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The stability of solutions for the Fornberg–Whitham equation in $L^1(\mathbb{R})$ space

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Abstract
The $L^2(\mathbb{R})$ conservation law of solutions for the nonlinear Fornberg–Whitham equation is derived. Making use of the Kruzkov's device of doubling the space variables, the stability of the solutions in $L^1(\mathbb{R})$ space is established under certain assumptions on the initial value.

MSC: 35G25; 35L05

Keywords: L^1 stability; Local strong solutions; The Fornberg–Whitham equation

1 Introduction

In this article, we investigate the Fornberg–Whitham(FW) equation

$$V_t - V_{txx} - V_x + \frac{3}{2}VV_x = \frac{9}{2}V_xV_{xx} + \frac{3}{2}VV_{xxx}, \tag{1}$$

which was first written down in Whitham [1]. The numerical and theoretical analysis of solutions for Eq. (1) are made in Fornberg and Whitham [2] in which the peakon solution

$$V(t,x) = \frac{8}{9}e^{-\frac{1}{2}|x-\frac{4}{3}t|} \tag{2}$$

is found.

Recently, Holmes and Thompson [3] have established the existence and uniqueness of the FW equation in the Besov space in both non-periodic and periodic cases and discussed the sharpness of continuity on the data-to-solution map. A Cauchy–Kowalevski type result, which guarantees the existence and uniqueness of real analytic solutions for Eq. (1), is given and the blow-up criterion for solutions is obtained in [3]. Haziot [4] employs the estimates derived from the FW equation itself and some conclusions in [5] to derive sufficient conditions on the initial value which lead to wave breaking of solutions. For the detailed discussion about the discovery of wave breaking, we refer the reader to [2, 5–8].

We know that the dynamic properties of the Fornberg–Whitham equation are related to those of the Camassa–Holm (CH) [9], Degasperis–Procesi (DP) [10], and Novikov equations [11]. The four types of equations possess the peakon solutions. Here, we recall several works on the study of the CH, DP, and Novikov equations. The well-posedness of the Cauchy problem for a generalized CH equation is established in Himonas and Holliman [12]. The nonuniform dependence of the periodic CH equation and the well-posedness of the DP equation are discussed in [13] and [14], respectively. The continuity properties of the data-to-solution map for the periodic b-family equation including the CH and DP equations are obtained in [15]. Coclite and Karlsen [16] discuss the existence and stability of the entropy solution for the DP equation. The existence and uniqueness of global solutions for the DP equation are studied in Liu and Yin [17] in the case that the initial data satisfy the sign condition. Escher et al. [18] investigate the global weak solutions and blow-up structure for the DP model under certain assumptions. Matsuno [19] finds out the multisoliton solutions of the DP equation and analyzes their peakon limits. The uniform stability of peakons for the Camassa–Holm model is established in Constantin and Strauss [7]. Using the conservation law and assuming that the initial data satisfy the sign condition, Lin and Liu [20] obtain the stability of peakons for the Degasperis–Procesi equation. The Cauchy problem for the Novikov equation is considered in [21]. A generalized Novikov model with peakon solutions is studied in [22]. For other studies of the CH, DP, and Novikov equations, the reader is referred to [21–29] and the references therein.

Motivated by the works made in Coclite and Karlsen [16], the aim of this article is to investigate the stability of local strong solutions for the Fornberg–Whitham equation (1). We find out the L^2 conservation law to the FW model. Assuming that the initial data belong to the space $L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ with $s > \frac{3}{2}$, we obtain the stability of local strong solution in the space $L^1(\mathbb{R})$. We state that the L^1 stability for Eq. (1) has never been established in the previous literature works. The main technique used in this work is the device of doubling the space variables presented in [30].

The structure of this paper is that several lemmas are given in Sect. 2 and the proof of our main result is presented in Sect. 3.

2 Several lemmas

Consider the Cauchy problem of Eq. (1)

$$\begin{cases} V_t - V_{txx} - V_x + \frac{3}{2} V V_x = \frac{9}{2} V_x V_{xx} + \frac{3}{2} V V_{xxx}, \\ V(0,x) = V_0(x). \end{cases} \tag{3}$$

Letting $\Lambda^2 = 1 - \partial_x^2$ and noting the expression $VV_{xxx} = \frac{1}{2}(V^2)_{xxx} - 3V_x V_{xx}$, multiplying both sides of the first equation of problem (3) by Λ^{-2}, we obtain the nonlocal form of problem (3) in the form

$$\begin{cases} V_t + \frac{3}{2} V V_x - (1 - \partial_x^2)^{-2} V_x = 0, \\ V(0,x) = V_0(x), \end{cases} \tag{4}$$

where $\Lambda^{-2} g = \frac{1}{2} \int_R e^{-|x-y|} g \, dy$ for any $g \in L^\infty$ or $g \in L^p(\mathbb{R})$ with $1 \le p \le \infty$.

Lemma 1 *If $V_0(x) \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ and $V(t,x)$ is the solution of problem (4), then*

$$\int_R V^2(t,x) \, dx = \int_R V_0^2(x) \, dx. \tag{5}$$

Proof Setting $(1 - \partial_x^2)^{-2} V = W$, we get $W - W_{xx} = V$ and

$$\int_R V \bigl(1 - \partial_x^2\bigr)^{-2} V_x \, dx = \int_R V W_x \, dx = \int_R (W - W_{xx}) W_x \, dx = 0, \tag{6}$$

from which we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_R V^2\,dx &= \int_R VV_t\,dx\\ &= \int_R\left[-\frac{3}{2}V^2V_x + V\left(1-\partial_x^2\right)^{-2}V_x\right)\right]dx\\ &= 0 + \int_R\left[V\left(1-\partial_x^2\right)^{-2}V_x\right)\right]dx\\ &= 0,\end{aligned}$$

which completes the proof. □

Lemma 2 ([3, 4, 23]) *Assume $V(0,x) = V_0(x) \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Then problem (3) or (4) has a unique strong solution V satisfying*

$$V \in C\big([0,T);H^s(\mathbb{R})\big) \cap C^1\big([0,T);H^{s-1}(\mathbb{R})\big),$$

where $T = T(V_0) > 0$ is the maximal existence time.

Consider the ordinary differential equation

$$\begin{cases} p_t = \frac{3}{2}V(t,p), & t \in [0,T),\\ p(0,x) = x. \end{cases} \tag{7}$$

Lemma 3 *Assume that $V_0 \in H^s$, $s \geq 3$, and $T > 0$ is the maximal existence time of the solution for problem (7). Then there exists a unique solution $p \in C^1([0,T) \times \mathbb{R})$ to problem (7) and the map $p(t,\cdot)$ is an increasing diffeomorphism of R with $p_x(t,x) > 0$ for $(t,x) \in [0,T) \times \mathbb{R}$.*

Proof Using Lemma 2, we have $V \in C^1([0,T);H^{s-1}(\mathbb{R}))$ and $H^s \in C^1(\mathbb{R})$. Therefore, we know that functions $V(t,x)$ and $V_x(t,x)$ are bounded, Lipschitz in space, and C^1 in time. Making use of the existence and uniqueness theorem of ordinary differential equations, we conclude that problem (7) has a unique solution $p \in C^1([0,T) \times \mathbb{R})$.

We differentiate (7) about the variable x and get

$$\begin{cases} \frac{d}{dt}p_x = \frac{3}{2}V_x(t,p)p_x, & t \in [0,T),\\ p_x(0,x) = 1, \end{cases} \tag{8}$$

which results in

$$p_x(t,x) = e^{\int_0^t \frac{3}{2}V_x(\tau,p(\tau,x))\,d\tau}. \tag{9}$$

For every $T' < T$, applying the Sobolev imbedding theorem gives rise to

$$\sup_{(\tau,x)\in[0,T')\times R}\left|V_x(\tau,x)\right| < \infty, \tag{10}$$

from which we know that there exists a constant $K_0 > 0$ to satisfy $p_x(t,x) \geq e^{-K_0 t} > 0$ for $(t,x) \in [0,T) \times \mathbb{R}$. The proof is finished. □

Lemma 4 *Suppose that T is the maximal existence time of the solution V to problem (4) and $V_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Then*

$$\|V(t,x)\|_{L^\infty} \leq t\|V_0\|_{L^2} + \|V_0\|_{L^\infty}, \quad \forall t \in [0,T], \tag{11}$$

$$|\Lambda^{-2}V_x| \leq \|V_0\|_{L^2}, \quad \forall t \in [0,T]. \tag{12}$$

Proof Using the density argument presented in [17], we only need to consider the case $s = 3$ to prove Lemma 4. If the initial value $V_0 \in H^3(\mathbb{R})$, we obtain $V \in C([0,T),H^3(\mathbb{R})) \cap C^1([0,T),H^2(\mathbb{R}))$. From (4), we have

$$\begin{aligned} V_t + \frac{3}{2}VV_x &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\frac{\partial}{\partial y}V(t,y)\,dy \\ &= \frac{1}{2}\int_{-\infty}^{x} e^{-x+y}\frac{\partial}{\partial y}V(t,y)\,dy + \frac{1}{2}\int_{x}^{\infty} e^{-y+x}\frac{\partial}{\partial y}V(t,y)\,dy \\ &= -\frac{1}{2}\int_{-\infty}^{x} e^{-x+y}V(t,y)\,dy + \frac{1}{2}\int_{x}^{\infty} e^{-y+x}V(t,y)\,dy \end{aligned} \tag{13}$$

and

$$\begin{aligned} \frac{dV(t,p(t,x))}{dt} &= V_t\big(t,p(t,x)\big) + V_x\big(t,p(t,x)\big)\frac{dp(t,x)}{dt} \\ &= \left(V_t + \frac{3}{2}VV_x\right)\big(t,p(t,x)\big). \end{aligned} \tag{14}$$

Using the identity $\int_{-\infty}^{\infty} e^{-2|x-y|}\,dy = 1$ and $\|V\|_{L^2} = \|V_0\|_{L^2}$ (see Lemma 1), we have

$$\begin{aligned} &\left|-\frac{1}{2}\int_{-\infty}^{x} e^{-x+y}V(t,y)\,dy + \frac{1}{2}\int_{x}^{\infty} e^{-y+x}V(t,y)\,dy\right| \\ &\quad\leq \frac{1}{2}\int_{-\infty}^{x} e^{-x+y}\big|V(t,y)\big|\,dy + \frac{1}{2}\int_{x}^{\infty} e^{-y+x}\big|V(t,y)\big|\,dy \\ &\quad\leq \left(\int_{-\infty}^{\infty} e^{-2|x-y|}\,dy\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty} V^2(t,y)\,dy\right)^{\frac{1}{2}} \\ &\quad\leq \|V\|_{L^2(\mathbb{R})} \\ &\quad= \|V_0\|_{L^2(\mathbb{R})}, \end{aligned} \tag{15}$$

from which together with (13) we derive that (12) holds.

From (13)–(15), we derive that

$$\begin{aligned} \left|\int_0^t \frac{dV(t,p(t,x))}{dt}\,dt\right| &\leq \frac{1}{2}\int_0^t\left|\int_{-\infty}^{\infty} e^{-|p(t,x)-y|}\frac{\partial}{\partial y}V(t,y)\,dy\right|dt \\ &\leq t\|V_0\|_{L^2(\mathbb{R})}, \end{aligned} \tag{16}$$

from which we obtain

$$\left|V\left(t,p(t,x)\right)\right| \le \left\|V\left(t,p(t,x)\right)\right\|_{L^\infty} \le t\|V_0\|_{L^2(\mathbb{R})} + \|V_0\|_{L^\infty}. \tag{17}$$

Using Lemma 3, for every $t \in [0,T')$, $T' < T$, we get that there exists a function $K(t) > 0$ such that

$$e^{-K(t)} \le p_x(t,x) \le e^{K(t)}, \quad x \in \mathbb{R}. \tag{18}$$

We deduce from (18) that the function $p(t,\cdot)$ is strictly increasing on $\mathbb{R}$ with $\lim_{x\to\pm\infty} p(t,x) = \pm\infty$ as long as $t \in [0,T')$. Applying (17) produces

$$\left\|V(t,x)\right\|_{L^\infty} = \left\|V\left(t,p(t,x)\right)\right\|_{L^\infty} \le t\|V_0\|_{L^2(\mathbb{R})} + \|V_0\|_{L^\infty}.$$

The proof is finished. □

Lemma 5 *Suppose that $V_1(t,x)$ and $V_2(t,x)$ are two solutions of problem (4) with initial data $V_{1,0}(x), V_{2,0}(x) \in H^s(\mathbb{R})$ $(s > \frac{3}{2})$, respectively. Assume $f(t,x) \in C_0^\infty([0,\infty) \times (-\infty,\infty)$. Then*

$$\int_{-\infty}^{\infty}\left|\Lambda^{-2}\frac{\partial}{\partial x}V_1(t,x) - \Lambda^{-2}\frac{\partial}{\partial x}V_2(t,x)\right|\left|f(t,x)\right|dx \\ \le c_0\int_{-\infty}^{\infty}\left|V_1(t,x) - V_2(t,x)\right|dx, \tag{19}$$

where $c_0 > 0$ depends on f.

Proof We have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left|\Lambda^{-2}\frac{\partial}{\partial x}V_1(t,x) - \Lambda^{-2}\frac{\partial}{\partial x}V_2(t,x)\right|\left|f(t,x)\right|dx \\
&\quad\le \int_{-\infty}^{\infty}\left|\partial_x\Lambda^{-2}(V_1 - V_2)\right|\left|f(t,x)\right|dx \\
&\quad\le \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} e^{-|x-y|}\operatorname{sign}(x-y)\left(V_1(t,y) - V_2(t,y)\right)dy\right|\left|f(t,x)\right|dx \\
&\quad\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-|x-y|}\left|V_1(t,y) - V_2(t,y)\right|\left|f(t,x)\right|dy\,dx \\
&\quad\le c_0\int_{-\infty}^{\infty}|V_1 - V_2|\,dy,
\end{aligned}$$

in which we have applied the Tonelli theorem. The proof is completed. □

Assume that $\delta(\sigma)$ is a function which is infinitely differentiable on $(-\infty,+\infty)$ such that $\delta(\sigma) \ge 0$, $\delta(\sigma) = 0$ for $|\sigma| \ge 1$ and $\int_{-\infty}^{\infty}\delta(\sigma)\,d\sigma = 1$. For an arbitrary $h > 0$, set $\delta_h(\sigma) = \frac{\delta(h^{-1}\sigma)}{h}$. We conclude that $\delta_h(\sigma)$ is a function in $C^\infty(-\infty,\infty)$ and

$$\begin{cases} \delta_h(\sigma) \ge 0, & \delta_h(\sigma) = 0 \quad \text{if } |\sigma| \ge h, \\ |\delta_h(\sigma)| \le \frac{c}{h}, & \int_{-\infty}^{\infty}\delta_h(\sigma)\,d\sigma = 1. \end{cases} \tag{20}$$

Suppose that the function $W_1(x)$ is locally integrable in $(-\infty, \infty)$. The approximation function of W_1 is defined by

$$W_1^h(x) = \frac{1}{h}\int_{-\infty}^{\infty} \delta\left(\frac{x-y}{h}\right) W_1(y)\,dy, \quad h > 0. \tag{21}$$

We call x_0 a Lebesgue point of function $W_1(x)$ if

$$\lim_{h\to 0}\frac{1}{h}\int_{|x-x_0|\le h}\bigl|W_1(x) - W_1(x_0)\bigr|\,dx = 0.$$

We introduce notation about the concept of a characteristic cone. For any $M > 0$, we define $M > N = \max_{t\in[0,T]}\|V\|_{L^\infty} < \infty$. Let $\mho$ designate the cone $\{(t,x) : |x| < M - Nt, 0 \le t \le T_0 = \min(T, MN^{-1})\}$. We let S_τ designate the cross section of the cone $\mho$ by the plane $t = \tau, \tau \in [0, T_0]$.

Let $K_{r+2\rho} = \{x : |x| \le r + 2\rho\}$ where $r > 0$, $\rho > 0$ and $\zeta_T = [0,T] \times \mathbb{R}$. The space of all infinitely differentiable functions $f(t,x)$ with compact support in $[0,T] \times \mathbb{R}$ is denoted by $C_0^\infty(\zeta_T)$.

Lemma 6 ([30]) *Let the function $U(t,x)$ be a bounded and measurable function in some cylinder $\Omega_T = [0,T] \times K_r$. If for some $\rho \in (0, \min[r,T])$ and any number $h \in (0,\rho)$, then the following function*

$$U_h = \frac{1}{h^2}\iiiint_{|\frac{t-\tau}{2}|\le h, \rho\le\frac{t+\tau}{2}\le T-\rho, |\frac{x-y}{2}|\le h, |\frac{x+y}{2}|\le r-\rho}\bigl|U(t,x) - U(\tau,y)\bigr|\,dx\,dt\,dy\,d\tau$$

satisfies $\lim_{h\to 0} U_h = 0$.

Lemma 7 ([30]) *If the function $G(U)$ satisfies a Lipschitz condition on the interval $[-N,N]$, then the function*

$$G_1(U_1,U_2) = \operatorname{sign}(U_1 - U_2)\bigl(G(U_1) - G(U_2)\bigr)$$

satisfies the Lipschitz condition in U_1 and U_2, respectively.

Lemma 8 *Suppose that V is the strong solution of problem* (4), $f(t,x) \in C_0^\infty(\zeta_T)$ *and* $f(0,x) = 0$. *Then*

$$\iint_{\zeta_T}\left\{|V-k|f_t + \frac{3}{4}\operatorname{sign}(V-k)\bigl[V^2 - k^2\bigr]f_x + \operatorname{sign}(V-k)\Lambda^{-2}V_x f\right\}dx\,dt = 0, \tag{22}$$

where k is an arbitrary constant.

Proof Here we mention that the method to prove this lemma comes from [30]. We assume that $\Phi(V)$ is an arbitrary twice differentiable function on the line $-\infty < V < \infty$. We multiply the first equation of Eq. (4) by the function $\Phi'(V)f(t,x)$, where $f(t,x) \in C_0^\infty(\zeta_T)$. Integrating over ζ_T and integrating by parts (transferring the derivatives with respect to t and x to function f), for any constant k, we have

$$\int_{-\infty}^{\infty}\left[\int_k^V \Phi'(z)z\,dz\right]f_x\,dx = -\int_{-\infty}^{\infty}\bigl[f\Phi'(V)VV_x\bigr]\,dx$$

and

$$\iint_{\zeta_T} \left\{ \Phi(V) f_t + \frac{3}{2} \left[\int_k^V \Phi'(z) z \, dz \right] f_x - \Phi'(V) \Lambda^{-2} V_x f \right\} dx \, dt = 0. \tag{23}$$

Integration by parts yields

$$\begin{aligned} \int_{-\infty}^{\infty} \left[\int_k^V \Phi'(z) z \, dz \right] f_x \, dx = \int_{-\infty}^{\infty} \biggl[& \frac{1}{2} \Phi'(V) V^2 - \frac{1}{2} \Phi'(k) k^2 \\ & - \frac{1}{2} \int_k^V (z^2 - k^2) \Phi''(z) \, dz \biggr] f_x \, dx. \end{aligned} \tag{24}$$

Choosing that $\Phi^h(V)$ is an approximation of the function $|V-k|$, setting $\Phi(V) = \Phi^h(V)$, and making use of the properties of the $\operatorname{sign}(V-k)$, (23), (24) and sending $h \to 0$, we notice that the last term in (24) becomes zero. Thus, we have

$$\iint_{\zeta_T} \left\{ |V-k| f_t + \frac{3}{4} \operatorname{sign}(V-k) \left[V^2 - k^2 \right] f_x + \operatorname{sign}(V-k) \Lambda^{-2} V_x f \right\} dx \, dt = 0. \tag{25}$$

The proof is finished. □

3 Main result

Now, we give the main result of this work.

Theorem 1 *Assume that V_1 and V_2 are two local strong solutions of Eq. (1) with initial data $V_{1,0}(x), V_{2,0}(x) \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$, $s > \frac{3}{2}$. Let T be the maximal existence time of the solutions. Then*

$$\left\| V_1(t,\cdot) - V_2(t,\cdot) \right\|_{L^1(\mathbb{R})} \le c_0 e^{c_0 t} \int_{-\infty}^{\infty} \left| V_{10}(x) - V_{20}(x) \right| dx, \quad t \in [0, T), \tag{26}$$

where $c_0 > 0$ is a constant.

Proof From Lemma 2, we know the existence of local strong solutions for Eq. (1). Let $f(t,x) \in C_0^\infty(\zeta_T)$. Assume $f(t,x) = 0$ outside the cylinder

$$\uplus = \left\{ (t,x) \right\} = [\rho, T - 2\rho] \times K_{r-2\rho}, \quad 0 < 2\rho \le \min(T, r). \tag{27}$$

We let

$$\xi = f\left(\frac{t+\tau}{2}, \frac{x+y}{2} \right) \delta_h \left(\frac{t-\tau}{2} \right) \delta_h \left(\frac{x-y}{2} \right) = f(\cdots) \lambda_h(*), \tag{28}$$

where $(\cdots) = (\frac{t+\tau}{2}, \frac{x+y}{2})$ and $(*) = (\frac{t-\tau}{2}, \frac{x-y}{2})$. The function $\delta_h(\sigma)$ is defined in (20). We obtain

$$\xi_t + \xi_\tau = f_t(\cdots) \lambda_h(*), \qquad \xi_x + \xi_y = f_x(\cdots) \lambda_h(*). \tag{29}$$

We apply the technique of Kruzkov's device of doubling the space variables [30]. In (22), we set $k = V_1(\tau, y)$ and $f = \xi(t,x,\tau,y)$ for a fixed point (τ, y). We note that $V_1(\tau, y)$ is defined almost everywhere in $\zeta_T = [0, T] \times \mathbb{R}$. We integrate (22) over ζ_T for variable (τ, y) and then get

$$\begin{aligned}\iiiint_{\zeta_T\times\zeta_T} \Big\{ & \big|V_1(t,x) - V_2(\tau,y)\big|\xi_t \\ & + \frac{3}{4}\operatorname{sign}\big(V_1(t,x) - V_2(\tau,y)\big)\left(\frac{V_1^2(t,x)}{2} - \frac{V_2^2(\tau,y)}{2}\right)\xi_x \\ & + \operatorname{sign}\big(V_1(t,x) - V_2(\tau,y)\big)\Lambda^{-2}\partial_x\big(V_1(t,x)\big)\xi\Big\}\,dt\,dx\,dy\,d\tau = 0. \end{aligned} \tag{30}$$

Similarly, it has

$$\begin{aligned}\iiiint_{\zeta_T\times\zeta_T} \Big\{ & \big|V_2(\tau,y) - V_1(t,x)\big|\xi_\tau \\ & + \frac{3}{4}\operatorname{sign}\big(V_2(\tau,y) - V_1(t,x)\big)\left(\frac{V_2^2(\tau,y)}{2} - \frac{V_1^2(t,x)}{2}\right)\xi_y \\ & + \operatorname{sign}\big(V_2(\tau,y) - V_1(t,x)\big)\Lambda^{-2}\partial_y\big(V_2(\tau,y)\big)\xi\Big\}\,dx\,dt\,dy\,d\tau = 0. \end{aligned} \tag{31}$$

Using (30) and (31), we acquire the inequality

$$\begin{aligned} 0 \le \iiiint_{\zeta_T\times\zeta_T} \Big\{ & \big|V_1(t,x) - V_2(\tau,y)\big|(\xi_t + \xi_\tau) \\ & + \frac{3}{4}\operatorname{sign}\big(V_1(t,x) - V_2(\tau,y)\big)\left(\frac{V_1^2(t,x)}{2} - \frac{V_2^2(\tau,y)}{2}\right)(\xi_x + \xi_y)\Big\}\,dx\,dt\,dy\,d\tau \\ & + \left|\iiiint_{\zeta_T\times\zeta_T} \operatorname{sign}\big(V_1(t,x) - V_2(t,x)\big)\right. \\ & \left. \times\big(\Lambda^{-2}\partial_x V_1(t,x) - \Lambda^{-2}\partial_y V_2(\tau,y)\big)\xi\,dx\,dt\,dy\,d\tau\right|. \\ = {} & L_1 + L_2 + \left|\iiiint_{\zeta_T\times\zeta_T} L_3\,dx\,dt\,dy\,d\tau\right|. \end{aligned} \tag{32}$$

We claim that the following inequality

$$\begin{aligned} 0 \le \iint_{\zeta_T} \Big\{ & \big|V_1(t,x) - V_2(t,x)\big|f_t \\ & + \frac{3}{4}\operatorname{sign}\big(V_1(t,x) - V_2(t,x)\big)\left(\frac{V_1^2(t,x)}{2} - \frac{V_2^2(t,x)}{2}\right)f_x\Big\}\,dx\,dt \\ & + \left|\iint_{\zeta_T} \operatorname{sign}\big(V_1(t,x) - V_2(t,x)\big)\Lambda^{-2}\partial_x\big[V_1(t,x) - V_2(t,x)\big]f\,dx\,dt\right| \end{aligned} \tag{33}$$

holds.

In fact, for the choice of ξ, the first two terms in the integrand of (32) can be represented in the form

$$D_h = D\big(t, x, \tau, y, V_1(t,x), V_2(\tau,y)\big)\lambda_h(*).$$

From Lemma 4, we know $\|V_1\|_{L^\infty} < C_T$ and $\|V_2\|_{L^\infty} < C_T$; from Lemma 7, we know D_h satisfies the Lipschitz condition in V_1 and V_2, respectively. By the choice of ξ, we derive that $D_h = 0$ outside the region

$$\begin{aligned}\{(t,x;\tau,y)\} = \biggl\{\rho \le \frac{t+\tau}{2} \le T-2\rho, \frac{|t-\tau|}{2} \le h,\\ \frac{|x+y|}{2} \le r-2\rho, \frac{|x-y|}{2} \le h\biggr\}.\end{aligned} \tag{34}$$

Furthermore, we get

$$\begin{aligned}&\iiiint_{\zeta_T\times\zeta_T} D_h\,dx\,dt\,dy\,d\tau\\ &\quad= \iiiint_{\zeta_T\times\zeta_T}\bigl[D\bigl(t,x,\tau,y,V_1(t,x),V_2(\tau,y)\bigr)\\ &\qquad - D\bigl(t,x,t,x,V_1(t,x),V_2(t,x)\bigr)\bigr]\lambda_h(*)\,dx\,dt\,dy\,d\tau\\ &\qquad + \iiiint_{\zeta_T\times\zeta_T} D\bigl(t,x,t,x,V_1(t,x),V_2(t,x)\bigr)\lambda_h(*)\,dx\,dt\,dy\,d\tau\\ &\quad= B_{11}(h)+B_{12}.\end{aligned} \tag{35}$$

Noticing $|\lambda(*)| \le \frac{c}{h^2}$ and the definition of D_h gives rise to

$$\begin{aligned}&\bigl|B_{11}(h)\bigr|\\ &\quad\le c\biggl[h+\frac{1}{h^2}\\ &\qquad\times\iiiint_{|\frac{t-\tau}{2}|\le h,\rho\le\frac{t+\tau}{2}\le T-\rho,|\frac{x-y}{2}|\le h,|\frac{x+y}{2}|\le r-\rho}\bigl|V_1(t,x)-V_2(\tau,y)\bigr|\,dx\,dt\,dy\,d\tau\biggr],\end{aligned} \tag{36}$$

where the constant c does not depend on h. Using Lemma 6, we get $B_{11}(h) \to 0$ as $h \to 0$. The integral B_{12} does not depend on h. Substituting $t = \alpha$, $\frac{t-\tau}{2} = \beta$, $x = \eta$, $\frac{x-y}{2} = \mu$ and noting the identity

$$\int_{-h}^{h}\int_{-\infty}^{\infty}\lambda_h(\beta,\mu)\,d\mu\,d\beta = 1, \tag{37}$$

we derive that

$$\begin{aligned}B_{12} &= 2^2\iint_{\zeta_T} D_h\bigl(\alpha,\eta,\alpha,\eta,V_1(\alpha,\eta),V_2(\alpha,\eta)\bigr)\biggl\{\int_{-h}^{h}\int_{-\infty}^{\infty}\lambda_h(\beta,\mu)\,d\mu\,d\beta\biggr\}\,d\eta\,d\alpha\\ &= 4\iint_{\zeta_T} D_h\bigl(t,x,t,x,V_1(t,x),V_2(t,x)\bigr)\,dx\,dt.\end{aligned} \tag{38}$$

Thus, we have

$$\lim_{h\to 0}\iiiint_{\zeta_T\times\zeta_T} D_h\,dx\,dt\,dy\,d\tau = 4\iint_{\zeta_T} D\bigl(t,x,t,x,V_1(t,x),V_2(t,x)\bigr)\,dx\,dt. \tag{39}$$

We write

$$\begin{aligned} L_3 &= \operatorname{sign}\bigl(u(t,x)-v(\tau,y)\bigr)\bigl(\Lambda^{-2}\partial_x V_1(t,x)-\Lambda^{-2}\partial_y V_2(\tau,y)\bigr)f(\cdots)\lambda_h(*) \\ &= \overline{L_3}(t.x,\tau,y)\lambda_h(*) \end{aligned} \tag{40}$$

and

$$\begin{aligned} \iiiint_{\zeta_T\times\zeta_T} L_3\,dx\,dt\,dy\,d\tau &= \iiiint_{\zeta_T\times\zeta_T} \bigl[\overline{L_3}(t.x,\tau,y)-\overline{L_3}(t.x,t,x)\bigr]\lambda_h(*)\,dx\,dt\,dy\,d\tau \\ &\quad + \iiiint_{\zeta_T\times\zeta_T} \overline{L_3}(t.x,t,x)\lambda_h(*)\,dx\,dt\,dy\,d\tau \\ &= B_{21}(h)+B_{22}, \end{aligned} \tag{41}$$

from which we have

$$\begin{aligned} &\bigl|B_{21}(h)\bigr| \\ &\quad\le c\biggl(h+\frac{1}{h^2}\iiiint_{|\frac{t-\tau}{2}|\le h,\rho\le\frac{t+\tau}{2}\le T-\rho,|\frac{x-y}{2}|\le h,|\frac{x+y}{2}|\le r-\rho} \bigl|\Lambda^{-2}\partial_x V_1(t,x) \\ &\qquad -\Lambda^{-2}\partial_y V_2(\tau,y)\bigr|\,dx\,dt\,dy\,d\tau\biggr). \end{aligned} \tag{42}$$

Using Lemmas 5 and 6, we have $B_{21}(h)\to 0$ as $h\to 0$. Using (37), we have

$$\begin{aligned} B_{22} &= 2^2\iint_{\zeta_T}\overline{L_3}\bigl(\alpha,\eta,\alpha,\eta,V_1(\alpha,\eta),V_2(\alpha,\eta)\bigr)\biggl\{\int_{\mathbb{R}}\int_{-h}^{h}\lambda_h(\beta,\mu)\,d\mu\,d\beta\biggr\}\,d\eta\,d\alpha \\ &= 4\iint_{\zeta_T}\overline{L_3}\bigl(t,x,t,x,V_1(t,x),V_2(t,x)\bigr)\,dx\,dt \\ &= 4\iint_{\zeta_T}\operatorname{sign}\bigl(V_1(t,x)-V_2(t,x)\bigr)(\Lambda^{-2}\partial_x\bigl[V_1(t,x)-V_2(t,x)\bigr]f(t,x)\,dx\,dt. \end{aligned} \tag{43}$$

From (36), (37), (42), and (43), we prove that inequality (33) holds.

Set

$$X(t)=\int_{-\infty}^{\infty}\bigl|V_1(t,x)-V_2(t,x)\bigr|\,dx. \tag{44}$$

Let

$$\gamma_h=\int_{-\infty}^{\sigma}\delta_h(\tau)\,d\tau \qquad \bigl(\gamma_h'(\sigma)=\delta_h(\sigma)\ge 0\bigr) \tag{45}$$

and choose two numbers ρ and $\tau\in(0,T_0)$, $\rho<\tau$. In (33), we choose

$$f=\bigl[\gamma_h(t-\rho)-\gamma_h(t-\tau)\bigr]\chi(t,x),\quad h<\min(\rho,T_0-\tau), \tag{46}$$

where

$$\chi(t,x)=\chi_\varepsilon(t,x)=1-\gamma_\varepsilon\bigl(|x|+Nt-M+\varepsilon\bigr),\quad \varepsilon>0. \tag{47}$$

We know that the function $\chi(t,x) = 0$ outside the cone $\mho$ and $f(t,x) = 0$ outside the set $\uplus$. If $(t,x) \in \mho$, we get the relations

$$0 = \chi_t + N|\chi_x| \geq \chi_t + N\chi_x. \tag{48}$$

Applying (46)–(48) and (33), we have

$$\begin{aligned} 0 \leq & \int_0^{T_0} \int_{-\infty}^{\infty} \{ [\delta_h(t-\rho) - \delta_h(t-\tau)] \chi_\varepsilon |V_1(t,x) - V_2(t,x)| \} \,dx\,dt \\ & + \int_0^{T_0} \int_{-\infty}^{\infty} [\gamma_h(t-\rho) - \gamma_h(t-\tau)] |(\Lambda^{-2}\partial_x[V_1(t,x) - V_2(t,x)]\chi(t,x)| \,dx\,dt. \end{aligned} \tag{49}$$

Using Lemma 5 and letting $\varepsilon \to 0$ and $M \to \infty$, we obtain

$$\begin{aligned} 0 \leq & \int_0^{T_0} \left\{ [\delta_h(t-\rho) - \delta_h(t-\tau)] \int_{-\infty}^{\infty} |V_1(t,x) - V_2(t,x)| \,dx \right\} dt \\ & + c_0(1+T_0) \int_0^{T_0} [\gamma_h(t-\rho) - \gamma_h(t-\tau)] \int_{-\infty}^{\infty} |V_1(t,x) - V_2(t,x)| \,dx\,dt. \end{aligned} \tag{50}$$

Using the properties of the function $\delta_h(\sigma)$ for $h \leq \min(\rho, T_0 - \rho)$ yields

$$\begin{aligned} \left| \int_0^{T_0} \delta_h(t-\rho) X(t)\,dt - X(\rho) \right| &= \left| \int_0^{T_0} \delta_h(t-\rho) |X(t) - X(\rho)| \,dt \right| \\ &\leq c\frac{1}{h} \int_{\rho-h}^{\rho+h} |X(t) - X(\rho)| \,dt \to 0 \quad \text{as } h \to 0, \end{aligned} \tag{51}$$

where c is independent of h. Denoting

$$L(\rho) = \int_0^{T_0} \gamma_h(t-\rho) X(t)\,dt = \int_0^{T_0} \int_{-\infty}^{t-\rho} \delta_h(\sigma)\,d\sigma X(t)\,dt, \tag{52}$$

we get

$$L'(\rho) = -\int_0^{T_0} \delta_h(t-\rho) X(t)\,dt \to -X(\rho) \quad \text{as } h \to 0, \tag{53}$$

and

$$L(\rho) \to L(0) - \int_0^{\rho} X(\sigma)\,d\sigma \quad \text{as } h \to 0. \tag{54}$$

Similarly, we obtain

$$L(\tau) \to L(0) - \int_0^{\tau} X(\sigma)\,d\sigma \quad \text{as } h \to 0. \tag{55}$$

It follows from (54) and (55) that

$$L(\rho) - L(\tau) \to \int_\rho^{\tau} X(\sigma)\,d\sigma \quad \text{as } h \to 0. \tag{56}$$

Send $\rho \to 0$, $\tau \to t$, and note that

$$\begin{aligned}\left|V_1(\rho,x) - V_2(\rho,x)\right| \le \left|V_1(\rho,x) - V_{10}(x)\right| \\ + \left|V_2(\rho,x) - V_{20}(x)\right| + \left|V_{10}(x) - V_{20}(x)\right|.\end{aligned} \tag{57}$$

Thus, from (50), (51), (56)–(57), we have

$$\begin{aligned}\int_{-\infty}^{\infty}\left|V_1(t,x) - V_2(t,x)\right| dx \le \int_{-\infty}^{\infty}|V_{10} - V_{20}|\,dx \\ + c_0\int_0^t\int_{-\infty}^{\infty}\left|V_1(t,x) - V_2(t,x)\right| dx\,dt.\end{aligned} \tag{58}$$

Using the Gronwall inequality and (58), we complete the proof. □

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