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Existence and stability of periodic solutions for a forced pendulum with time-dependent damping

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Abstract

In this paper, we study the existence, multiplicity and stability of periodic solutions for a forced pendulum with time-dependent damping. The proof is based on the third order approximation method and a suitable version of the Poincaré–Birkhoff fixed point theorem.

MSC: 34C25; 34D20

Keywords: Damped pendulum; Third order approximation method; Poincaré–Birkhoff fixed point theorem

1 Introduction

During the last few decades, the dynamics of the forced pendulum equation

$$u'' + \kappa u' + \omega \sin u = p(t) \tag{1.1}$$

has attracted much attention of many researchers, where κ, ω are constants and $p \in \mathbb{C}(\mathbb{R}/T\mathbb{Z})$. We refer the reader to [1, 2, 14, 18, 26–28, 32] for the existence and nonexistence of periodic solutions of (1.1), [5, 12, 29, 30] for the stability of periodic solutions of (1.1) and [17, 19, 23, 34, 35] for its chaotic behaviors.

If the damping coefficient depends on time and is a continuous T-periodic function, then Eq. (1.1) becomes the following nonlinear damped equation:

$$u'' + h(t)u' + \omega \sin u = p(t), \tag{1.2}$$

in which we consider the case that $h \in \mathbb{C}(\mathbb{R}/T\mathbb{Z})$ has mean value equal to zero. Such an equation can be regard as a model on which not the inertial resistance but the viscous resistance acts predominantly. As far as we know, the study on the dynamics of Eq. (1.2) is few in the literature. Recently, Sugie in [37] studied the stability of the origin of Eq. (1.2) when the driving force $p(t) \equiv 0$. Based on Lyapunov's stability theory and phase plane analysis of the positive orbits of an equivalent planar system to Eq. (1.2), a necessary and sufficient condition of the asymptotic stability for the origin of Eq. (1.2) was obtained.

However, up to now, the problem on the existence and stability of periodic solutions for Eq. (1.2) has not attracted attention in the literature. The purpose of this paper is to fill this gap. In this paper, we study the existence, multiplicity and stability of periodic solutions for Eq. (1.2).

In the third section, we prove that Eq. (1.2) has a twist T-periodic solution if the driving force is not too large. Such twist periodic solution is stable in the sense of Lyapunov [36]. The proof is based on the third order approximation method for nonlinear damped equations, which was developed by Chu et al. [6]. The third order approximation method for general time-periodic Lagrangian equations was developed by Ortega [31] and Zhang [40] and has been applied in [8, 10, 11, 20–22, 24, 38, 39] for different kinds of equations. Recently, in [9], Chu, Liang, and Torres used the Poincaré–Birkhoff fixed point theorem and the third order approximation method to study the existence and stability of periodic oscillations of a rigid dumbbell satellite around its center of mass. We will apply them to study the existence, multiplicity and stability of periodic solutions for a forced pendulum with time-dependent damping.

In the fourth section, we prove that Eq. (1.2) has at least two geometrically distinct T-periodic solutions. Moreover, at least one of them is unstable. Here, we say that Eq. (1.2) has at least two geometrically distinct T-periodic solutions, if such solutions are not differing by a multiple of 2π. Furthermore, we also study the existence of periodic and subharmonic solutions with winding number for Eq. (1.2). The proof is based on a suitable version of the Poincaré–Birkhoff fixed point theorem, which was originally conjectured by Poincaré [33] in 1912 when he studied the restricted three body problems, and was first proved by Birkhoff [3, 4] in 1913. During the last century, different proofs and developments were given. We refer the reader to [13, Sect. 1] for a short review of the Poincaré–Birkhoff fixed point theorem.

2 Preliminaries

2.1 A stability criterion

Given a function $a(t)$, we denote $a_+(t) = \max\{a(t), 0\}$ and $a_-(t) = \max\{-a(t), 0\}$ the positive and the negative parts of $a(t)$.

Consider the nonlinear damped equation

$$u'' + h(t)u' + g(t,u) = 0, \tag{2.1}$$

where $g : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is T-periodic in t and of class $\mathbb{C}^{0,4}$ in (t,u), $h \in L^1(\mathbb{R}/T\mathbb{Z})$ with zero mean value. Let ψ be a T-periodic solution of Eq. (2.1), by translating it to the origin, we obtain the third order approximation

$$u'' + h(t)u' + a(t)u + b(t)u^2 + c(t)u^3 + \cdots = 0, \tag{2.2}$$

where

$$a(t) = g_u\bigl(t,\psi(t)\bigr), \qquad b(t) = \frac{1}{2}g_{uu}\bigl(t,\psi(t)\bigr), \qquad c(t) = \frac{1}{6}g_{uuu}\bigl(t,\psi(t)\bigr). \tag{2.3}$$

Based on the method of third order approximation for damped differential equations [6], the following stability criterion was proved in [9].

Theorem 2.1 ([9, Theorem 2.2]) *Assume there exist two constants* σ_1, σ_2 *such that* $0 \leq \sigma_1 \leq \sigma_2 \leq \frac{\pi}{2\hat{T}}$ $(\hat{T} = \tau(T))$ *and*

$$\sigma_1^2 \leq a(t)\sigma(2h)(t) \leq \sigma_2^2, \quad \forall t \in \mathbb{R}. \tag{2.4}$$

Suppose further that

$$\frac{\|B_+\|_1 \|B_-\|_1}{\min\{\sin(\frac{3\hat{T}\sigma_1}{2}), \sin(\frac{3\hat{T}\sigma_2}{2})\}} < \frac{3}{10}\sigma_1^3 \left[\frac{\|C_-\|_1}{\sigma_2^2} - \frac{\|C_+\|_1}{\sigma_1^2}\right], \tag{2.5}$$

where $B(t) = b(t)\sigma(h)(t)$, $C(t) = c(t)\sigma(h)(t)$. *Then the trivial solution* $u = 0$ *of Eq.* (2.2) *is twist and therefore is stable.*

2.2 The Poincaré–Birkhoff fixed point theorem

Set $A = \mathbb{R} \times [-a, a]$ and $B = \mathbb{R} \times [-b, b]$, where $a > b > 0$. We will work with a $\mathbb{C}^k$-diffeomorphism $S : A \to B$ defined by

$$S(\theta, r) = \big(Q(\theta, r), P(\theta, r)\big),$$

where $Q, P : \mathbb{R}^2 \to \mathbb{R}$ are functions of class $\mathbb{C}^k$ satisfying the periodicity conditions

$$\begin{cases} Q(\theta + 2\pi, r) = Q(\theta, r) + 2\pi, \\ P(\theta + 2\pi, r) = P(\theta, r). \end{cases}$$

We say that S is isotopic to the inclusion, if there exists a function $H : A \times [0, 1] \to B$ such that, for every $\lambda \in [0, 1]$, $H_\lambda(x) = H(x, \lambda)$ is a homeomorphism with $H_0(x) = S(x)$ and $H_1(x) = x$. The class of the maps satisfying the above characteristics will be indicated by $\varepsilon^k(A)$.

We say that $S \in \varepsilon^k(A)$ is exact symplectic if there exists a smooth function $V = V(\theta, r)$ with $V(\theta + 2\pi, r) = V(\theta, r)$ such that

$$\mathrm{d}V = P\,\mathrm{d}Q - r\,\mathrm{d}\theta. \tag{2.6}$$

The following theorem is a slight modified version of the Poincaré–Birkhoff fixed point theorem proved by Franks in [15, 16] and the statement on the instability was proved by Marò in [25].

Theorem 2.2 ([15, 25]) *Let* $S : A \to B$ *be an exact symplectic diffeomorphism belonging to* $\varepsilon^2(A)$ *such that* $S(A) \subset \operatorname{int}(B)$. *Suppose that there exists* $\epsilon > 0$ *such that*

$$\begin{cases} Q(\theta, \alpha) - \theta > \epsilon, & \forall \theta \in [0, 2\pi), \\ Q(\theta, -\alpha) - \theta < -\epsilon, & \forall \theta \in [0, 2\pi). \end{cases} \tag{2.7}$$

Then S *has at least two distinct fixed points* p_1 *and* p_2 *in* A *such that* $p_1 - p_2 \neq (2k\pi, 0)$ *for every* $k \in \mathbb{Z}$. *Moreover, at least one of the fixed points is unstable if* S *is analytic.*

3 Stable periodic solutions

In this section, we prove that Eq. (1.2) has a twist T-periodic solution u, which has the smallest L^∞ normal among all of T-periodic solutions of Eq. (1.2). Throughout this section, we assume that

$$\frac{1}{T}\int_0^T h(s)\,\mathrm{d}s = 0. \tag{3.1}$$

Consider the periodic problem of the linear damped equation

$$\begin{cases} u'' + h(t)u' + \omega u = 0, \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{cases} \tag{3.2}$$

where $\omega > 0$ and $h \in L^1(\mathbb{R}/2\pi\mathbb{Z})$. It was proved in [7, Corollary 2.6] that if

$$\int_0^T \sigma(-h)(s)\,\mathrm{d}s \int_0^T \sigma(h)(s)\,\mathrm{d}s < \frac{4}{\omega}, \tag{3.3}$$

then the Green function $G(t,s)$ of the periodic problem (3.2) is positive for all $(t,s) \in [0,T] \times [0,T]$. In this case, the nonhomogeneous damped equation

$$u'' + h(t)u' + \omega u = p(t)$$

has a unique T-periodic solution, which can be written as

$$u(t) = \int_0^T G(t,s)p(s)\,\mathrm{d}s.$$

Lemma 3.1 ([20], Lemma 2.1) *Let γ and η be positive parameters. Then the cubic equation*

$$\gamma x^3 + \eta = x$$

has a positive root if and only if $27\gamma\eta^2 < 4$. In this case, the minimal positive root is given by

$$x = \Phi(\gamma,\eta) = \frac{2}{\sqrt{3\gamma}}\cos\frac{\pi + y}{y}\left(y = \arccos\left(\frac{3\eta\sqrt{3\gamma}}{2}\right)\right),$$

which satisfies

$$\Phi(\gamma,\eta) \le \frac{3\eta}{2}. \tag{3.4}$$

Theorem 3.2 *Assume that (3.3) is satisfied and*

$$\|p\|_\infty \le \frac{2\sqrt{2}\omega}{3}. \tag{3.5}$$

Then Eq. (1.2) *has a unique T-periodic solution u such that $\|u\|_\infty$ is the smallest among all of T-periodic solutions of Eq.* (1.2). *Moreover, u satisfies*

$$\|u\|_\infty \le \Phi\left(\frac{1}{6}, \frac{\|p\|_\infty}{\omega}\right) \le \frac{3\|p\|_\infty}{2\omega}. \tag{3.6}$$

Proof It is obvious that u is a T-periodic solution of Eq. (1.2) if and only if it is a fixed point of the operator equation

$$u(t) = \int_0^T G(t,s)\bigl[\omega\bigl(u(s) - \sin u(s)\bigr) + p(s)\bigr]\,\mathrm{d}s := (\mathbb{T}u)(t).$$

Since (3.3) holds, the Green function $G(t,s) > 0$ for all $(t,s) \in [0,T] \times [0,T]$. Furthermore, it is easy to verify that the operator $\mathbb{T}$ is a completely continuous operator from $\mathbb{C}(\mathbb{R}/T\mathbb{Z})$ to itself.

For any $u \in \mathbb{C}(\mathbb{R}/T\mathbb{Z})$, it follows from the basic estimate $|u - \sin u| \le \frac{|u|^3}{6}$ and the fact $\int_0^T G(t,s)\,\mathrm{d}s = \frac{1}{\omega}$ that

$$\begin{aligned}
\bigl|(\mathbb{T}u)(t)\bigr| &= \left|\int_0^T G(t,s)\bigl[\omega\bigl(u(s) - \sin u(s)\bigr) + p(s)\bigr]\,\mathrm{d}s\right| \\
&\le \int_0^T \omega G(t,s)\bigl|u(s) - \sin u(s)\bigr|\,\mathrm{d}s + \int_0^T G(t,s)\bigl|p(s)\bigr|\,\mathrm{d}s \\
&\le \frac{\|u\|_\infty^3}{6}\int_0^T \omega G(t,s)\,\mathrm{d}s + \frac{\|p\|_\infty}{\omega} \\
&\le \frac{\|u\|_\infty^3}{6} + \frac{\|p\|_\infty}{\omega},
\end{aligned}$$

which yields

$$\|\mathbb{T}u\|_\infty \le \frac{\|u\|_\infty^3}{6} + \frac{\|p\|_\infty}{\omega}. \tag{3.7}$$

Now we define the closed ball

$$\mathbb{B} = \left\{u \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}) : \|u\|_\infty \le \Phi\left(\frac{1}{6}, \frac{\|p\|_\infty}{\omega}\right)\right\}.$$

By Lemma 3.1 and (3.7), we know that $\mathbb{T}(\mathbb{B}) \subset \mathbb{B}$.

We next prove that the operator $\mathbb{T} : \mathbb{B} \to \mathbb{B}$ is a strict contraction map. Let $u, v \in \mathbb{B}$, then

$$\begin{aligned}
\bigl|(\mathbb{T}u)(t) - (\mathbb{T}v)(t)\bigr| &= \left|\int_0^T \omega G(t,s)\bigl[\bigl(u(s) - \sin u(s)\bigr) - \bigl(v(s) - \sin v(s)\bigr)\bigr]\,\mathrm{d}s\right| \\
&\le \int_0^T \omega G(t,s)\bigl|\bigl(u(s) - \sin u(s)\bigr) - \bigl(v(s) - \sin v(s)\bigr)\bigr|\,\mathrm{d}s.
\end{aligned} \tag{3.8}$$

Using the estimate (3.4), we have

$$\begin{aligned}
\bigl|\bigl(u(s) - \sin u(s)\bigr) - \bigl(v(s) - \sin v(s)\bigr)\bigr| &\le \frac{1}{2}\Phi^2\left(\frac{1}{6}, \frac{\|p\|_\infty}{\omega}\right)\bigl|u(s) - v(s)\bigr| \\
&\le \frac{9}{8}\left(\frac{\|p\|_\infty}{\omega}\right)^2\bigl|u(s) - v(s)\bigr|.
\end{aligned} \tag{3.9}$$

By (3.5), (3.8) and (3.9), we have

$$\begin{aligned}\|\mathbb{T}u-\mathbb{T}v\|_\infty &\le \frac{9}{8}\left(\frac{\|p\|_\infty}{\omega}\right)^2\|u-v\|_\infty\int_0^T \omega G(t,s)\,\mathrm{d}s\\ &\le \|u-v\|_\infty,\end{aligned}$$

for all $u,v\in\mathbb{B}$. Thus, if the strict inequality in condition (3.5) is satisfied, then the operator $\mathbb{T}:\mathbb{B}\to\mathbb{B}$ is a strict contraction map.

Finally, by using the Banach contraction mapping theorem, we know that the operator $\mathbb{T}$ has a unique fixed point u in $\mathbb{B}$ if the strict inequality in condition (3.5) is satisfied. Note that if the equality in (3.5) holds, we can also obtain the uniqueness from the proof above, although $\mathbb{T}$ may not be a strict contraction map.

By the uniqueness of the T-periodic solution of Eq. (1.2) in $\mathbb{B}$, we know that $\|u\|_\infty$ is smaller than other possible T-periodic solutions of Eq. (1.2). Moreover, (3.6) holds. □

The main result of this section reads as follows.

Theorem 3.3 *Assume that* (3.1) *and* (3.3) *are satisfied. Then there exists a constant* $\rho\in(0,\frac{2\sqrt{2}\omega}{3}]$, *such that the T-periodic solution u of Eq.* (1.2) *obtained in Theorem* 3.2 *is of twist type if* $\|p\|_\infty\le\rho$.

Proof By (3.5) and (3.6), we know

$$\|u\|_\infty\le\frac{3\|p\|_\infty}{2\omega}\le\sqrt{2}<\frac{\pi}{2}. \tag{3.10}$$

Let us fix $g(t,u)=\omega\sin u$. Then the coefficients of the expansion (2.3) gives

$$a(t)=\omega\cos u(t),\qquad b(t)=-\frac{\omega\sin u(t)}{2},\qquad c(t)=-\frac{\omega\cos u(t)}{6}.$$

If (3.3) holds, then the linearized damped equation

$$x''+h(t)x'+\omega\cos u(t)x=0$$

is in the first elliptic region. By (3.10), we have the following estimates:

$$\omega e^{2H_*}\cos\left(\frac{3\|p\|_\infty}{2\omega}\right)\le a(t)\sigma(2h)(t)\le\omega e^{2H^*},$$

where $H_*=\min_{t\in[0,T]}\int_0^t h(s)\,\mathrm{d}s$ and $H^*=\max_{t\in[0,T]}\int_0^t h(s)\,\mathrm{d}s$. Thus (2.4) is satisfied if we take

$$\sigma_1=e^{H_*}\sqrt{\omega\cos\left(\frac{3\|p\|_\infty}{2\omega}\right)},\qquad\sigma_2=\sqrt{\omega}e^{H^*}.$$

By (3.10), we obtain $C_+=0$ and

$$\|C_-\|_1=\int_0^T\frac{\omega\cos u(t)}{6}\sigma(2h)(t)\,\mathrm{d}t\ge\frac{\omega T}{6}e^{2H_*}\cos\left(\frac{3\|p\|_\infty}{2\omega}\right).$$

Therefore, we note that

$$\begin{aligned} \frac{3}{10}\sigma_1^3\left[\frac{\|C_-\|_1}{\sigma_2^2}-\frac{\|C_+\|_1}{\sigma_1^2}\right] &\geq \frac{\omega T}{20}e^{2H_*}\cos\left(\frac{3\|p\|_\infty}{2\omega}\right)\cdot\frac{\sigma_1^3}{\sigma_2^2} \\ &= \frac{T\omega^{3/2}e^{(5H_*-2H^*)}}{20}\left(\cos\left(\frac{3\|p\|_\infty}{2\omega}\right)\right)^{5/2} \\ &:= \Gamma_1\left(\|p\|_\infty\right) \end{aligned}$$

and

$$\begin{aligned} \|B_+\|_1\|B_-\|_1 &\leq \left(\int_0^T\left|\frac{\omega\sin u(t)}{2}\right|\sigma(2h)(t)\,\mathrm{d}t\right)^2 \\ &\leq \frac{T^2\omega^2}{4}e^{4H^*}\sin^2\left(\frac{3\|p\|_\infty}{2\omega}\right) \\ &:= \Gamma_2\left(\|p\|_\infty\right). \end{aligned}$$

Then we can see that (2.5) holds if the following inequality holds:

$$\Gamma_1\left(\|p\|_\infty\right) > \frac{\Gamma_2(\|p\|_\infty)}{\min\{\sin(\frac{3T\sigma_1}{2}),\sin(\frac{3T\sigma_2}{2})\}}. \tag{3.11}$$

By straightforward computations, we obtain

$$\Gamma_1(0) = \frac{T\omega^{3/2}e^{(5H_*-2H^*)}}{20} > \Gamma_2(0) = 0.$$

Therefore, by continuity there exists a constant $\rho \in (0, \frac{2\sqrt{2\omega}}{3}]$ such that (3.11) holds whenever $\|p\|_\infty \leq \rho$. Then the proof is finished by using Theorem 2.1. □

Example 3.4 Consider the following forced damped pendulum equation:

$$u'' - \frac{2\alpha\sin t}{1+\alpha\cos t}u' + \omega\sin u = \eta\sin t, \tag{3.12}$$

where α, ω and η are positive constants with $\alpha < 1$. Moreover, assume that

$$\omega \leq \frac{2(1-\alpha^2)^{\frac{3}{2}}}{(2+\alpha^2)\pi^2}. \tag{3.13}$$

Then there exists a constant $\rho \in (0, \frac{2\sqrt{2\omega}}{3}]$, such that Eq. (3.12) has a 2π-periodic solution which is twist and therefore stable in the sense of Lyapunov if $\eta \leq \rho$.

Proof Equation (3.12) can be regarded as a problem of the form Eq. (1.2), where

$$h(t) = -\frac{2\alpha\sin t}{1+\alpha\cos t}, \qquad p(t) = \eta\sin t.$$

By calculating, if (3.13) holds, then we have

$$\int_0^{2\pi}\sigma(-h)(s)\,\mathrm{d}s\int_0^{2\pi}\sigma(h)(s)\,\mathrm{d}s = \int_0^{2\pi}\left(\frac{1+\alpha}{1+\alpha\cos t}\right)^2\mathrm{d}s\int_0^{2\pi}\left(\frac{1+\alpha\cos t}{1+\alpha}\right)^2\mathrm{d}s$$

$$= \frac{2(2+\alpha^2)\pi^2}{(1-\alpha^2)^{\frac{3}{2}}}$$
$$\le \frac{4}{\omega}.$$

Now the result follows directly from Theorem 3.3. □

4 Multiplicity of periodic solutions

The aim of this section is to prove that the Poincaré map associated to Eq. (1.2) fits the hypotheses of Theorem 2.2 exposed in Sect. 2.1.

Multiplying both sides of Eq. (1.2) by $\sigma(h)(t)$, we have

$$\bigl(\sigma(h)(t)u'\bigr)' + \omega\sigma(h)(t)\sin u = \sigma(h)(t)p(t). \tag{4.1}$$

Note that Eq. (4.1) is equivalent to the planar system

$$\begin{cases} u' = f_1(t,v) = \sigma(-h)(t)v, \\ v' = f_2(t,u) = -\omega\sigma(h)(t)\sin u + \sigma(h)(t)p(t). \end{cases} \tag{4.2}$$

Let $(u,v)^\top = (u(t,\theta,r), v(t,\theta,r))^\top$ be the solution of the system (4.2) satisfying the initial condition

$$\begin{cases} u(0) = \theta, \\ v(0) = r. \end{cases} \tag{4.3}$$

It is easy to verify that there exist two positive constants q_1 and q_2 such that

$$\sqrt{f_1^2(t,v) + f_2^2(t,u)} \le q_1\sqrt{u^2+v^2} + q_2,$$

which guarantees that the solution $(u,v)^\top$ of the initial value problem (4.2)–(4.3) is unique and globally defined. Then we can define the Poincaré map associated to the system (4.2) as

$$S(\theta,r) = \bigl(Q(\theta,r), P(\theta,r)\bigr) = \bigl(u(T,\theta,r), v(T,\theta,r)\bigr).$$

Obviously, the fixed points of the Poincaré map S correspond to the T-periodic solutions of the system (4.2). It follows from 2π-periodicity of the function $\sin u$ and the uniqueness of $(u(t,\theta,r), v(t,\theta,r))^\top$ that

$$u(t,\theta+2\pi,r) = u(t,\theta,r) + 2\pi, \qquad v(t,\theta+2\pi,r) = v(t,\theta,r). \tag{4.4}$$

Then we have

$$Q(\theta+2\pi,r) = Q(\theta,r) + 2\pi, \qquad P(\theta+2\pi,r) = P(\theta,r),$$

which implies that the Poincaré map S is defined on the cylinder. Based on the theorem of differentiability with respect to the initial conditions, it is easy to see that the Poincaré map $S \in C^2(A)$. Since $(u(t,\theta,r), v(t,\theta,r))^\top$ is unique and globally defined, the Poincaré map S is a diffeomorphism of A. The isotopy to the identity is given by the flow

$$\begin{aligned}\Psi_\lambda(\theta,r) &= \Psi\big((1-\lambda)T,\theta,r\big)\\ &= \big(x\big((1-\lambda)T,\theta,r\big), y\big((1-\lambda)T,\theta,r\big)\big), \quad \lambda \in [0,1].\end{aligned}$$

Note that $\Psi_0(\theta,r) = S(\theta,r)$, $\Psi_1(\theta,r) = (\theta,r)$ and this isotopy is valid on the cylinder. Therefore, the Poincaré map $S \in \varepsilon^2(A)$.

Theorem 4.1 *Assume that*

$$\int_0^T \sigma(h)(s)p(s)\,\mathrm{d}s = 0. \tag{4.5}$$

Then Eq. (1.2) *has at least two geometrically distinct T-periodic solutions, and at least one of them is unstable.*

Proof In order to apply Theorem 2.2, we need to prove that the Poincaré map S is exact symplectic and satisfies the boundary twist condition (2.7).

Let us first prove that the Poincaré map S is exact symplectic. Consider the $\mathbb{C}^1$ function

$$\begin{aligned}V(\theta,r) = {}&\int_0^T \left[\frac{(\sigma(h)(t)u'(t,\theta,r))^2}{2} + \omega\sigma(h)(t)\cos u(t,\theta,r)\right]\mathrm{d}t\\ &+ \int_0^T \sigma(h)(t)p(t)u(t,\theta,r)\,\mathrm{d}t.\end{aligned}$$

By (4.4) and (4.5), we have

$$\begin{aligned}V(\theta+2\pi,r) &= \int_0^T \left[\frac{(\sigma(h)(t)u'(t,\theta+2\pi,r))^2}{2} + \omega\sigma(h)(t)\cos u(t,\theta+2\pi,r)\right]\mathrm{d}t\\ &\quad + \int_0^T \sigma(h)(t)p(t)u(t,\theta+2\pi,r)\,\mathrm{d}t\\ &= \int_0^T \left[\frac{(\sigma(h)(t)u'(t,\theta,r))^2}{2} + \omega\sigma(h)(t)\cos u(t,\theta,r)\right]\mathrm{d}t\\ &\quad + \int_0^T \sigma(h)(t)p(t)u(t,\theta,r)\,\mathrm{d}t + 2\pi\int_0^T \sigma(h)(t)p(t)\,\mathrm{d}t\\ &= V(\theta,r).\end{aligned}$$

Let us compute the partial derivatives of $V(\theta,r)$

$$\begin{aligned}V_\theta(\theta,r) = {}&\int_0^T \sigma(h)(t)\left[u'(t,\theta,r)\frac{\partial u'(t,\theta,r)}{\partial\theta} - \omega\sin u(t,\theta,r)\frac{\partial u(t,\theta,r)}{\partial\theta}\right]\mathrm{d}t\\ &+ \int_0^T \sigma(h)(t)p(t)\frac{\partial u(t,\theta,r)}{\partial\theta}\,\mathrm{d}t.\end{aligned}$$

Then, by the second equation of the system (4.2), we have

$$V_\theta(\theta,r) = \int_0^T \sigma(h)(t)\left[u'(t,\theta,r)\frac{\partial u'(t,\theta,r)}{\partial\theta} + v'(t,\theta,r)\frac{\partial u(t,\theta,r)}{\partial\theta}\right]\mathrm{d}t. \tag{4.6}$$

Integrating by parts and using the first equation of the system (4.2), we have

$$\begin{aligned}
&\int_0^T v'(t,\theta,r)\frac{\partial u(t,\theta,r)}{\partial\theta}\,\mathrm{d}t \\
&= \left(\frac{\partial u(t,\theta,r)}{\partial\theta}v(t,\theta,r)\right)\Big|_0^T - \int_0^T \frac{\partial u'(t,\theta,r)}{\partial\theta}v(t,\theta,r)\,\mathrm{d}t \\
&= v(T,\theta,r)\frac{\partial u(T,\theta,r)}{\partial\theta} - r - \int_0^T \sigma(h)(t)u'(t,\theta,r)\frac{\partial u'(t,\theta,r)}{\partial\theta}\,\mathrm{d}t.
\end{aligned}$$

Substituting the above equality into (4.6) gives

$$V_\theta(\theta,r) = v(T,\theta,r)\frac{\partial u(T,\theta,r)}{\partial\theta} - r. \tag{4.7}$$

Analogously, we have

$$V_r(\theta,r) = v(T,\theta,r)\frac{\partial u(T,\theta,r)}{\partial r}. \tag{4.8}$$

By (4.7) and (4.8), we have

$$\begin{aligned}
\mathrm{d}V &= V_\theta(\theta,r)\,\mathrm{d}\theta + V_r(\theta,r)\,\mathrm{d}r \\
&= \left[v(T,\theta,r)\frac{\partial u(T,\theta,r)}{\partial\theta} - r\right]\mathrm{d}\theta + v(T,\theta,r)\frac{\partial u(T,\theta,r)}{\partial r}\,\mathrm{d}r \\
&= v(T,\theta,r)\left[\frac{\partial u(T,\theta,r)}{\partial\theta}\,\mathrm{d}\theta + \frac{\partial u(T,\theta,r)}{\partial r}\,\mathrm{d}r\right] - r\,\mathrm{d}\theta \\
&= v(T,\theta,r)\,\mathrm{d}u(T,\theta,r) - r\,\mathrm{d}\theta \\
&= P\,\mathrm{d}Q - r\,\mathrm{d}\theta,
\end{aligned}$$

which means that the Poincaré map $S(\theta,r)$ is exact symplectic.

We next prove that the Poincaré map S satisfies the boundary twist condition (2.7). Integrating the second equation of the system (4.2) from 0 to t with $t \in [0,t]$, we obtain

$$\begin{aligned}
v(t) &= r + \int_0^t \left[-\omega\sigma(h)(s)\sin u + \sigma(h)(s)p(s)\right]\mathrm{d}s \\
&\geq r - \int_0^T \left(\omega\sigma(h)(s) + \left|\sigma(h)(s)p(s)\right|\right)\mathrm{d}s \\
&= r - T\left(\omega\overline{\sigma(h)} + \overline{\left|\sigma(h)p\right|}\right),
\end{aligned}$$

where $\overline{\xi} = \frac{1}{T}\int_0^T \xi(s)\,\mathrm{d}s$. Thus we can find a positive constant $\rho_1 \geq T(\omega\overline{\sigma(h)} + \overline{|\sigma(h)p|}) > 0$ such that $v(t) > 0$ if $r > \rho_1$, $\forall t \in [0,T]$. By the first equation of (4.2), we know that

$$u'(t) = \sigma(-h)(t)v(t) > 0,$$

which means that u is increasing for $t \in [0, T]$. So we can choose a positive constant ρ with $\rho > \rho_1$, then we have

$$Q(\theta, \rho) - \theta = u(T, \theta, \rho) - u(0, \theta, \rho) > 0.$$

By a standard compactness argument, we can conclude that there exists $\epsilon > 0$ such that

$$Q(\theta, \rho) - \theta > \epsilon, \quad \theta \in [0, 2\pi).$$

Analogously, we have

$$Q(\theta, -\rho) - \theta < -\epsilon, \quad \theta \in [0, 2\pi).$$

In order to apply Theorem 2.2, we take $A = \mathbb{R} \times [-\rho, \rho]$. By the solutions of the system (4.2) are globally defined, one can find a larger B such that $S(A) \subset \operatorname{int} B$. Since the right-hand side of the system (4.2) is analytic with respect to the variables (u, v), the Poincaré map S is also analytic, as follows from the analytic dependence on the initial conditions.

Up to now, all the conditions of Theorem 2.2 are satisfied, thus we see that the Poincaré map

$$S(\theta, r) = \bigl(Q(\theta, r), P(\theta, r)\bigr) = \bigl(u(T, \theta, r), v(T, \theta, r)\bigr)$$

has at least two fixed points, and at least one of them is unstable. That is, Eq. (1.2) has at least two geometrically distinct T-periodic solutions and at least one of them is unstable. □

Now we consider the existence of the so-called T-periodic solutions with winding number of Eq. (1.2), i.e., solutions u such that

$$u(t + T) = u(t) + 2N\pi, \quad N \in \mathbb{Z}, \forall t \in \mathbb{R}.$$

Such solutions are also called running solutions. Obviously, we get the usual T-periodic solutions when $N = 0$.

Let u be a T-periodic solution of Eq. (1.2) with winding number N. Taking the change of variables

$$y(t) = u(t) - \frac{2N\pi}{T} t.$$

Obviously, we have

$$\begin{aligned} y(t + T) &= u(t + T) - \frac{2N\pi}{T}(t + T) \\ &= u(t) - \frac{2N\pi}{T} t \\ &= y(t), \end{aligned}$$

which implies that T-periodic solutions with winding number N of Eq. (1.2) correspond to T-periodic solutions of the equation

$$y'' + h(t)y' + \omega \sin\left(y + \frac{2N\pi}{T}t\right) = p(t) + \frac{2N\pi}{T}h(t). \tag{4.9}$$

Proceeding as in the proof of Theorem 4.1, we can prove the following result.

Theorem 4.2 *Assume that* (4.5) *holds and*

$$\int_0^T h(s)\sigma(h)(s)\,\mathrm{d}s = 0. \tag{4.10}$$

Then for every integer N, Eq. (1.2) *has at least two geometrically distinct T-periodic solutions with winding number N, and at least one of them is unstable.*

Let u is a periodic solution of (1.2) and take its minimal period $\tau > 0$. Since T is the minimal period of p, we have $\tau = kT, k \in \mathbb{N}$ with $k \geq 1$. Therefore, if $k = 1$, x is a harmonic (periodic) solution of Eq. (1.2); if $k \in \mathbb{N}$ with $k > 1$, u is a subharmonic solution of Eq. (1.2).

Finally, we study the existence of k-order subharmonic solutions with winding number N of Eq. (1.2), that is,

$$u(t + kT) = u(t) + 2N\pi, \quad k \in \mathbb{N} \text{ with } k > 1, \forall t \in \mathbb{R}. \tag{4.11}$$

Theorem 4.3 *Assume that* (4.5) *and* (4.10) *hold. Then, for each couple of relatively prime natural numbers N and k, Eq.* (1.2) *has at least two geometrically distinct k-order subharmonic solutions with winding number N and kT is the minimal period. Moreover, at least one of them is unstable.*

Proof By Theorem 4.2, with T replaced by kT, we see that Eq. (1.2) has at least two geometrically distinct k-order subharmonic solutions with winding number N. Moreover, at least one of them is unstable. We only need to verify that kT is the minimal period of periodic solutions with winding number N for Eq. (1.2).

Conversely, suppose that mT is the minimal period, where $m \in \{1, 2, \ldots, k-1\}$. So there exists an integer i with $i \neq 0$, such that

$$u(t + mT) = u(t) + 2i\pi, \quad \forall t \in \mathbb{R}. \tag{4.12}$$

Notice that there exist positive integers l_1 and l_2 such that

$$l_1 m = l_2 k. \tag{4.13}$$

By (4.12), we have

$$u(t + l_1 mT) = u(t) + 2l_1 i\pi, \quad \forall t \in \mathbb{R}.$$

By (4.11), we have

$$u(t + l_2 kT) = u(t) + 2l_2 N\pi, \quad \forall t \in \mathbb{R}.$$

According to the above two equalities and the uniqueness of the solution u, we have

$$l_1 i = l_2 N. \tag{4.14}$$

By (4.13) and (4.14), we have

$$\frac{N}{k} = \frac{i}{m},$$

which is impossible because N and k are relatively prime and i is a nonzero integer and $m \in \{1, 2, \dots, k-1\}$. □

Example 4.4 Consider the following damped pendulum equation:

$$u'' - \frac{2\alpha \sin t}{1 + \alpha \cos t} u' + \omega \sin u = \frac{2\alpha \sin t}{1 + \alpha \cos t}, \tag{4.15}$$

where α and ω are positive constants with $\alpha < 1$. Then the following conclusions hold:

(I) Equation (4.15) has at least two geometrically distinct 2π-periodic solutions, and at least one of them is unstable.

(II) For every integer N, Eq. (4.15) has at least two geometrically distinct 2π-periodic solutions with winding number N, and at least one of them is unstable.

(III) For each couple of relatively prime natural numbers N and k, Eq. (4.15) has at least two geometrically distinct k-order subharmonic solutions with winding number N and kT is the minimal period. Moreover, at least one of them is unstable.

Proof Equation (4.15) can be regarded as a problem of the form Eq. (1.2), where

$$h(t) = -\frac{2\alpha \sin t}{1 + \alpha \cos t}, \qquad p(t) = \frac{2\alpha \sin t}{1 + \alpha \cos t}.$$

By calculating, we have

$$\begin{aligned} \int_0^{2\pi} p(s)\sigma(h)(s)\,\mathrm{d}s &= \int_0^T \frac{2\alpha \sin t}{1 + \alpha \cos t} \cdot \left(\frac{1 + \alpha \cos t}{1 + \alpha}\right)^2 \mathrm{d}t = 0, \\ \int_0^{2\pi} h(s)\sigma(h)(s)\,\mathrm{d}s &= -\int_0^T \frac{2\alpha \sin t}{1 + \alpha \cos t} \cdot \left(\frac{1 + \alpha \cos t}{1 + \alpha}\right)^2 \mathrm{d}t = 0. \end{aligned}$$

Now the results (I)–(III) follow directly from Theorems 4.1–4.3. □

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