-\rho}}, \tag{2.2}$$

where $\alpha \in \mathbb{C}$, $\Re(\alpha) \geq 0$.

Definition 2.2 ([31]) The left conformable derivative of order $\alpha \in \mathbb{C}$, $\Re(\alpha) \geq 0$ in the Riemann–Liouville setting is defined by

$$\begin{aligned} {}_a\mathfrak{D}^{\alpha,\rho} f(x) &= {}_a^mD^\rho \left({}_a\mathfrak{J}^{m-\alpha,\rho}\right) f(x) \\ &= \frac{{}_a^mD^\rho}{\Gamma(m-\alpha)} \int_a^x \left(\frac{(x-a)^\rho - (t-a)^\rho}{\rho} \right)^{m-\alpha-1} \frac{f(t)\,dt}{(t-a)^{1-\rho}}, \end{aligned} \tag{2.3}$$

where $m = \lceil \Re(\alpha) \rceil$, ${}_a^mD^\rho = \underbrace{{}_aD^\rho\,{}_aD^\rho \cdots {}_aD^\rho}_{m \text{ times}}$, and ${}_aD^\rho f$ is the left conformable differential operator presented in (2.1).

Definition 2.3 ([31]) The left Caputo conformable derivative of order $\alpha \in \mathbb{C}$, $\Re(\alpha) \geq 0$ is defined by

$$\begin{aligned} {}_a^C\mathfrak{D}^{\alpha,\rho} f(x) &= {}_a\mathfrak{J}^{m-\alpha,\rho} \left({}_a^mD^\rho f(x)\right) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_a^x \left(\frac{(x-a)^\rho - (t-a)^\rho}{\rho} \right)^{m-\alpha-1} \frac{{}_a^mD^\rho f(t)\,dt}{(t-a)^{1-\rho}}, \end{aligned} \tag{2.4}$$

where $m = \lceil \Re(\alpha) \rceil$, ${}_a^mD^\rho = \underbrace{{}_aD^\rho\,{}_aD^\rho \cdots {}_aD^\rho}_{m \text{ times}}$, and ${}_aD^\rho f$ is the left conformable differential operator presented in (2.1).

Lemma 2.1 ([31]) *Let $\alpha \in \mathbb{C}$ and ${}_a\mathfrak{I}^{j-\alpha,\rho}x(t)$ be the conformable integral (2.2) of order $j-\alpha$, then*

$$ {}_a\mathfrak{I}^{\alpha,\rho}\big({}_a\mathfrak{D}^{\alpha,\rho}f(x)\big) = f(x) - \sum_{j=1}^{m} \frac{{}_a\mathfrak{I}^{j-\alpha,\rho}f(a)}{\rho^{\alpha-j}\Gamma(\alpha-j+1)}(x-a)^{\rho(\alpha-j)}. \tag{2.5} $$

3 Forced oscillation of conformable differential equations in the frame of Riemann

In this section we study the oscillation theory for equation (1.4). We prove our result under the following assumption:

(H) $p \in \mathbb{C}(\mathbb{R}^+,\mathbb{R})$, $q \in \mathbb{C}(\mathbb{R}^+,\mathbb{R}^+)$, $g \in \mathbb{C}(\mathbb{R}^+,\mathbb{R})$, $f \in \mathbb{C}(\mathbb{R},\mathbb{R})$ and $f(u)/u > 0$ for all $u \neq 0$.

We set

$$ \Phi(t) = \Gamma(\alpha)\sum_{j=1}^{m}\frac{b_j(t-a)^{\rho(\alpha-j)}}{\rho^{\alpha-j}\Gamma(\alpha-j+1)} \tag{3.1} $$

and

$$ \Lambda(t,T) = \int_a^T \left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left(\frac{M+{}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} \tag{3.2} $$

for $a \le t \le T$, where $M = {}_a\mathfrak{D}^{\alpha,\rho}x(t_1)V(t_1)$.

Theorem 3.1 *Suppose that (H) and for every sufficiently large T the following conditions hold*:

$$ \liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_T^t\left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left(\frac{M+{}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} = -\infty \tag{3.3} $$

and

$$ \limsup_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_T^t\left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left(\frac{M+{}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} = \infty, \tag{3.4} $$

where $V(t) = \exp\int_{t_1}^t (s-a)^{\rho-1}p(s)\,ds$, $t_1 > a$, and M is an arbitrary constant. Then every solution of problem (1.4) is oscillatory.

Proof Let x be a nonoscillatory solution of problem (1.4). Without loss of generality, suppose that $T > a$ is large enough and $t_1 > T$ so that $x(t) > 0$ for $t > t_1$. According to (1.4) and (H), the following inequality is satisfied:

$$ \begin{aligned} {}_aD^\rho\big[{}_a\mathfrak{D}^{\alpha,\rho}x(t)V(t)\big] &= (t-a)^{1-\rho}\frac{d}{dt}\big[{}_a\mathfrak{D}^{\alpha,\rho}x(t)V(t)\big] \\ &= (t-a)^{1-\rho}\frac{d}{dt}\big({}_a\mathfrak{D}^{\alpha,\rho}x(t)\big)V(t) + (t-a)^{1-\rho}\,{}_a\mathfrak{D}^{\alpha,\rho}x(t)\frac{d}{dt}\big(V(t)\big) \end{aligned} $$

$$= {}_a\mathfrak{D}^{\alpha+1,\rho}x(t)V(t) + p(t){}_a\mathfrak{D}^{\alpha,\rho}x(t)V(t)$$
$$= -q(t)f\big(x(t)\big)V(t) + g(t)V(t)$$
$$< g(t)V(t).$$

Taking the left conformable integral order ρ for the above inequality from t_1 to t, we can obtain

$$\begin{aligned} {}_a\mathfrak{D}^{\alpha,\rho}x(t)V(t) &< {}_a\mathfrak{D}^{\alpha,\rho}x(t_1)V(t_1) + {}_{t_1}I^{\rho}\big(g(t)V(t)\big) \\ &= M + {}_{t_1}I^{\rho}\big(g(t)V(t)\big). \end{aligned} \tag{3.5}$$

From Lemma 2.1 and (3.5) we get

$${}_a\mathfrak{D}^{\alpha,\rho}x(t) < \frac{M + {}_{t_1}I^{\rho}(g(t)V(t))}{V(t)},$$

which leads to

$$x(t) - \sum_{j=1}^{m} \frac{{}_a\mathfrak{J}^{j-\alpha,\rho}x(a)(t-a)^{\rho(\alpha-j)}}{\rho^{\alpha-j}\Gamma(\alpha-j+1)} < {}_a\mathfrak{J}^{\alpha,\rho}\left[\frac{M + {}_{t_1}I^{\rho}(g(t)V(t))}{V(t)}\right].$$

So, we have

$$\begin{aligned} x(t) &< \sum_{j=1}^{m} \frac{b_j(t-a)^{\rho(\alpha-j)}}{\rho^{\alpha-j}\Gamma(\alpha-j+1)} \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{(t-a)^{\rho}-(w-a)^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{M + {}_{t_1}I^{\rho}(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} \end{aligned}$$

for every sufficiently large T. Multiplying both sides of the above inequality by $\Gamma(\alpha)$, we can obtain

$$\begin{aligned} \Gamma(\alpha)x(t) &< \Gamma(\alpha)\sum_{j=1}^{m} \frac{b_j(t-a)^{\rho(\alpha-j)}}{\rho^{\alpha-j}\Gamma(\alpha-j+1)} \\ &\qquad + \int_a^T \left(\frac{(t-a)^{\rho}-(w-a)^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{M + {}_{t_1}I^{\rho}(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} \\ &\qquad + \int_T^t \left(\frac{(t-a)^{\rho}-(w-a)^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{M + {}_{t_1}I^{\rho}(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} \\ &= \Phi(t) + \Lambda(t,T) \\ &\qquad + \int_T^t \left(\frac{(t-a)^{\rho}-(w-a)^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{M + {}_{t_1}I^{\rho}(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}}, \end{aligned} \tag{3.6}$$

where Φ and Λ are defined in (3.1) and (3.2), respectively.

Multiplying (3.6) by $(\frac{t^{\rho}}{\rho})^{1-\alpha}$, we get

$$0 < \left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha}\Gamma(\alpha)x(t)$$

$$
\begin{aligned}
&< \left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\Phi(t) + \left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\Lambda(t,T) \\
&\quad + \left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_T^t \left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left(\frac{M + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}}. \qquad (3.7)
\end{aligned}
$$

Taking $T_1 > T$, we consider two cases as follows.

Case (1): Let $0 < \alpha \le 1$. Then $m = 1$ and $(\frac{t^\rho}{\rho})^{1-\alpha}\Phi(t) = b_1 t^{\rho-\rho\alpha}(t-a)^{\rho\alpha-\rho}$. Since the function $h_1(t) = t^{\rho-\rho\alpha}(t-a)^{\rho\alpha-\rho}$ is decreasing for $\rho > 0$ and $\alpha < 1$, we get for $t \ge T_1$ (see [21])

$$
\left|\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\Phi(t)\right| \le |b_1| T_1^{\rho-\rho\alpha}(T_1-a)^{\rho\alpha-\rho} := c_1(T_1). \qquad (3.8)
$$

The function $h_2(t) = t^{\rho-\rho\alpha}[(t-a)^\rho - (w-a)^\rho]^{\alpha-1}$ is decreasing for $\rho > 0$ and $\alpha < 1$. Thus, we get

$$
\begin{aligned}
&\left|\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\Lambda(t,T)\right| \\
&= \left|\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_a^T \left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left(\frac{M + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}}\right| \\
&\le \int_a^T \left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left|\frac{M + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right|\frac{dw}{(w-a)^{1-\rho}} \\
&\le \int_a^T \left(\frac{T_1^\rho}{\rho}\right)^{1-\alpha}\left(\frac{(T_1-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left|\frac{M + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right|\frac{dw}{(w-a)^{1-\rho}} \\
&:= c_2(T,T_1). \qquad (3.9)
\end{aligned}
$$

Then, from equation (3.7) and $t \ge T_1$, we get

$$
\begin{aligned}
&\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_T^t \left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left(\frac{M + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} \\
&\ge -\left[c_1(T_1) + c_2(T,T_1)\right],
\end{aligned}
$$

hence

$$
\begin{aligned}
&\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_T^t \left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left(\frac{M + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} \\
&\ge -\left[c_1(T_1) + c_2(T,T_1)\right] \\
&> -\infty,
\end{aligned}
$$

which is a contradiction to condition (3.3).

Case (2): Let $\alpha > 1$. Then $m \ge 2$. Also $(\frac{t-a}{t})^{\rho\alpha-\rho} < 1$ for $\alpha > 1$ and $\rho > 0$. The function $h_3(t) = (t-a)^{\rho-\rho j}$ is decreasing for $j > 1$ and $\rho > 0$. Thus, for $t \ge T_1$, we have (see [21])

$$
\left|\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\Phi(t)\right| \le \Gamma(\alpha)\sum_{j=1}^m \frac{|b_j|(T_1-a)^{\rho-\rho j}}{\rho^{1-j}\Gamma(\alpha-j+1)} := c_3(T_1). \qquad (3.10)
$$

Also, since $(\frac{t^\rho}{\rho})^{1-\alpha} < 1$ and $(\frac{(t-a)^\rho-(w-a)^\rho}{t^\rho})^{\alpha-1} < 1$ for $\alpha > 1$ and $\rho > 0$, we get

$$\begin{aligned}
&\left|\left(\frac{t^\rho}{\rho}\right)^{1-\alpha} \Lambda(t,T)\right| \\
&\quad = \left|\left(\frac{t^\rho}{\rho}\right)^{1-\alpha} \int_a^T \left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1} \left(\frac{M + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right) \frac{dw}{(w-a)^{1-\rho}}\right| \\
&\quad \le \int_a^T \left(\frac{(t-a)^\rho-(w-a)^\rho}{t^\rho}\right)^{\alpha-1} \left|\frac{M + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right| \frac{dw}{(w-a)^{1-\rho}} \\
&\quad \le \int_a^T \left|\left(\frac{M + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\right| \frac{dw}{(w-a)^{1-\rho}} := c_4(T).
\end{aligned} \tag{3.11}$$

From (3.7), (3.10), and (3.11), we conclude that

$$\begin{aligned}
&\left(\frac{t^\rho}{\rho}\right)^{1-\alpha} \int_T^t \left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1} \left(\frac{M + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right) \frac{dw}{(w-a)^{1-\rho}} \\
&\quad \ge -\left[c_3(T_1) + c_4(T)\right]
\end{aligned}$$

for $t \ge T_1$. Hence

$$\begin{aligned}
&\liminf_{t\to\infty} \left(\frac{t^\rho}{\rho}\right)^{1-\alpha} \int_T^t \left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1} \left(\frac{M + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right) \frac{dw}{(w-a)^{1-\rho}} \\
&\quad \ge -\left[c_3(T_1) + c_4(T)\right] \\
&\quad > -\infty,
\end{aligned}$$

which is a contradiction to condition (3.3). Therefore, we get that $x(t)$ is oscillatory. In case x is eventually negative, similar arguments lead to a contradiction with condition (3.4). The proof is completed. □

4 Forced oscillation of conformable differential equations in the frame of Caputo

In this section, we study the forced oscillation of conformable initial value problem (1.5).

We set

$$\Psi(t) = \Gamma(\alpha) \sum_{k=0}^{m-1} \frac{b_k (t-a)^{\rho k}}{\rho^k k!} \tag{4.1}$$

and

$$\Omega(t,T) = \int_a^T \left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1} \left(\frac{M^* + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right) \frac{dw}{(w-a)^{1-\rho}} \tag{4.2}$$

for $a \le t \le T$, where $M^* = {}_a^C\mathfrak{D}^{\alpha,\rho} x(t_1)V(t_1)$.

Lemma 4.1 [31] *Let* $f \in C^m_{\rho,a}[a,b]$ *and* $\alpha \in \mathbb{C}$, *then*

$${}_a\mathfrak{I}^{\alpha,\rho}\left({}_a^C\mathfrak{D}^{\alpha,\rho} f(x)\right) = f(x) - \sum_{k=0}^{m-1} \frac{{}_a^k D^\rho f(a)(x-a)^{\rho k}}{\rho^k k!}. \tag{4.3}$$

Lemma 4.2 [31] *Let $\alpha, \beta \in \mathbb{C}$. If the conformable derivatives ${}_a^C\mathfrak{D}^{\alpha,\rho}f(x)$ and ${}_a^C\mathfrak{D}^{\alpha+\beta,\rho}f(x)$ exist, then*

$$ {}_a^C\mathfrak{D}^{\alpha,\rho}\big({}_a^C\mathfrak{D}^{\beta,\rho}f(x)\big) = {}_a^C\mathfrak{D}^{\alpha+\beta,\rho}f(x). \tag{4.4}$$

Lemma 4.3 [31] *Let $\alpha \in \mathbb{C}$, $m = \lceil \Re(\alpha) \rceil$. If $\alpha \in \mathbb{N}$, then*

$$ {}_a^C\mathfrak{D}^{\alpha,\rho}f(x) = {}_a^m D^{\rho}f(x). \tag{4.5}$$

Theorem 4.1 *Suppose that (H) and for every sufficiently large T the following conditions hold*:

$$\begin{aligned} &\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_T^t\left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1} \\ &\quad\times\left(\frac{M^*+{}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}}=-\infty \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\limsup_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_T^t\left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1} \\ &\quad\times\left(\frac{M^*+{}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}}=\infty, \end{aligned} \tag{4.7}$$

where $V(t) = \exp\int_{t_1}^t (s-a)^{\rho-1}p(s)\,ds$, $t_1 > a$, and M^* is an arbitrary constant. Then every solution of problem (1.5) is oscillatory.

Proof Let x be a nonoscillatory solution of problem (1.5). Without loss of generality, suppose that $T > a$ is large enough and $t_1 > T$ so that $x(t) > 0$ for $t > t_1$. According to (1.5) and (H), the following inequality is satisfied:

$$\begin{aligned} {}_aD^\rho\big[{}_a^C\mathfrak{D}^{\alpha,\rho}x(t)V(t)\big] &= (t-a)^{1-\rho}\frac{d}{dt}\big[{}_a^C\mathfrak{D}^{\alpha,\rho}x(t)V(t)\big] \\ &= (t-a)^{1-\rho}\frac{d}{dt}\big({}_a^C\mathfrak{D}^{\alpha,\rho}x(t)\big)V(t) + (t-a)^{1-\rho}{}_a^C\mathfrak{D}^{\alpha,\rho}x(t)\frac{d}{dt}\big(V(t)\big) \\ &= {}_a^C\mathfrak{D}^{\alpha+1,\rho}x(t)V(t) + p(t){}_a^C\mathfrak{D}^{\alpha,\rho}x(t)V(t) \\ &= -q(t)f\big(x(t)\big)V(t) + g(t)V(t) \\ &< g(t)V(t). \end{aligned}$$

Taking the left conformable integral of order ρ to the above inequality from t_1 to t, we can obtain

$$\begin{aligned} {}_a^C\mathfrak{D}^{\alpha,\rho}x(t)V(t) &< {}_a^C\mathfrak{D}^{\alpha,\rho}x(t_1)V(t_1) + {}_{t_1}I^\rho\big(g(t)V(t)\big) \\ &= M^* + {}_{t_1}I^\rho\big(g(t)V(t)\big). \end{aligned} \tag{4.8}$$

From Lemma 4.1 and (4.8) we have

$$ {}_{a}^{C}\mathfrak{D}^{\alpha,\rho}x(t) < \frac{M^{*} + {}_{t_1}I^{\rho}(g(t)V(t))}{V(t)}. $$

Then we get

$$ x(t) - \sum_{k=0}^{m-1} \frac{{}_{a}^{C}D^{\rho}x(a)(t-a)^{\rho k}}{\rho^{k}k!} < {}_{a}\mathfrak{J}^{\alpha,\rho}\left[\frac{M^{*} + {}_{t_1}I^{\rho}(g(t)V(t))}{V(t)}\right]. $$

So, we have

$$ \begin{aligned} x(t) < {} & \sum_{k=0}^{m-1} \frac{{}_{a}^{C}D^{\rho}x(a)(t-a)^{\rho k}}{\rho^{k}k!} \\ & + \frac{1}{\Gamma(\alpha)}\int_{a}^{t}\left(\frac{(t-a)^{\rho}-(w-a)^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{M^{*} + {}_{t_1}I^{\rho}(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} \end{aligned} $$

for every sufficiently large T. Multiplying both sides of the above inequality by a constant $\Gamma(\alpha)$, we have

$$ \begin{aligned} \Gamma(\alpha)x(t) < {} & \Gamma(\alpha)\sum_{k=0}^{m-1} \frac{{}_{a}^{C}D^{\rho}x(a)(t-a)^{\rho k}}{\rho^{k}k!} \\ & + \int_{a}^{T}\left(\frac{(t-a)^{\rho}-(w-a)^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{M^{*} + {}_{t_1}I^{\rho}(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} \\ & + \int_{T}^{t}\left(\frac{(t-a)^{\rho}-(w-a)^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{M^{*} + {}_{t_1}I^{\rho}(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} \\ = {} & \Psi(t) + \Omega(t,T) \\ & + \int_{T}^{t}\left(\frac{(t-a)^{\rho}-(w-a)^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{M^{*} + {}_{t_1}I^{\rho}(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}}, \end{aligned} \tag{4.9} $$

where Ψ and Ω are defined in (4.1) and (4.2), respectively.

Multiplying (4.9) by $(\frac{t^{\rho}}{\rho})^{1-m}$, we get

$$ \begin{aligned} 0 & < \left(\frac{t^{\rho}}{\rho}\right)^{1-m}\Gamma(\alpha)x(t) \\ & < \left(\frac{t^{\rho}}{\rho}\right)^{1-m}\Psi(t) + \left(\frac{t^{\rho}}{\rho}\right)^{1-m}\Omega(t,T) \\ & \quad + \left(\frac{t^{\rho}}{\rho}\right)^{1-m}\int_{T}^{t}\left(\frac{(t-a)^{\rho}-(w-a)^{\rho}}{\rho}\right)^{\alpha-1} \\ & \quad \times \left(\frac{M^{*} + {}_{t_1}I^{\rho}(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}}. \end{aligned} \tag{4.10} $$

Take $T_1 > T$. We consider two cases as follows.

Case (1): Let $0 < \alpha \leq 1$. Then $m = 1$ and $(\frac{t^\rho}{\rho})^{1-m}\Psi(t) = \Gamma(\alpha)b_0$.

The function $h_4(t) = (\frac{(t-a)^\rho-(w-a)^\rho}{\rho})^{\alpha-1}$ is decreasing for $\rho > 0$, $t > T_1 > w$, and $\alpha < 1$. Thus, we get

$$\begin{aligned}
&\left|\left(\frac{t^\rho}{\rho}\right)^{1-m}\Omega(t,T)\right| \\
&\quad = \left|\int_a^T \left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left(\frac{M^* + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}}\right| \\
&\quad \leq \int_a^T \left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left|\frac{M^* + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right|\frac{dw}{(w-a)^{1-\rho}} \\
&\quad \leq \int_a^T \left(\frac{(T_1-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left|\frac{M^* + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right|\frac{dw}{(w-a)^{1-\rho}} \\
&\quad := c_5(T,T_1).
\end{aligned} \tag{4.11}$$

Then, from equation (4.10) and $t \geq T_1$, we get

$$\begin{aligned}
&\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_T^t \left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left(\frac{M^* + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} \\
&\quad \geq -\left[\Gamma(\alpha)b_0 + c_5(T,T_1)\right],
\end{aligned}$$

hence

$$\begin{aligned}
&\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_T^t \left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left(\frac{M^* + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} \\
&\quad \geq -\left[\Gamma(\alpha)b_0 + c_5(T,T_1)\right] \\
&\quad > -\infty,
\end{aligned}$$

which contradicts condition (4.6).

Case (2): Let $\alpha > 1$. Then $m \geq 2$. Also $(\frac{t-a}{t})^{\rho m-\rho} < 1$ for $m \geq 2$ and $\rho > 0$. The function $h_5(t) = (t-a)^{\rho(k-m+1)}$ is decreasing for $k < m-1$ and $\rho > 0$. Thus, for $t \geq T_1$, we have

$$\begin{aligned}
\left|\left(\frac{t^\rho}{\rho}\right)^{1-m}\Psi(t)\right| &= \left|\left(\frac{t^\rho}{\rho}\right)^{1-m}\Gamma(\alpha)\sum_{k=0}^{m-1}\frac{b_k(t-a)^{\rho k}}{\rho^k k!}\right| \\
&= \left|\left(\frac{t-a}{t}\right)^{\rho m-\rho}\Gamma(\alpha)\sum_{k=0}^{m-1}\frac{b_k(t-a)^{\rho(k-m+1)}}{\rho^{k-m+1}k!}\right| \\
&\leq \Gamma(\alpha)\sum_{k=0}^{m-1}\frac{|b_k|(t-a)^{\rho(k-m+1)}}{\rho^{k-m+1}k!} \\
&\leq \Gamma(\alpha)\sum_{k=0}^{m-1}\frac{|b_k|(T_1-a)^{\rho(k-m+1)}}{\rho^{k-m+1}k!} := c_6(T_1).
\end{aligned} \tag{4.12}$$

Also, since $(\frac{t^\rho}{\rho})^{1-m} < 1$ and $(\frac{(t-a)^\rho-(w-a)^\rho}{t^\rho})^{\alpha-1} < 1$ for $\alpha > 1$ and $\rho > 0$, we get

$$\begin{aligned}
&\left|\left(\frac{t^\rho}{\rho}\right)^{1-m}\Omega(t,T)\right| \\
&\quad = \left|\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_a^T\left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left(\frac{M^* + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}}\right| \\
&\quad \le \int_a^T\left(\frac{(t-a)^\rho-(w-a)^\rho}{t^\rho}\right)^{\alpha-1}\left|\frac{M^* + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right|\frac{dw}{(w-a)^{1-\rho}} \\
&\quad \le \int_a^T\left|\left(\frac{M^* + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\right|\frac{dw}{(w-a)^{1-\rho}} := c_7(T).
\end{aligned} \tag{4.13}$$

From (4.10), (4.12), and (4.13), we conclude that

$$\begin{aligned}
&\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_T^t\left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left(\frac{M^* + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} \\
&\quad \ge -\left[c_6(T_1)+c_7(T)\right]
\end{aligned}$$

for $t \ge T_1$. Hence

$$\begin{aligned}
&\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_T^t\left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left(\frac{M^* + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} \\
&\quad \ge -\left[c_6(T_1)+c_7(T)\right] \\
&\quad > -\infty,
\end{aligned}$$

which contradicts condition (4.6). Therefore, we conclude that x is oscillatory. In case x is eventually negative, similar arguments lead to a contradiction with condition (4.7). The proof is completed. □

5 Examples

In this section, we present examples to illustrate our results.

Example 5.1 Consider the conformable initial value problem

$$\begin{cases}
{}_0\mathfrak{D}^{\frac{3}{2},1}x(t) - {}_0\mathfrak{D}^{\frac{1}{2},1}x(t) + (t+5)^2(2x+5)e^{\sin 2x} = e^{2t}\cos t, & t > 0, \\
\lim_{t\to 0^+} {}_0\mathfrak{I}^{\frac{1}{2},1}x(t) = 0.
\end{cases} \tag{5.1}$$

Here $\alpha = 1/2$, $\rho = 1$, $a = 0$, $p(t) = -1$, $q(t) = (t+5)^2$, $f(x) = (2x+5)e^{\sin 2x}$, $g(t) = e^{2t}\cos t$, and $V(s) = e^{t_1-s}$. It is easy to verify that assumption (H) is satisfied if $x(t) > 0$. Then

$$\begin{aligned}
{}_{t_1}I^1\big(g(w)V(w)\big) &= \int_{t_1}^w g(s)V(s)\,ds \\
&= \int_{t_1}^w e^{2s}\cos s e^{t_1-s}\,ds \\
&= \frac{e^{t_1+w}}{2}(\sin w + \cos w) - \frac{e^{2t_1}}{2}(\sin t_1 + \cos t_1)
\end{aligned}$$

$$= \frac{e^{t_1+w}}{2}\sqrt{2}\sin\left(w+\frac{\pi}{4}\right) - \frac{e^{2t_1}}{2}(\sin t_1 + \cos t_1).$$

Set $t_1 = \pi/2$. Hence, we can obtain

$$\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_0^t\left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}\left(\frac{M+{}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}}$$
$$= t^{\frac{1}{2}}\int_0^t (t-w)^{-\frac{1}{2}}e^{w-\frac{\pi}{2}}\left(\left(M-\frac{e^\pi}{2}\right)+\frac{\sqrt{2}}{2}e^{\frac{\pi}{2}+w}\sin\left(w+\frac{\pi}{4}\right)\right)dw.$$

Set $t-w=s^2$, then the above integral can be written as the following form:

$$t^{\frac{1}{2}}\int_{\sqrt{t}}^0\frac{1}{s}e^{t-s^2-\frac{\pi}{2}}\left(\left(\frac{2M-e^\pi}{2}\right)+\frac{\sqrt{2}}{2}e^{\frac{\pi}{2}+t-s^2}\sin\left(t-s^2+\frac{\pi}{4}\right)\right)(-2s)\,ds$$
$$= t^{\frac{1}{2}}\left(2M-e^\pi\right)e^{t-\frac{\pi}{2}}\int_0^{\sqrt{t}}e^{-s^2}\,ds + t^{\frac{1}{2}}\sqrt{2}e^{2t}\int_0^{\sqrt{t}}e^{-2s^2}\sin\left(t-s^2+\frac{\pi}{4}\right)ds$$
$$= t^{\frac{1}{2}}\left(2M-e^\pi\right)e^{t-\frac{\pi}{2}}\int_0^{\sqrt{t}}e^{-s^2}\,ds + t^{\frac{1}{2}}\sqrt{2}e^{2t}\sin\left(t+\frac{\pi}{4}\right)\int_0^{\sqrt{t}}e^{-2s^2}\cos s^2\,ds$$
$$- t^{\frac{1}{2}}\sqrt{2}e^{2t}\cos\left(t+\frac{\pi}{4}\right)\int_0^{\sqrt{t}}e^{-2s^2}\sin s^2\,ds.$$

Let $t\to+\infty$, as the result of $|e^{-2s^2}\cos s^2|\le e^{-2s^2}$, $|e^{-2s^2}\sin s^2|\le e^{-2s^2}$ and $\lim_{t\to+\infty}\int_0^{\sqrt{t}}e^{-2s^2}\,ds=\frac{\sqrt{2\pi}}{4}$. So, we know that

$$\lim_{t\to+\infty}\int_0^{\sqrt{t}}e^{-2s^2}\cos s^2\,ds \quad\text{and}\quad \lim_{t\to+\infty}\int_0^{\sqrt{t}}e^{-2s^2}\sin s^2\,ds$$

are convergent.

Thus, we can set $\lim_{t\to+\infty}\int_0^{\sqrt{t}}e^{-2s^2}\cos s^2\,ds=A$, $\lim_{t\to+\infty}\int_0^{\sqrt{t}}e^{-2s^2}\sin s^2\,ds=B$. Select the sequence $\{t_k\}=\{\frac{7\pi}{2}-\frac{\pi}{4}+2k\pi-\arctan\frac{-B}{A}\}$, $\lim_{k\to\infty}t_k=\infty$, then we calculate the following term:

$$\lim_{k\to\infty}\left\{t_k^{\frac{1}{2}}e^{t_k}\left[\left(2M-e^\pi\right)e^{-\frac{\pi}{2}}\int_0^{\sqrt{t_k}}e^{-s^2}\,ds+\sqrt{2}e^{t_k}\left(\sin\left(t_k+\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}}e^{-2s^2}\cos s^2\,ds\right.\right.\right.$$
$$\left.\left.\left.-\cos\left(t_k+\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}}e^{-2s^2}\sin s^2\,ds\right)\right]\right\}. \tag{5.2}$$

Firstly, we consider the following limit:

$$\lim_{k\to\infty}\left(\sin\left(t_k+\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}}e^{-2s^2}\cos s^2\,ds-\cos\left(t_k+\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}}e^{-2s^2}\sin s^2\,ds\right)$$
$$= A\cdot\lim_{k\to\infty}\sin\left(\frac{7\pi}{2}+2k\pi-\arctan\frac{-B}{A}\right)-B\cdot\lim_{k\to\infty}\cos\left(\frac{7\pi}{2}+2k\pi-\arctan\frac{-B}{A}\right)$$
$$= A\cdot\sin\left(\frac{7\pi}{2}-\arctan\frac{-B}{A}\right)-B\cdot\cos\left(\frac{7\pi}{2}-\arctan\frac{-B}{A}\right)$$
$$= -\sqrt{A^2+B^2}.$$

Secondly, we know that $\lim_{k\to\infty} t_k^{\frac{1}{2}} e^{t_k} = +\infty$ and $\lim_{k\to\infty}(2M-e^{\pi})e^{-\frac{\pi}{2}} \int_0^{\sqrt{t_k}} e^{-s^2}\,ds = (2M-e^{\pi})e^{-\frac{\pi}{2}} \frac{\sqrt{\pi}}{2}$. Hence, for (5.2), we have

$$\begin{aligned}
&\lim_{k\to\infty}\left\{t_k^{\frac{1}{2}} e^{t_k}\left[(2M-e^{\pi})e^{-\frac{\pi}{2}}\int_0^{\sqrt{t_k}} e^{-s^2}\,ds + \sqrt{2}e^{t_k}\left(\sin\left(t_k+\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{-2s^2}\cos s^2\,ds\right.\right.\right.\\
&\qquad\left.\left.\left.-\cos\left(t_k+\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{-2s^2}\sin s^2\,ds\right)\right]\right\}\\
&= (+\infty)\cdot\left[(2M-e^{\pi})e^{-\frac{\pi}{2}}\frac{\sqrt{\pi}}{2} + (+\infty)\left(-\sqrt{A^2+B^2}\right)\right]\\
&= -\infty.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\lim_{t\to\infty}\inf\left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha}\int_0^t\left(\frac{(t-a)^{\rho}-(w-a)^{\rho}}{\rho}\right)^{\alpha-1}\\
&\quad\times\left(\frac{M+{}_{t_1}I^{\rho}(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} = -\infty.
\end{aligned}$$

Similarly, selecting the sequence $\{t_l\} = \{\frac{5\pi}{2} - \frac{\pi}{4} + 2l\pi - \arctan\frac{-B}{A}\}$, we can obtain

$$\lim_{t\to\infty}\sup\left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha}\int_0^t\left(\frac{(t-a)^{\rho}-(w-a)^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{M+{}_{t_1}I^{\rho}(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} = \infty.$$

Hence, by Theorem 3.1 all solutions of (5.1) are oscillatory.

Example 5.2 Consider the Caputo conformable initial value problem

$$\begin{cases} {}_0^C\mathfrak{D}^{\frac{3}{2},1}x(t) - {}_0^C\mathfrak{D}^{\frac{1}{2},1}x(t) + e^{t^2}\ln(x+e) = e^{2t}\sin t, & t>0,\\ x(0)=0. \end{cases} \tag{5.3}$$

Here $\alpha = 1/2$, $\rho = 1$, $a = 0$, $m = 1$, $p(t) = -1$, $q(t) = e^{t^2}$, $f(x) = \ln(x+e)$, $g(t) = e^{2t}\sin t$, and $V(s) = e^{t_1-s}$. Thus assumption (H) is satisfied. Then we have

$${}_{t_1}I^1\left(g(w)V(w)\right) = \frac{e^{t_1+w}}{2}\sqrt{2}\sin\left(w-\frac{\pi}{4}\right) - \frac{e^{2t_1}}{2}(\sin t_1 - \cos t_1).$$

By setting $t_1 = \pi/4$ and $t-w = s^2$, we obtain

$$\begin{aligned}
&\left(\frac{t^{\rho}}{\rho}\right)^{1-m}\int_0^t\left(\frac{(t-a)^{\rho}-(w-a)^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{M^*+{}_{t_1}I^{\rho}(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}}\\
&\quad= \int_0^t (t-w)^{-\frac{1}{2}}e^{w-\frac{\pi}{4}}\left(M^* + \frac{\sqrt{2}}{2}e^{\frac{\pi}{4}+w}\sin\left(w-\frac{\pi}{4}\right)\right)dw
\end{aligned}$$

and

$$\int_{\sqrt{t}}^0 \frac{1}{s}e^{t-s^2-\frac{\pi}{4}}\left(M^* + \frac{\sqrt{2}}{2}e^{\frac{\pi}{4}+t-s^2}\sin\left(t-s^2-\frac{\pi}{4}\right)\right)(-2s)\,ds$$

$$= 2M^* e^{t-\frac{\pi}{4}} \int_0^{\sqrt{t}} e^{-s^2}\,ds + \sqrt{2}e^{2t}\sin\left(t-\frac{\pi}{4}\right)\int_0^{\sqrt{t}} e^{-2s^2}\cos s^2\,ds$$
$$- \sqrt{2}e^{2t}\cos\left(t-\frac{\pi}{4}\right)\int_0^{\sqrt{t}} e^{-2s^2}\sin s^2\,ds,$$

respectively. Using the method in Example 5.1, we choose a sequence

$$\{t_k\} = \left\{\frac{3\pi}{2} + \frac{\pi}{4} + 2k\pi - \arctan\frac{-B}{A}\right\},$$

where the constants A and B are defined in Example 5.1. Then we calculate

$$\lim_{k\to\infty}\left(\sin\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{-2s^2}\cos s^2\,ds - \cos\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{-2s^2}\sin s^2\,ds\right)$$
$$= A\cdot\lim_{k\to\infty}\sin\left(\frac{3\pi}{2} + 2k\pi - \arctan\frac{-B}{A}\right) - B\cdot\lim_{k\to\infty}\cos\left(\frac{3\pi}{2} + 2k\pi - \arctan\frac{-B}{A}\right)$$
$$= -\sqrt{A^2+B^2}$$

and

$$\lim_{k\to\infty}\left\{e^{t_k}\left[2M^* e^{-\frac{\pi}{4}}\int_0^{\sqrt{t_k}} e^{-s^2}\,ds + \sqrt{2}e^{t_k}\left(\sin\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{-2s^2}\cos s^2\,ds\right.\right.\right.$$
$$\left.\left.\left.- \cos\left(t_k-\frac{\pi}{4}\right)\int_0^{\sqrt{t_k}} e^{-2s^2}\sin s^2\,ds\right)\right]\right\}$$
$$= (+\infty)\cdot\left[2M^* e^{-\frac{\pi}{4}}\frac{\sqrt{\pi}}{2} + (+\infty)\left(-\sqrt{A^2+B^2}\right)\right] = -\infty.$$

Then we obtain

$$\lim_{t\to\infty}\inf\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_0^t\left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}$$
$$\times\left(\frac{M^* + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} = -\infty.$$

Similarly, by selecting the sequence $\{t_l\} = \{\frac{\pi}{2} + \frac{\pi}{4} + 2l\pi - \arctan\frac{-B}{A}\}$, we can obtain

$$\lim_{t\to\infty}\sup\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_0^t\left(\frac{(t-a)^\rho-(w-a)^\rho}{\rho}\right)^{\alpha-1}$$
$$\times\left(\frac{M^* + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right)\frac{dw}{(w-a)^{1-\rho}} = \infty.$$

Hence, by Theorem 4.1 all the solutions of (5.3) are oscillatory.

Example 5.3 By direct computation, we can find that the function $x(t) = -t^2$ is a nonoscillatory solution of problem

$$\begin{cases} {}_0\mathfrak{D}^{\frac{3}{2},1}x(t) + \sqrt{t}\left(\frac{4}{\sqrt{\pi}} + \frac{e^{\sqrt{x}}}{x^{\frac{1}{4}}}\right) = e^t, & t > 0, \\ \lim_{t\to 0^+} {}_0\mathfrak{I}^{\frac{1}{2},1}x(t) = 0. \end{cases} \tag{5.4}$$

Next we will show that condition (3.3) does not hold by setting $\alpha = 1/2$, $\rho = 1$, $a = 0$, $p(t) = 0$, $q(t) = \sqrt{t}$, $f(x) = ((4/\sqrt{\pi}) + (e^{\sqrt{x}}/x^{1/4}))$, $g(t) = e^t$, and $V(s) = 1$. It is obvious that (H) is satisfied. Therefore, we get

$$ {}_{t_1}I^1\bigl(g(w)V(w)\bigr) = e^w - e^{t_1}. $$

By setting $t_1 = 1$, we obtain

$$ \begin{aligned} &\left(\frac{t^\rho}{\rho}\right)^{1-\alpha} \int_0^t \left(\frac{(t-a)^\rho - (w-a)^\rho}{\rho}\right)^{\alpha-1} \left(\frac{M + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right) \frac{dw}{(w-a)^{1-\rho}} \\ &\quad = 2t^{\frac{1}{2}} \left((M-e)\sqrt{t} + e^t \int_0^{\sqrt{t}} e^{-s^2}\,ds\right), \end{aligned} $$

which yields

$$ \begin{aligned} &\lim_{t\to\infty} \inf \left(\frac{t^\rho}{\rho}\right)^{1-\alpha} \int_0^t \left(\frac{(t-a)^\rho - (w-a)^\rho}{\rho}\right)^{\alpha-1} \left(\frac{M + {}_{t_1}I^\rho(g(w)V(w))}{V(w)}\right) \frac{dw}{(w-a)^{1-\rho}} \\ &\quad = \lim_{t\to\infty} \inf \left\{ 2t^{\frac{1}{2}} \left((M-e)\sqrt{t} + e^t \int_0^{\sqrt{t}} e^{-s^2}\,ds\right)\right\} \\ &\quad = (+\infty) \cdot \left[(M-e)(+\infty) + (+\infty)\frac{\sqrt{\pi}}{2}\right] = \infty. \end{aligned} $$

6 Conclusion

In this paper force oscillatory properties of solutions of conformable differential equations with damping term are established. The cases of conformable differential equations in the frame of Riemann and Caputo type are considered. A sufficient condition for oscillation of all solutions is given. The obtained results are illustrated by numerical examples. Moreover, a counterexample is presented to show the existence of a nonoscillatory solution in case the conditions do not hold.

Acknowledgements
The authors express their deep gratitude to the referees for their valuable suggestions and comments for improvement of the paper.

Funding
A. Aphithana is supported by the Thailand Research Fund through the Royal Golden Jubilee PhD Program (Grant No. PHD/0134/2558).

Availability of data and materials
Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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Publisher's Note

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Received: 11 November 2018 Accepted: 27 February 2019 Published online: 04 March 2019

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