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Infinitely many periodic solutions of planar Hamiltonian systems via the Poincaré–Birkhoff theorem

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Abstract

In this paper, we study the multiplicity of periodic solutions of one kind of planar Hamiltonian systems with a nonlinear term satisfying semilinear conditions. Using a generalized Poincaré–Birkhoff fixed point theorem, we prove that the system has infinitely many periodic solutions, provided that the time map tends to zero.

MSC: Primary 34C25; secondary 34C15

Keywords: Planar Hamiltonian system; Time map; Periodic solution

1 Introduction

We are concerned with the multiplicity of periodic solutions of planar Hamiltonian systems of the type

$$\begin{cases} x' = f(y) + p_1(t, x, y), \\ y' = -g(x) + p_2(t, x, y), \end{cases} \quad (1.1)$$

where $f, g : \mathbf{R} \rightarrow \mathbf{R}$ are continuous, and $p_i : \mathbf{R}^3 \rightarrow \mathbf{R}$ ($i = 1, 2$) are continuous and 2π -periodic with respect to the first variable t .

The periodic problem for planar Hamiltonian systems is a classical topic in nonlinear analysis and ordinary differential equations, which has been widely studied in literature by using various different methods such as phase plane analysis, topological degree, fixed point theorems, variational methods (see [1–5, 10, 11, 14, 19, 20] and the references therein). For instance, using the Poincaré–Bohl fixed point theorem, Fonda and Sfecci [11] studied the existence of periodic solutions of planar Hamiltonian systems

$$Jz' = \nabla_z H(t, z), \quad z = (x, y) \in \mathbf{R}^2, \quad (1.2)$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard symplectic matrix, and $H : [0, T] \times \mathbf{R}^2 \rightarrow \mathbf{R}$ is differentiable with respect to the second variable. When $\nabla_z H(t, z)$ satisfies some semilinear conditions at infinity, it was proved in [11] that (1.2) has at least one T -periodic solution. Using a generalized Poincaré–Birkhoff fixed point theorem, Boscaggin [4] studied the multiplicity of periodic solutions of (1.2), provided that $\nabla_z H(t, z)$ satisfies some superlinear condition

at infinity. It was pointed out that the main theorem (Theorem 2.3) in [4] applies to the forced Duffing equation

$$x'' + g(x) = p(t, x) \tag{1.3}$$

when g satisfies the superlinear condition

$$\lim_{|x| \rightarrow +\infty} \frac{g(x)}{x} = +\infty. \tag{1.4}$$

We note that Ding and Zanolin [7] proved the multiplicity of periodic solutions of Eq. (1.3) by replacing (1.4) with a weaker assumption on the time map of the autonomous equation $x'' + g(x) = 0$, namely that the limit of the time map equals zero. Clearly, this condition is not included in [4].

In the present paper, we study the multiplicity of periodic solutions of (1.1) in terms of the time map. Assume that g satisfies the condition

$$\lim_{|x| \rightarrow +\infty} \operatorname{sgn}(x)g(x) = +\infty \tag{h_1}$$

and f satisfies the semilinear condition at infinity

$$0 < \alpha = \liminf_{|y| \rightarrow +\infty} \frac{f(y)}{y} \leq \limsup_{|y| \rightarrow +\infty} \frac{f(y)}{y} = \beta < +\infty. \tag{h_2}$$

Let $G(x) = \int_0^x g(s) ds$. Define the function

$$\tau(c) = \left| \int_0^c \frac{dx}{\sqrt{2(G(c) - G(x))}} \right|.$$

From condition (h₁) we know that $\tau(c)$ is continuous for $|c|$ large enough; $\tau(c)$ is usually called the time map related to the autonomous equation $x'' + g(x) = 0$. The properties of $\tau(c)$ were studied deeply in [6, 7, 18]. Assume that τ satisfies

$$\lim_{|c| \rightarrow +\infty} \tau(c) = 0. \tag{h_3}$$

From [18] we know that (h₃) holds if g satisfies the superlinear condition (1.4). Throughout the paper, we always assume that there exists a constant $M > 0$ such that

$$|p_i(t, x, y)| \leq M \quad \text{for all } t, x, y \in \mathbf{R} \text{ and } i = 1, 2. \tag{h_4}$$

Moreover, there is a function $\mathcal{U} : \mathbf{R}^3 \rightarrow \mathbf{R}$ such that

$$\frac{\partial \mathcal{U}}{\partial y} = p_1(t, x, y), \quad \frac{\partial \mathcal{U}}{\partial x} = -p_2(t, x, y).$$

In this case, system (1.1) is a Hamiltonian system. We can give simple examples of such functions. For example,

$$\mathcal{U}(t, x, y) = p(t) \sin x \sin y.$$

Clearly, if $p_1(t, x, y) = p_1(t, y)$ and $p_2(t, x, y) = p_2(t, x)$, then (1.1) is a Hamiltonian system.

Note that we can write system (1.1) in the form (1.2). Let

$$H(t, z) = H(t, x, y) = G(x) + F(y) + \mathcal{U}(t, x, y), \quad z = (x, y) \in \mathbf{R}^2,$$

where $F(y) = \int_0^y f(s) ds$. Then we have that $\nabla_z H(t, z) = (g(x) - p_2(t, x, y), f(y) + p_1(t, x, y))$. Since g only satisfies condition (h_1) , we know that $\nabla_z H(t, z)$ does not satisfy the semilinear condition as in [11] or the superlinear condition as in [4].

Using a generalized Poincaré–Birkhoff fixed point theorem and the phase-plane analysis method, we prove the following results.

Theorem 1.1 *Assume that conditions (h_i) ($i = 1, \dots, 4$) hold. Then system (1.1) has infinitely many 2π -periodic solutions $\{(x_j(t), y_j(t))\}_{j=1}^\infty$ that satisfy*

$$\lim_{j \rightarrow \infty} \left(\min_{t \in [0, 2\pi]} (x_j^2(t) + y_j^2(t)) \right) = +\infty.$$

Theorem 1.2 *Assume that conditions (h_i) ($i = 1, \dots, 4$) hold. Then for any given integer $m \geq 2$, system (1.1) has infinitely many $2m\pi$ -periodic solutions $\{(x_j(t), y_j(t))\}_{j=1}^\infty$ that are not $2k\pi$ -periodic for $1 \leq k \leq m - 1$ and satisfy*

$$\lim_{j \rightarrow \infty} \left(\min_{t \in [0, 2m\pi]} (x_j^2(t) + y_j^2(t)) \right) = +\infty.$$

Corollary 1.3 *Assume that conditions (h_2) , (h_4) , and (1.4) hold. Then the conclusions of Theorems 1.1 and 1.2 still hold.*

Remark 1.4 Ding and Zanolin [7] proved the multiplicity of periodic solutions of Eq. (1.3) when conditions (h_1) and (h_3) hold and $p(t, x)$ is bounded. Note that Eq. (1.3) is equivalent to the planar Hamiltonian system $x' = y, y' = -g(x) + p(t, x)$, which is a particular form of (1.1). Therefore our conclusions generalize the main results in [7].

Remark 1.5 We will prove the above results under the additional assumption that the solutions to Cauchy problems of (1.1) are unique. It is shown in Sect. 4 that this requirement is not restrictive and that our results are valid when the uniqueness of the solutions to Cauchy problems is not satisfied.

Throughout this paper, by \mathbf{R} and \mathbf{N} we denote the sets of real and natural numbers, respectively.

2 Several lemmas

In this section, we perform some phase-plane analysis for system (1.1). Let $(x(t), y(t)) = (x(t, x_0, y_0), y(t, x_0, y_0))$ be the solution of (1.1) satisfying the initial value

$$x(0, x_0, y_0) = x_0, \quad y(0, x_0, y_0) = y_0.$$

We denote

$$G(x) = \int_0^x g(s) ds, \quad F(y) = \int_0^y f(s) ds.$$

Lemma 2.1 *Assume that conditions (h_i) ($i = 1, 2, 4$) hold. Then every solution $(x(t), y(t))$ of (1.1) exists on the whole t -axis.*

Proof In view of (h_1) , there exists a constant $c_0 > 0$ such that

$$\operatorname{sgn}(x)g(x) > 0 \quad \text{for } |x| \geq c_0. \tag{2.1}$$

Let us define two Lyapunov-like functions $V_{\pm} : \mathbf{R}^2 \rightarrow \mathbf{R}$:

$$V_{\pm}(x, y) = G(x) + F(y) \pm yh(x),$$

where $h : \mathbf{R} \rightarrow \mathbf{R}$ is a continuously differentiable function such that $h(x) = M \operatorname{sgn}(x)$ for $|x| \geq c_0$ with M from (h_4) . We will prove that V_{\pm} are coercive, that is, $V_{\pm}(x, y) \rightarrow +\infty$ as $|x| + |y| \rightarrow +\infty$. From (h_1) we know that $\lim_{|x| \rightarrow +\infty} G(x) = +\infty$. From (h_2) we get that

$$\frac{\alpha}{2} \leq \liminf_{|y| \rightarrow +\infty} \frac{F(y)}{y^2} \leq \limsup_{|y| \rightarrow +\infty} \frac{F(y)}{y^2} \leq \frac{\beta}{2}.$$

Since $h(x)$ is bounded, we further see that the inequalities

$$\frac{\alpha}{2} \leq \liminf_{|y| \rightarrow +\infty} \frac{F(y) \pm yh(x)}{y^2} \leq \limsup_{|y| \rightarrow +\infty} \frac{F(y) \pm yh(x)}{y^2} \leq \frac{\beta}{2} \tag{2.2}$$

hold uniformly with respect to $x \in \mathbf{R}$. From (2.2) we have that the limits

$$\lim_{|y| \rightarrow +\infty} (F(y) \pm yh(x)) = +\infty$$

hold uniformly with respect to $x \in \mathbf{R}$. Therefore $V_{\pm}(x, y) \rightarrow +\infty$ as $|x| + |y| \rightarrow +\infty$.

Next, we show that every solution $(x(t), y(t))$ of (1.1) is defined on the interval $[0, +\infty)$. Set $V_+(t) = V_+(x(t), y(t))$. We first prove that there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$V'_+(t) \leq c_1 V_+(t) + c_2. \tag{2.3}$$

For simplicity, we denote $p_1(t) = p_1(t, x(t), y(t))$, $p_2(t) = p_2(t, x(t), y(t))$. By (1.1) we have

$$\begin{aligned} V'_+(t) &= (p_1(t) - h(x(t)))g(x(t)) + y(t)h'(x(t))(f(y(t)) + p_1(t)) \\ &\quad + p_2(t)(f(y(t)) + h(x(t))). \end{aligned}$$

If $|x(t)| \geq c_0$, then we infer from (h_4) , the definition of $h(x)$, and (2.1) that

$$(p_1(t) - h(x(t)))g(x(t)) \leq 0.$$

If $|x(t)| \leq c_0$, then it follows from (h_4) and the continuity of $g(x)$ and $h(x)$ that there exists $\alpha_0 > 0$ such that

$$(p_1(t) - h(x(t)))g(x(t)) \leq \alpha_0.$$

In view of (h_2) , we conclude that there exist $l_0 > 0$ and $\alpha_1 > 0$ such that

$$|f(y)| \leq l_0|y| + \alpha_1 \quad \text{for } y \in \mathbf{R}. \tag{2.4}$$

Since $h'(x) = 0$ for $|x| \geq c_0$, we know that $h'(x)$ is bounded, and then there exists $\beta_0 > 0$ such that $|h'(x)| \leq \beta_0, x \in \mathbf{R}$. It follows from (2.4) that

$$|y(t)h'(x(t))f(y(t))| \leq \beta_0 l_0 y^2(t) + \alpha_1 \beta_0 |y(t)|.$$

Meanwhile, in view of (h_4) , we have

$$|y(t)h'(x(t))p_1(t)| \leq M\beta_0 |y(t)|.$$

From (h_4) and (2.4) we get that

$$|p_2(t)f(y(t))| \leq Ml_0 |y(t)| + M\alpha_1.$$

Since $h(x)$ is bounded for $x \in \mathbf{R}$, there exists $\beta'_0 > 0$ such that $|h(x)| \leq \beta'_0, x \in \mathbf{R}$. Consequently, we have

$$|p_2(t)h(x(t))| \leq M\beta'_0.$$

Therefore, we obtain

$$\begin{aligned} V'_+(t) &\leq \beta_0 l_0 y(t)^2 + (\alpha_1 \beta_0 + M\beta_0 + Ml_0) |y(t)| + (\alpha_0 + M\alpha_1 + M\beta'_0) \\ &\leq \beta_1 y(t)^2 + \beta'_1 \end{aligned} \tag{2.5}$$

with $\beta_1 = \beta_0 l_0 + \frac{1}{2}$ and $\beta'_1 = \frac{1}{2}(\alpha_1 \beta_0 + M\beta_0 + Ml_0)^2 + (\alpha_0 + M\alpha_1 + M\beta'_0)$. From (2.2) we know that there exist $l_1 > 0$ and $\beta_2 > 0$ such that

$$y^2 \leq l_1(F(y) + yh(x)) + \beta_2 \quad \text{for } (x, y) \in \mathbf{R}^2. \tag{2.6}$$

Combining (2.5) and (2.6), we get

$$V'_+(t) \leq \beta_1 l_1 (F(y(t)) + y(t)h(x(t))) + \beta_1 \beta_2 + \beta'_1.$$

Since $\lim_{|x| \rightarrow +\infty} G(x) = +\infty$, there exists $G_0 > 0$ such that $G(x) + G_0 \geq 0$ for $x \in \mathbf{R}$. We conclude that

$$V'_+(t) \leq \beta_1 l_1 (G(x(t)) + F(y(t)) + y(t)h(x(t))) + \beta_1 l_1 G_0 + \beta_1 \beta_2 + \beta'_1.$$

Set $c_1 = \beta_1 l_1$ and $c_2 = \beta_1 l_1 G_0 + \beta_1 \beta_2 + \beta'_1$. We get that $V'_+(t) \leq c_1 V_+(t) + c_2$. Then, for any finite $\omega > 0$, we have

$$V_+(t) \leq V_+(0)e^{c_1 \omega} + \frac{c_2}{c_1} (e^{c_1 \omega} - 1) \quad \text{for } t \in [0, \omega).$$

Since V_+ is coercive, there is no blow-up for $(x(t), y(t))$ on any finite interval $[0, \omega)$. Therefore, $(x(t), y(t))$ exists on the interval $[0, +\infty)$.

To prove that $(x(t), y(t))$ exists on the interval $(-\infty, 0]$, we consider another Lyapunov-like function $V_-(x, y)$. Set $V_-(t) = V_-(x(t), y(t))$. Using the same methods as before, we can show that there exist two positive constants d_1, d_2 such that

$$V'_-(t) \geq -d_1 V_-(t) - d_2. \tag{2.7}$$

Then, for any $\omega > 0$, we have

$$V_-(t) \leq V_-(0)e^{d_1\omega} + \frac{d_2}{d_1}(e^{d_1\omega} - 1) \quad \text{for } t \in (-\omega, 0].$$

Since V_- is coercive, there is also no blow-up for $(x(t), y(t))$ on any finite interval $(-\omega, 0]$. Therefore, $(x(t), y(t))$ exists on the interval $(-\infty, 0]$. The proof is complete. \square

Since the uniqueness of the solutions to Cauchy problems of (1.1) is assumed, we can define the Poincaré map $P : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ as follows:

$$P : (x_0, y_0) \rightarrow (x_1, y_1) = (x(2\pi, x_0, y_0), y(2\pi, x_0, y_0)).$$

It is well known that the Poincaré map P is an area-preserving homeomorphism. The fixed points of P correspond to the 2π -periodic solutions of (1.1).

On the basis of the global existence of solutions of (1.1), we can get the elasticity property of solutions of (1.1) by using a classical result (Theorem 6.5 in [15]).

Lemma 2.2 *Assume that conditions (h_i) ($i = 1, 2, 4$) hold. Then, for any $T > 0$ and $R_1 > 0$, there exists $R_2 > R_1$ such that:*

- (1) *If $x_0^2 + y_0^2 \leq R_1^2$, then $x(t)^2 + y(t)^2 \leq R_2^2$, $t \in [0, T]$.*
- (2) *If $x_0^2 + y_0^2 \geq R_2^2$, then $x(t)^2 + y(t)^2 \geq R_1^2$, $t \in [0, T]$.*

From Lemma 2.2 we know that if $x_0^2 + y_0^2$ is large enough, then $x^2(t) + y^2(t) \neq 0$, $t \in [0, T]$. Thus we can take the polar coordinate transformation

$$x(t) = r(t) \cos \theta(t), \quad y(t) = r(t) \sin \theta(t).$$

Under this transformation, system (1.1) becomes

$$\begin{cases} \frac{dr}{dt} = -g(r \cos \theta) \sin \theta + f(r \sin \theta) \cos \theta + p_1(t, r, \theta) \cos \theta + p_2(t, r, \theta) \sin \theta, \\ \frac{d\theta}{dt} = -\frac{1}{r}g(r \cos \theta) \cos \theta - \frac{1}{r}f(r \sin \theta) \sin \theta - \frac{1}{r}p_1(t, r, \theta) \sin \theta + \frac{1}{r}p_2(t, r, \theta) \cos \theta, \end{cases} \tag{2.8}$$

where $p_1(t, r, \theta) = p_1(t, r \cos \theta, r \sin \theta)$ and $p_2(t, r, \theta) = p_2(t, r \cos \theta, r \sin \theta)$. Let us denote by $(r(t), \theta(t)) = (r(t, r_0, \theta_0), \theta(t, r_0, \theta_0))$ the solution of (2.8) satisfying the initial value

$$r(0, r_0, \theta_0) = r_0, \quad \theta(0, r_0, \theta_0) = \theta_0$$

with $x_0 = r_0 \cos \theta_0, y_0 = r_0 \sin \theta_0$. We can rewrite the Poincaré map P in the polar coordinate form $P : (r_0, \theta_0) \rightarrow (r_1, \theta_1)$,

$$r_1 = r(2\pi, r_0, \theta_0), \quad \theta_1 = \theta(2\pi, r_0, \theta_0) + 2l\pi,$$

where l is an arbitrary integer.

Lemma 2.3 *Assume that conditions (h_i) ($i = 1, 2, 4$) hold. Then, for any $T > 0$, there exists a constant $R > 0$ such that if $r(t) \geq R, t \in [0, T]$, then*

$$\theta'(t) < 0, \quad t \in [0, T].$$

Proof It follows from (h_1) that there exists $a_1 > 0$ such that

$$\operatorname{sgn}(x)g(x) > M \quad \text{for } |x| \geq a_1,$$

which, together with (h_4) , implies that

$$\operatorname{sgn}(x)(g(x) - p_2(t, x, y)) > 0 \quad \text{for } |x| \geq a_1, t, y \in \mathbf{R}.$$

From (h_2) and (h_4) we know that there exist two constants $\gamma > 0$ and $a_2 > 0$ such that

$$\frac{f(y) + p_1(t, x, y)}{y} \geq \gamma \quad \text{for } |y| \geq a_2, t, x \in \mathbf{R}.$$

Therefore, if $|x(t)| \geq a_1$ and $|y(t)| \geq a_2$, then we have

$$\frac{d\theta}{dt} = -\frac{1}{r}(g(r \cos \theta) - p_2(t, r, \theta)) \cos \theta - \frac{1}{r}(f(r \sin \theta) + p_1(t, r, \theta)) \sin \theta < 0. \tag{2.9}$$

If $|x(t)| \leq a_1$ and $|y(t)| \geq a_2$, then we have

$$\frac{1}{r}(f(r \sin \theta) + p_1(t, r, \theta)) \sin \theta \geq \gamma \sin^2 \alpha,$$

where $\alpha = \arctan \frac{a_2}{a_1}$. On the other hand, there exists $A_1 > 0$ such that

$$|g(x)| + |p_2(t, x, y)| \leq A_1 \quad \text{for } |x| \leq a_1, t, y \in \mathbf{R}.$$

It follows that if $r(t)$ is large enough and $|x(t)| \leq a_1$, then

$$\left| \frac{1}{r}(g(r \cos \theta) - p_2(t, r, \theta)) \right| \leq \frac{A_1}{r(t)} \leq \frac{\gamma}{2} \sin^2 \alpha.$$

Consequently, if $r(t)$ is large enough and $|x(t)| \leq a_1, |y(t)| \geq a_2$, then we get

$$\begin{aligned} \frac{d\theta}{dt} &\leq \left| \frac{1}{r}(g(r \cos \theta) - p_2(t, r, \theta)) \cos \theta \right| - \frac{1}{r}[f(r \sin \theta) + p_1(t, r, \theta)] \sin \theta \\ &\leq -\frac{\gamma}{2} \sin^2 \alpha. \end{aligned} \tag{2.10}$$

Finally, we know that there exists $A_2 > 0$ such that

$$|f(y)| + |p_1(t, x, y)| \leq A_2 \quad \text{for } |y| \leq a_2, t, x \in \mathbf{R}. \tag{2.11}$$

If $|y(t)| \leq a_2$ and $r(t)$ is large enough, then we have that $|x(t)|$ is also large enough. Therefore we get from (h_1) , (h_4) , and (2.11) that, for $r(t)$ large enough,

$$\begin{aligned} & [g(r \cos \theta) - p_2(t, r, \theta)] \cos \theta + [f(r \sin \theta) + p_1(t, r, \theta)] \sin \theta \\ & \geq g(r \cos \theta) \cos \theta - (M + A_2) > 0. \end{aligned}$$

Furthermore

$$\frac{d\theta}{dt} = -\frac{1}{r} \{ [g(r \cos \theta) - p_2(t, r, \theta)] \cos \theta + [f(r \sin \theta) + p_1(t, r, \theta)] \sin \theta \} < 0. \tag{2.12}$$

Combining (2.9), (2.10), and (2.12), we get the conclusion of Lemma 2.3. □

Lemma 2.4 *Assume that conditions (h_i) ($i = 1, \dots, 4$) hold. Let m be a positive integer. Then, for any given integer $n \geq 1$, there exists a constant $\varrho_n > 0$ such that, for $r_0 \geq \varrho_n$,*

$$\theta(2m\pi, r_0, \theta_0) - \theta_0 < -2n\pi.$$

Proof From conditions (h_1) and (h_2) we know that there exists $d > 0$ such that

$$\text{sgn}(x)g(x) \geq M, \quad |x| \geq d \quad \text{and} \quad \text{sgn}(y)f(y) \geq M, \quad |y| \geq d. \tag{2.13}$$

From Lemma 2.3 we know that there is a constant $c_m \geq d$ such that

$$r(t) \geq c_m, \quad t \in [0, 2m\pi]$$

and

$$\theta'(t) < 0, \quad t \in [0, 2m\pi].$$

Then the solution $(r(t), \theta(t))$ moves clockwise in the region

$$D = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \geq c_m^2\}.$$

We now decompose the set D into eight regions as follows:

$$\begin{aligned} D_1 &= \{(x, y) \in D : |x| \leq c_m, y > 0\}, \\ D_2 &= \{(x, y) \in D : x \geq c_m, y \geq d\}, \\ D_3 &= \{(x, y) \in D : x \geq c_m, |y| \leq d\}, \\ D_4 &= \{(x, y) \in D : x \geq c_m, y \leq -d\}, \\ D_5 &= \{(x, y) \in D : |x| \leq c_m, y < 0\}, \end{aligned}$$

$$D_6 = \{(x, y) \in D : x \leq -c_m, y \leq -d\},$$

$$D_7 = \{(x, y) \in D : x \leq -c_m, |y| \leq d\},$$

$$D_8 = \{(x, y) \in D : x \leq -c_m, y \geq d\}.$$

Next, we will estimate the time needed for the solution $(x(t), y(t))$ to pass through the regions D_i ($i = 1, \dots, 8$), respectively. Without loss of generality, we assume that $(x_0, y_0) \in D_1$. Then, $(x(t), y(t))$ rotates following the cycle

$$D_1 \rightarrow D_2 \rightarrow D_3 \rightarrow D_4 \rightarrow D_5 \rightarrow D_6 \rightarrow D_7 \rightarrow D_8 \rightarrow D_1.$$

Given k ($k = 1, \dots, 8$), let $[t_1, t_2] \subset [0, 2m\pi]$ be such that

$$(x(t), y(t)) \in D_k, \quad t \in [t_1, t_2],$$

and

$$(x(t_i), y(t_i)) \in \partial D_k \quad (i = 1, 2).$$

We first treat the case $(x(t), y(t)) \in D_1, t \in [t_1, t_2]$. It follows from (h_2) that there exist constants $\beta_0 \geq \alpha_0 > 0$ and $M_0 > 0$ such that

$$\alpha_0 y - M_0 \leq f(y) \leq \beta_0 y + M_0, \quad y \geq 0. \tag{2.14}$$

Therefore, if $(x(t), y(t)) \in D_1$, then we have

$$x'(t) = f(y(t)) + p_1(t, x(t), y(t)) \geq \alpha_0 y(t) - M_1$$

with $M_1 = M_0 + M$. From Lemma 2.2 we know that, for any constant $l (> \sqrt{c_m^2 + \frac{M_1^2}{\alpha_0^2}})$, there is a constant $l_0 > l$ such that, for $r_0 \geq l_0$,

$$r(t) \geq l, \quad t \in [0, 2m\pi].$$

As a result, we get that, for $r_0 \geq l_0$ and $(x(t), y(t)) \in D_1, t \in [t_1, t_2]$,

$$y(t) = \sqrt{r^2(t) - x^2(t)} \geq \sqrt{l^2 - c_m^2}.$$

Consequently,

$$x'(t) \geq \alpha_0 \sqrt{l^2 - c_m^2} - M_1 > 0,$$

which implies that, for any sufficiently small $\varepsilon > 0$,

$$t_2 - t_1 \leq \frac{2c_m}{\alpha_0 \sqrt{l^2 - c_m^2} - M_1} < \varepsilon, \tag{2.15}$$

provided that l is sufficiently large. According to Lemma 2.2, we further know that (2.15) holds when r_0 is sufficiently large.

Similarly, we have that if $(x(t), y(t)) \in D_5, t \in [t_1, t_2]$, then

$$t_2 - t_1 < \varepsilon,$$

provided that r_0 is sufficiently large.

We next treat the case $(x(t), y(t)) \in D_2, t \in [t_1, t_2]$. Let us define $W_+ : \mathbf{R}^2 \rightarrow \mathbf{R}$ as follows:

$$W_+(x, y) = F(y) + G(x) - M(x - y).$$

Set

$$W_+(t) = F(y(t)) + G(x(t)) - M(x(t) - y(t)).$$

If $x(t) \geq c_m$ and $y(t) \geq d$, then we get from (h₄) and (2.14) that

$$\begin{aligned} W'_+(t) &= f(y(t))(p_2(t, x, y) - M) + g(x(t))(p_1(t, x, y) - M) + M(p_2(t, x, y) - p_1(t, x, y)) \\ &\leq f(y(t))(p_2(t, x, y) - M) + g(x(t))(p_1(t, x, y) - M) + M(M - p_1(t, x, y)) \\ &\leq f(y(t))(p_2(t, x, y) - M) + (g(x(t)) - M)(p_1(t, x, y) - M) \leq 0, \end{aligned}$$

which implies that $W_+(t)$ is decreasing when $(x(t), y(t))$ lies in the field D_2 . Hence, we get that, for $t \in [t_1, t_2]$,

$$W_+(t) \geq W_+(t_2).$$

Consequently,

$$\begin{aligned} &F(y(t)) + G(x(t)) - M(x(t) - y(t)) \\ &\geq F(y(t_2)) + G(x(t_2)) - M(x(t_2) - y(t_2)), \quad t \in [t_1, t_2]. \end{aligned} \tag{2.16}$$

Since $y(t_2) = d$, there is a constant $B > 0$ such that $|F(y(t_2))| \leq B$. It follows from (2.16) that

$$F(y(t)) + My(t) \geq (G(x(t_2)) - G(x(t))) - M(x(t_2) - x(t)) - M_1, \quad t \in [t_1, t_2], \tag{2.17}$$

where $M_1 = B + Md$. According to (2.14), we have that, for $y \geq 0$,

$$F(y) \leq \frac{1}{2}\beta_0 y^2 + M_0 y. \tag{2.18}$$

Hence we get that, if $t \in [t_1, t_2]$, then we infer from (2.17) and (2.18) that

$$\beta_0 y^2(t) + 2(M + M_0)y(t) \geq 2(G(x(t_2)) - G(x(t))) - 2M(x(t_2) - x(t)) - 2M_1.$$

Let us take $\eta > \beta_0$ such that, for $y \geq d$,

$$\eta y^2 \geq \beta_0 y^2 + 2(M + M_0)y + 2M_1.$$

Then we obtain

$$\eta y^2(t) \geq 2(G(x(t_2)) - G(x(t))) - 2M(x(t_2) - x(t)). \tag{2.19}$$

Using the mean value theorem, we get

$$\begin{aligned} &G(x(t_2)) - G(x(t)) - M(x(t_2) - x(t)) \\ &= \frac{1}{2}(G(x(t_2)) - G(x(t))) + \left(\frac{1}{2}g(\xi) - M\right)(x(t_2) - x(t)), \end{aligned}$$

where $\xi \in [x(t), x(t_2)]$. Since $x(t) \geq c_m$, $t \in [t_1, t_2]$, we can take c_m large enough such that $g(\xi) \geq 2M$. Therefore, we obtain that, for $t \in [t_1, t_2]$,

$$G(x(t_2)) - G(x(t)) - M(x(t_2) - x(t)) \geq \frac{1}{2}(G(x(t_2)) - G(x(t))),$$

which, together with (2.19), implies that

$$y(t) \geq \sqrt{\frac{1}{\eta}(G(x(t_2)) - G(x(t)))}. \tag{2.20}$$

Since $x'(t) = f(y(t)) + p_1(t, x(t), y(t))$, we infer from (2.14) and (2.20) that

$$x'(t) \geq \frac{\alpha_0}{\sqrt{\eta}}\sqrt{G(x(t_2)) - G(x(t))} - (M_0 + M).$$

Let us take a fixed positive constant L . Then we have that, for $x(t) \in [c_m, x(t_2) - L]$,

$$G(x(t_2)) - G(x(t)) \geq G(x(t_2)) - G(x(t_2) - L) = g(\xi')L \rightarrow +\infty, \quad x(t_2) \rightarrow +\infty,$$

which implies that there exists a positive constant $\eta_0 < \frac{\alpha_0}{\sqrt{\eta}}$ such that, for $x(t) \in [c_m, x(t_2) - L]$,

$$\frac{\alpha_0}{\sqrt{\eta}}\sqrt{G(x(t_2)) - G(x(t))} - (M_0 + M) \geq \eta_0\sqrt{G(x(t_2)) - G(x(t))}.$$

Consequently, for $x(t) \in [c_m, x(t_2) - L]$, we get

$$x'(t) \geq \eta_0\sqrt{G(x(t_2)) - G(x(t))}.$$

Let $t_* \in [t_1, t_2]$ be such that $x(t_*) = x(t_2) - L$. Then we have that, for any sufficiently small $\varepsilon > 0$,

$$\begin{aligned} t_* - t_1 &\leq \frac{1}{\eta_0} \int_{c_0}^{x(t_2)-L} \frac{dx}{\sqrt{G(x(t_2)) - G(x)}} \\ &\leq \frac{1}{\eta_0} \int_0^{x(t_2)} \frac{dx}{\sqrt{G(x(t_2)) - G(x)}} \\ &= \frac{\sqrt{2}}{\eta_0} \tau(x(t_2)) < \frac{\varepsilon}{2}, \end{aligned} \tag{2.21}$$

provided that $x(t_2)$ is large enough. Consequently, we have that $t_* - t_1 < \frac{\varepsilon}{2}$, provided that r_0 is sufficiently large. We next estimate $t_2 - t_*$. If $t \in [t_*, t_2]$, then we have

$$\begin{aligned}
 y(t) &= d + \int_t^{t_2} g(x(s)) \, ds + \int_t^{t_2} p_2(s, x(s), y(s)) \, ds \\
 &\geq d + v(x(t_2))(t_2 - t) - M(t_2 - t),
 \end{aligned}$$

where $v(x(t_2)) = \min\{g(x) : x(t_2) - L \leq x \leq x(t_2)\}$. Obviously, $v(x(t_2)) \rightarrow +\infty$ as $x(t_2) \rightarrow +\infty$. On the other hand, it follows from (2.14) that

$$x'(t) = f(y) + p_1(t, x(t), y(t)) \geq \alpha_0 y(t) - (M + M_0).$$

Therefore, we get that, for $x(t_2)$ large enough,

$$\begin{aligned}
 L &= \int_{t_*}^{t_2} x'(s) \, ds \geq \alpha_0 \int_{t_*}^{t_2} y(s) \, ds - (M + M_0)(t_2 - t_*) \\
 &\geq \alpha_0 \left[d(t_2 - t_*) + \frac{1}{2}v(x(t_2))(t_2 - t_*)^2 - \frac{1}{2}M(t_2 - t_*)^2 \right] - (M + M_0)(t_2 - t_*) \\
 &\geq (\alpha_0 d - M - M_0)(t_2 - t_*) + \frac{1}{4}\alpha_0 v(x(t_2))(t_2 - t_*)^2,
 \end{aligned}$$

which, together with $v(x(t_2)) \rightarrow +\infty$ as $x(t_2) \rightarrow +\infty$, implies that, for any sufficiently small $\varepsilon > 0$,

$$t_2 - t_* < \frac{\varepsilon}{2}, \tag{2.22}$$

provided that $x(t_2)$ is large enough or r_0 is large enough. From (2.21) and (2.22) we know that, for any sufficiently small $\varepsilon > 0$,

$$t_2 - t_1 < \varepsilon,$$

provided that r_0 is large enough.

Similarly, we have that, if $(x(t), y(t)) \in D_i, i = 4, 6, 8, t \in [t_1, t_2]$, then

$$t_2 - t_1 < \varepsilon,$$

provided that r_0 is large enough.

We now consider the case $(x(t), y(t)) \in D_3, t \in [t_1, t_2]$. In this case, we have

$$|y(t)| \leq d, \quad t \in [t_1, t_2]. \tag{2.23}$$

Integrating both sides of $y' = -g(x(t)) + p_2(t, x(t), y(t))$ over $[t_1, t_2]$ and using $y(t_1) = d$ and $y(t_2) = -d$, we get

$$\begin{aligned}
 2d &= \int_{t_1}^{t_2} g(x(s)) \, ds - \int_{t_1}^{t_2} p_2(s, x(s), y(s)) \, ds \\
 &\geq (\mu_* - M)(t_2 - t_1),
 \end{aligned}$$

where $\mu_* = \min\{g(x(t)) : t_1 \leq t \leq t_2\}$. From (2.23), (h_1) , and Lemma 2.2 we get that $\mu_* \rightarrow +\infty$ as $r_0 \rightarrow \infty$. Therefore, we have that, for any sufficiently small $\varepsilon > 0$,

$$t_2 - t_1 < \varepsilon,$$

provided that r_0 is large enough.

Similarly, we have that, if $(x(t), y(t)) \in D_7, t \in [t_1, t_2]$, then

$$t_2 - t_1 < \varepsilon,$$

provided that r_0 is large enough.

From the previous conclusion we get that, for any sufficiently small $\varepsilon > 0$, there is $\varrho_1 > 0$ such that if $r_0 \geq \varrho_1$, then $(x(t), y(t)) \in D$, and if

$$\theta(s_2) - \theta(s_1) = -2\pi,$$

then

$$0 < s_2 - s_1 < 8\varepsilon.$$

Consequently, the motion $(x(t), y(t))$ rotates clockwise a turn in a period less than 8ε . Therefore, for any integer $n \geq 1$, there is $\varrho_n > 0$ such that, for $r_0 \geq \varrho_n$, the motion $(x(t), y(t))$ can rotate more than n turns during the period $2m\pi$.

The proof of Lemma 2.4 is thus complete. □

3 Proof of main theorems

First, we recall a generalized version of the Poincaré–Birkhoff fixed point theorem by Rebelo [19].

A generalized form of the Poincaré–Birkhoff fixed point theorem. Let \mathcal{A} be an annular region bounded by two strictly star-shaped curves around the origin, Γ_1 and $\Gamma_2, \Gamma_1 \subset \text{int}(\Gamma_2)$, where $\text{int}(\Gamma_2)$ denotes the interior domain bounded by Γ_2 . Suppose that $F : \overline{\text{int}(\Gamma_2)} \rightarrow \mathbb{R}^2$ is an area-preserving homeomorphism and $F|_{\mathcal{A}}$ admits a lifting, with the standard covering projection $\Pi : (r, \theta) \rightarrow z = (r \cos \theta, r \sin \theta)$, of the form

$$\tilde{F}|_{\mathcal{A}} : (r, \theta) \rightarrow (w(r, \theta), \theta + h(r, \theta)),$$

where w and h are continuous functions of period 2π in the second variable. Correspondingly, for $\tilde{\Gamma}_1 = \Pi^{-1}(\Gamma_1)$ and $\tilde{\Gamma}_2 = \Pi^{-1}(\Gamma_2)$, assume the twist condition

$$h(r, \theta) > 0 \text{ on } \tilde{\Gamma}_1; \quad h(r, \theta) < 0 \text{ on } \tilde{\Gamma}_2.$$

Then, F has two fixed points z_1, z_2 in the interior of \mathcal{A} such that

$$h(\Pi^{-1}(z_1)) = h(\Pi^{-1}(z_2)) = 0.$$

Remark 3.1 The assumption on the star-shaped boundaries of the annulus is a delicate hypothesis. Martins and Ureña [17] showed that the star-shapedness assumption on the interior boundary is not eliminable. Le Calvez and Wang [16] then proved that the star-shapedness of the exterior boundary is also necessary, although this assumption was not made in Ding’s theorem [8].

Proof of Theorem 1.1 Let us take $m = 1$ in Lemma 2.4. From Lemmas 2.2 and 2.3 we have that there is $c'_1 > 0$ such that, for $r_0 > c'_1$,

$$\theta'(t) < 0, \quad t \in [0, 2\pi].$$

Let $a_1 > c'_1$ be a fixed constant. Then there exists an integer $n_1 > 0$ such that, for $r_0 = a_1$,

$$\theta(2\pi, r_0, \theta_0) - \theta_0 > -2n_1\pi. \tag{3.1}$$

According to Lemma 2.4, there exists a constant $b_1 > a_1$ such that, for $r_0 = b_1$,

$$\theta(2\pi, r_0, \theta_0) - \theta_0 < -2n_1\pi. \tag{3.2}$$

Let us define the annulus

$$D_1 = \{(x, y) \in \mathbb{R}^2 : a_1 \leq \sqrt{x^2 + y^2} \leq b_1\}.$$

Consider the Poincaré map $P : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Write the map P in the form

$$(r_0, \theta_0) \rightarrow (r_1, \theta_1)$$

with

$$r_1 = r(2\pi, r_0, \theta_0), \quad \theta_1 = \theta(2\pi, r_0, \theta_0) + 2n_1\pi.$$

Set

$$\Theta(r_0, \theta_0) = \theta(2\pi, r_0, \theta_0) - \theta_0 + 2n_1\pi.$$

Then we have

$$P : \quad r_1 = r(2\pi, r_0, \theta_0), \quad \theta_1 = \theta_0 + \Theta(r_0, \theta_0).$$

From (3.1) and (3.2) we have that

$$\Theta(r_0, \theta_0) > 0 \quad \text{for } r_0 = a_1, \tag{3.3}$$

$$\Theta(r_0, \theta_0) < 0 \quad \text{for } r_0 = b_1. \tag{3.4}$$

Since system (1.1) is conservative, P is an area-preserving mapping. It follows from (3.3) and (3.4) that the area-preserving map P is twisting on the annulus D_1 . According to the

generalized Poincaré–Birkhoff fixed point theorem, P has at least two fixed points in D_1 . Similarly, we can find a sequence

$$(a_1 < b_1 <) a_2 < b_2 < \dots < a_j < b_j < \dots (\rightarrow \infty)$$

such that the area-preserving map P is twisting on the annuluses

$$D_j = \{(x, y) \in \mathbb{R}^2 : a_j \leq \sqrt{x^2 + y^2} \leq b_j\}, \quad j = 2, 3, \dots$$

Using the generalized Poincaré–Birkhoff fixed point theorem, we know that P has at least two fixed points in each annulus $D_j, j = 2, 3, \dots$. Thus we have obtained the existence of a sequence of 2π -periodic solutions $\{(x_j(t), y_j(t))\}$ of system (1.1) satisfying

$$\lim_{j \rightarrow \infty} \left(\min_{t \in [0, 2\pi]} (x_j^2(t) + y_j^2(t)) \right) = +\infty.$$

The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2 Let $m \geq 2$ in Lemma 2.4. From Lemmas 2.2 and 2.3 we have that there is $c'_m > 0$ such that, for $r_0 \geq c'_m$,

$$\theta'(t) < 0, \quad t \in [0, 2m\pi].$$

Let $a'_1 > c'_m$ be a sufficiently large constant. Then there is a prime $q_1 \geq 2$ such that, for $r_0 = a'_1$,

$$\theta(2m\pi, r_0, \theta_0) - \theta_0 > -2q_1\pi.$$

According to Lemma 2.4, we know that there is $b'_1 > a'_1$ such that, for $r_0 = b'_1$,

$$\theta(2m\pi, r_0, \theta_0) - \theta_0 < -2q_1\pi.$$

Set

$$D'_1 = \{(x, y) \in \mathbb{R}^2 : a'_1 \leq \sqrt{x^2 + y^2} \leq b'_1\}.$$

Then the area-preserving iteration map P^m is twisting on the annulus D'_1 . Using the generalized Poincaré–Birkhoff fixed point theorem, P^m has at least two fixed points $(x_{1i}, y_{1i}) (i = 1, 2)$ in D'_1 . Hence

$$(x_{1i}(t), y_{1i}(t)) = (x(t, x_{1i}, y_{1i}), y(t, x_{1i}, y_{1i})) \quad (i = 1, 2)$$

are two $2m\pi$ -periodic solutions of system (1.1). Since q_1 is a prime, we can further prove that $(x_{1i}(t), y_{1i}(t))$ are not $2k\pi$ -periodic for $1 \leq k \leq m - 1$ by the same method as in [7]. Similarly, we can find two sequences

$$(a'_1 < b'_1 <) a'_2 < b'_2 < \dots < a'_j < b'_j < \dots (\rightarrow \infty)$$

and

$$q_1 < q_2 < \dots < q_j < \dots (\rightarrow \infty)$$

with q_j ($j = 1, 2, \dots$) being prime numbers such that, for $r_0 = a'_j$,

$$\theta(2m\pi, r_0, \theta_0) - \theta_0 > -2q_j\pi$$

and, for $r_0 = b'_j$,

$$\theta(2m\pi, r_0, \theta_0) - \theta_0 < -2q_j\pi.$$

Therefore, P^m is twisting on the annuluses

$$D'_j = \{(x, y) : a'_j \leq \sqrt{x^2 + y^2} \leq b'_j\} \quad (j = 2, 3, \dots).$$

It follows that P^m has at least two fixed points in each annulus $D'_j, j = 2, 3, \dots$, which correspond to two $2m\pi$ -periodic solutions of system (1.1). In the same way, these $2m\pi$ -periodic solutions are not $2k\pi$ -periodic for $1 \leq k \leq m - 1$. Consequently, system (1.1) has infinitely many $2m\pi$ -periodic solutions $\{(x_j(t), y_j(t))\}_{j=1}^\infty$ that are not $2k\pi$ -periodic for $1 \leq k \leq m - 1$ and satisfy

$$\lim_{j \rightarrow \infty} \left(\min_{t \in [0, 2m\pi]} (x_j^2(t) + y_j^2(t)) \right) = +\infty.$$

The proof of Theorem 1.2 is thus complete. □

4 Remarks

The assumption on the uniqueness of the solutions to Cauchy problems of (1.1) made in the proofs of the previous sections can be removed. In fact, Lemmas 2.3 and 2.4 guarantee the applicability of the nonuniqueness version of the Poincaré–Birkhoff theorem, which was proved by Fonda and Ureña [13]. We now state this theorem for a general Hamiltonian system in \mathbf{R}^{2N} . Let us consider the Hamiltonian system

$$\begin{cases} x' = \nabla_y H(t, x, y), \\ y' = -\nabla_x H(t, x, y), \end{cases} \tag{4.1}$$

where the continuous function $H : \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N \rightarrow \mathbf{R}, H = H(t, x, y)$ is T -periodic in its first variable t and continuously differentiable with respect to $(x, y), x = (x_1, \dots, x_N), y = (y_1, \dots, y_N)$.

We next introduce the definition of the rotation number of a continuous path in \mathbf{R}^2 . Let $\sigma : [t_1, t_2] \rightarrow \mathbf{R}^2$ be a continuous path such that $\sigma(t) \neq (0, 0)$ for every $t \in [t_1, t_2]$. The rotation number of σ around the origin is defined as

$$\text{Rot}(\sigma(t); [t_1, t_2]) = \frac{\theta(t_2) - \theta(t_1)}{2\pi},$$

where $\theta : [t_1, t_2] \rightarrow \mathbf{R}$ is a continuous determination of the argument along σ , that is, $\sigma(t) = |\sigma(t)|(\cos \theta(t), \sin \theta(t))$.

Assume that, for each $i = 1, \dots, N$, there are two strictly star-shaped Jordan closed curves around the origin $\Gamma_1^i, \Gamma_2^i \subset \mathbf{R}^2$ such that

$$o \in \mathcal{D}(\Gamma_1^i) \subset \overline{\mathcal{D}(\Gamma_1^i)} \subset \mathcal{D}(\Gamma_2^i),$$

where $\mathcal{D}(\Gamma)$ is the open region bounded by the Jordan closed curve Γ . Consider the generalized annular region

$$\mathcal{A} = [\overline{\mathcal{D}(\Gamma_2^1)} \setminus \mathcal{D}(\Gamma_1^1)] \times \dots \times [\overline{\mathcal{D}(\Gamma_2^N)} \setminus \mathcal{D}(\Gamma_1^N)] \subset \mathbf{R}^{2N}.$$

Theorem 4.1 ([13]) *Under the framework above, denoting $z_i(t) = (x_i(t), y_i(t))$, assume that every solution $z(t) = (z_1(t), \dots, z_N(t))$ of (4.1) departing from $z(0) \in \partial\mathcal{A}$ is defined on $[0, T]$ and satisfies*

$$z_i(t) \neq (0, 0) \quad \text{for all } t \in [0, T] \text{ and } i = 1, \dots, N.$$

Assume that there are integer numbers $v_1, \dots, v_N \in \mathbf{Z}$ such that, for each $i = 1, \dots, N$, either

$$\text{Rot}(z_i(t); [0, T]) \begin{cases} < v_i, & \text{if } z_i(0) \in \Gamma_1^i, \\ > v_i, & \text{if } z_i(0) \in \Gamma_2^i, \end{cases}$$

or

$$\text{Rot}(z_i(t); [0, T]) \begin{cases} > v_i, & \text{if } z_i(0) \in \Gamma_1^i, \\ < v_i, & \text{if } z_i(0) \in \Gamma_2^i. \end{cases}$$

Then Hamiltonian system (4.1) has at least $N + 1$ distinct T -periodic solutions $z^0(t), \dots, z^N(t)$, with $z^0(0), \dots, z^N(0) \in \mathcal{A}$, such that

$$\text{Rot}(z_i^k(t); [0, T]) = v_i \quad \text{for all } k = 0, \dots, N \text{ and } i = 1, \dots, N.$$

Remark 4.2 Note that there is no requirement of uniqueness of the solutions to Cauchy problems of (4.1) in this higher-dimensional Poincaré–Birkhoff theorem for Hamiltonian flows. Theorem 4.1 can be applied to deal with the multiplicity of periodic solutions of higher-dimensional Hamiltonian systems [9, 12]. Fonda and Sfecci [12] studied the multiplicity of periodic solutions of weakly coupled Hamiltonian systems

$$\begin{cases} -x_1'' = x_1[h_1(t, x_1) + p_1(t, x_1, \dots, x_N)], \\ \vdots \\ -x_N'' = x_N[h_N(t, x_N) + p_N(t, x_1, \dots, x_N)], \end{cases} \tag{4.2}$$

where all the functions $h_i : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ are continuous, T -periodic in the first variable t , and such that

$$\lim_{|x| \rightarrow \infty} h_i(t, x) = +\infty \quad \text{uniformly in } t \in [0, T].$$

The functions $p_i : \mathbf{R} \times \mathbf{R}^N$ are continuous, T -periodic in the first variable t , and bounded. Moreover, there is a function $\mathcal{U} : \mathbf{R} \times \mathbf{R}^N \rightarrow \mathbf{R}$ such that

$$\frac{\partial \mathcal{U}}{\partial x_i} = x_i p_i(t, x_1, \dots, x_N) \quad \text{for all } (t, x_1, \dots, x_N) \in [0, T] \times \mathbf{R}^N \text{ and } i = 1, \dots, N.$$

In this case, (4.2) is a superlinear Hamiltonian system. Under these conditions, it was proved in [12] that (4.2) has infinitely many periodic solutions by using Theorem 4.1. Theorems 1.1 and 1.2 in the present paper can also be extended to a weakly coupled Hamiltonian system of the type

$$\begin{cases} x'_i = f_i(y_i) + p_{1i}(t, x, y), \\ y'_i = -g_i(x_i) + p_{2i}(t, x, y) \end{cases} \quad (i = 1, \dots, N), \tag{4.3}$$

where $x = (x_1, \dots, x_N)$, $y = (y_1, \dots, y_N)$, $f_i, g_i : \mathbf{R} \rightarrow \mathbf{R}$ are continuous, $p_{ji} : \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N \rightarrow \mathbf{R}$ ($j = 1, 2, i = 1, \dots, N$) are continuous and 2π -periodic in the first variable t . Assume that there is a function $\mathcal{W} : \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N \rightarrow \mathbf{R}$ such that

$$\frac{\partial \mathcal{W}}{\partial y_i} = p_{1i}(t, x, y), \quad \frac{\partial \mathcal{W}}{\partial x_i} = -p_{2i}(t, x, y) \quad (i = 1, \dots, N).$$

It follows that system (4.3) is a Hamiltonian system. Assume that the following conditions hold:

(h'_1) $\lim_{|s| \rightarrow \infty} \operatorname{sgn}(s)g_i(s) = +\infty$ ($i = 1, \dots, N$).

(h'_2) There are positive constants α_i, β_i such that

$$0 < \alpha_i = \liminf_{|s| \rightarrow +\infty} \frac{f_i(s)}{s} \leq \limsup_{|s| \rightarrow +\infty} \frac{f_i(s)}{s} = \beta_i < +\infty.$$

(h'_3) There are positive constants M_i such that

$$|p_{1i}(t, x, y)| \leq M_i, \quad |p_{2i}(t, x, y)| \leq M_i \quad \text{for all } t, x, y \in \mathbf{R} \text{ and } i = 1, \dots, N.$$

(h'_4) The time maps $\tau_i(c)$ satisfy $\lim_{|c| \rightarrow \infty} \tau_i(c) = 0$, where $\tau_i(c)$ are defined like $\tau(c)$ in Sect. 1.

Using Theorem 4.1, we can prove that (4.3) has infinitely many 2π -periodic solutions and, for any integer $m \geq 2$, (4.3) has infinitely many $2m\pi$ -periodic solutions that are not $2k\pi$ -periodic for $1 \leq k \leq m - 1$, provided that conditions (h'_i) ($i = 1, \dots, 4$) hold. For brevity, we omit the details.

Acknowledgements

The authors are grateful to the referees for many valuable suggestions to make the paper more readable.

Funding

Research supported by the National Nature Science Foundation of China, No. 11501381.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

ZW proved the global existence of the solution of any Cauchy problem. TM proved the other conclusions and helped to draft the manuscript. Both authors read and approved the final manuscript.

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Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 26 December 2017 Accepted: 23 June 2018 Published online: 04 July 2018

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