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On stability with respect to boundary conditions for anisotropic parabolic equations with variable exponents

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Abstract

The anisotropic parabolic equations with variable exponents are considered. If some of diffusion coefficients $\{b_i(x)\}$ are degenerate on the boundary, the others are always positive, then how to impose a suitable boundary value condition is researched. The existence of weak solutions is proved by the parabolically regularized method. The stability of weak solutions, based on the partial boundary value condition, is established by choosing a suitable test function.

MSC: 35L65; 35L85; 35K92

Keywords: Parabolic equation; Variable exponent; Partial boundary value condition; Stability

1 Introduction and the main results

Recently, the anisotropic parabolic equations with the variable exponents

$$v_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|v_{x_i}|^{p_i(x)-2} v_{x_i}\right), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

were studied by Antontsev and Shmarev [1], Tersenov [2, 3], and some essential characteristics different from the evolutionary p-Laplacian equations were revealed. Zhan [4, 5] studied the equations

$$v_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(b_i(x)|v_{x_i}|^{p_i(x)-2} v_{x_i}\right) \tag{1.2}$$

and showed some essential characteristics different from equation (1.1). Here,

$$b_i(x) > 0, \quad x \in \Omega; \qquad b_i(x) = 0, \quad x \in \partial\Omega. \tag{1.3}$$

In this paper, we study the equation

$$v_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(b_i(x)|v_{x_i}|^{p_i(x)-2} v_{x_i}\right) + \sum_{i=1}^{N} g^i(x)\frac{\partial a(v)}{\partial x_i} \tag{1.4}$$

with the initial value condition

$$v(x,0) = v_0(x), \quad x \in \Omega, \tag{1.5}$$

and with a partial boundary value condition

$$v(x,t) = 0, \quad (x,t) \in \Sigma_1 \times (0,T), \tag{1.6}$$

where $\Sigma_1 \subseteq \partial\Omega$ is a relatively open subset. A similar partial boundary value condition was imposed on the equation

$$\frac{\partial v}{\partial t} - \operatorname{div}\bigl(a(x)|\nabla v|^{p-2}\nabla v\bigr) - \sum_{i=1}^{N} b^i(x)D_i v + c(x,t)v = f(x,t), \quad (x,t) \in Q_T, \tag{1.7}$$

and a new approach to prescribe the boundary value condition rather than define the Fichera function was formulated by Yin and Wang [6]. However, since equation (1.4) is anisotropic and with the variable exponents, the method of [6] seems difficult to be applied to equation (1.4). In what follows, we will try to depict Σ_1 in another way. Moreover, instead of depicting the explicit formula of Σ_1, we will try to find the other conditions to substitute the boundary value condition.

Instead of condition (1.3), we assume that $x \in \Omega$, $b_i(x) > 0$, and

$$b_{i_1}(x) > 0, b_{i_2}(x) > 0, \ldots, b_{i_k}(x) > 0, \quad x \in \overline{\Omega}, \tag{1.8}$$

$$b_{j_1}(x) = 0, b_{j_2}(x) = 0, \ldots, b_{j_l}(x) = 0, \quad x \in \partial\Omega. \tag{1.9}$$

Here, $\{i_1, i_2, \ldots, i_k\} \cup \{j_1, j_2, \ldots, j_l\} = \{1, 2, \ldots, N\}$, $k + l = N$. For the sake of simplicity, we denote that

$$p_0 = \min_{x\in\overline{\Omega}}\bigl\{p_1(x), p_2(x), \ldots, p_{N-1}(x), p_N(x)\bigr\},$$

$$p^0 = \max_{x\in\overline{\Omega}}\bigl\{p_1(x), p_2(x), \ldots, p_{N-1}(x), p_N(x)\bigr\},$$

and assume that $p_0 > 1$.

Let us introduce the basic definition and the main results. First of all, for any small constant $\eta > 0$, we define

$$\Omega_\eta = \left\{x \in \Omega : \sum_{s=1}^{l} b_{j_s}(x) > \eta\right\}.$$

Conditions (1.8)–(1.9) assure that this set is an open subset of Ω.

Definition 1.1 If a function $v(x,t)$ satisfies

$$v \in L^\infty(Q_T), \quad v_t \in L^{p^{0'}}\bigl(0,T; W^{-1,p^{0'}}(\Omega)\bigr), \quad b_i(x)|v_{x_i}|^{p_i(x)} \in L^2\bigl(0,T; L^1(\Omega)\bigr), \tag{1.10}$$

and for $\varphi \in L^2(0,T;W^{1,p^0}(\Omega)), \varphi|_{x\in\partial\Omega}=0$,

$$\iint_{Q_T}\left\{\frac{\partial v}{\partial t}\varphi+\sum_{i=1}^{N}\left[b_i(x)|v_{x_i}|^{p_i(x)-2}v_{x_i}\varphi_{x_i}+a(v)g^i(x)\varphi_{x_i}+a(v)g^i_{x_i}\varphi\right]\right\}dx\,dt=0, \tag{1.11}$$

then we say that $v(x,t)$ is a weak solution of equation (1.1) with initial value condition (1.5), provided that

$$\lim_{t\to 0}\int_\Omega\left|v(x,t)-v_0(x)\right|dx=0. \tag{1.12}$$

Besides, if the partial boundary value condition (1.6) is satisfied in the sense of the trace, then we say that $v(x,t)$ is a weak solution of the initial-boundary value problem (1.4)–(1.6).

Here and in what follows, $p'=\frac{p}{p-1}$ as usual.

Theorem 1.2 *If* $p_0>1$, $b_i(x)$ *satisfies conditions* (1.8), (1.9), $g^i(x)\in C^1(\overline{\Omega})$, $a(s)$ *is a continuous function,*

$$v_0\in L^\infty(\Omega), \left(b_i(x)\right)^{\frac{1}{p_i(x)}}v_{0x_i}\in L^{p_i(x)}(\Omega), \tag{1.13}$$

then equation (1.4) *with initial value* (1.5) *has a solution. If*

$$\int_\Omega b_{j_r}^{-\frac{1}{p_{j_r}-1}}(x)\,dx<\infty, \quad 1\le r\le l,$$

then there exists a solution of the initial-boundary value problem (1.4)–(1.6).

Theorem 1.3 *Let* $p_0>1$, $b_i(x)$ *satisfy conditions* (1.8), (1.9), $g^i(x)\in C^1(\overline{\Omega})$, $a(s)$ *be a Lipschitz function and for every* $1\le r\le l$, $\int_\Omega b_{j_r}^{-\frac{1}{p_{j_r}-1}}(x)\,dx<\infty$. *If* $v(x,t)$ *and* $u(x,t)$ *are two solutions of equation* (1.4),

$$v(x,t)=u(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T), \tag{1.14}$$

then

$$\int_\Omega\left|v(x,t)-u(x,t)\right|dx\le\int_\Omega\left|v(x,0)-u(x,0)\right|dx. \tag{1.15}$$

Theorem 1.4 *If* $p_0>1$, $b_i(x)$ *satisfies conditions* (1.8), (1.9), $g^i(x)\in C^1(\overline{\Omega})$, $a(s)$ *is a Lipschitz function. Let* $v(x,t)$ *and* $u(x,t)$ *be two solutions of equation* (1.4). *If*

$$v(x,t)=u(x,t)=0, \quad (x,t)\in\Sigma_1\times(0,T), \tag{1.16}$$

and for every $1\le r\le l$,

$$\frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_\eta}b_{j_r}(x)\left|\left(\sum_{s=1}^{l}b_{j_s}(x)\right)_{x_{j_r}}\right|^{p_{j_r}}dx\right)^{\frac{1}{p_{j_r}^+}}\le c, \tag{1.17}$$

then the stability (1.15) *is true*. *Here*,

$$\Sigma_1 = \left\{ x \in \partial\Omega : \frac{|(\sum_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\sum_{j=1}^{l} b_j(x)]^{p_{i_r}(x)-1}} \neq 0 \right\}, \quad r = 1, 2, \ldots, k. \tag{1.18}$$

If $a(s) \equiv 0$ *in equation* (1.4), *the similar conclusion as Theorem* 1.4 *was obtained in* [5], *where the partial boundary* Σ_1 *was depicted as follows*:

$$\Sigma_1 = \left\{ x \in \partial\Omega : \frac{|(\prod_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\prod_{j=1}^{l} b_j(x)]^{p_{i_r}(x)-1}} \neq 0 \right\}, \quad r = 1, 2, \ldots, k. \tag{1.19}$$

In fact, *letting* φ *be a nonnegative* C^1 *function*, *satisfying*

$$\varphi(x) > 0, \quad x \in \Omega, \qquad \varphi(x) = 0, \quad x \in \partial\Omega, \tag{1.20}$$

the partial boundary Σ_1 *can be depicted by* φ *as*

$$\Sigma_\varphi = \left\{ x \in \partial\Omega : \frac{|\varphi_{x_{i_r}}(x)|^{p_{i_r}(x)}}{[\varphi(x)]^{p_{i_r}(x)-1}} \neq 0 \right\}, \quad r = 1, 2, \ldots, k. \tag{1.21}$$

By this token, *the exact partial boundary* Σ_1, *such that the partial boundary value condition* (1.6) *matches up the nonlinear degenerate parabolic equation*, *should satisfy that*

$$\Sigma_1 \subseteq \Sigma_\varphi,$$

and we can depict it as

$$\Sigma_1 = \bigcap_{\varphi} \Sigma_\varphi \tag{1.22}$$

for any φ *satisfying* (1.20). *However*, *if we really choose* Σ_1 *as* (1.22), *it lacks the technical support to obtain the stability of the weak solutions for the time being*. *Anyway*, *by adopting some ideas and techniques in* [4, 5], *in some special cases*, *we can prove the stability of the weak solutions independent of the boundary value condition*.

Theorem 1.5 *If* $p_0 > 1$, $b_i(x)$ *satisfies conditions* (1.8), (1.9), $g^i(x) \in C^1(\overline{\Omega})$, $a(s)$ *is a Lipschitz function*. *Let* $v(x,t)$ *and* $u(x,t)$ *be two solutions of equation* (1.4) *only with the initial values* $v_0(x)$ *and* $u_0(x)$, *respectively*, *but without any boundary value condition*. *If condition* (1.17) *is true*, *and*

$$\frac{1}{\eta} \left(\int_{\Omega \setminus \Omega_\eta} \left| \left(\sum_{s=1}^{l} b_{j_s}(x) \right)_{x_{i_r}} \right|^{p_{i_r}} dx \right)^{\frac{1}{p_{i_r}^{+}}} \leq c \tag{1.23}$$

for every $1 \leq r \leq k$, *then the stability* (1.15) *is true*.

One can see that no boundary value condition is required in Theorem 1.5. *From my own perspective*, *condition* (1.23) *is an alternative of the partial boundary value condition* (1.6).

By the way, for the following reaction–diffusion equation

$$\frac{\partial v}{\partial t} = \operatorname{div}\big(a(v,x,t)\nabla v\big) + \operatorname{div}\big(b(v)\big), \quad (x,t) \in \Omega \times (0,T), \tag{1.24}$$

with

$$a(v,x,t)|_{x\in\partial\Omega} = 0, \tag{1.25}$$

we had conjectured that a partial boundary value condition should be imposed. This conjecture was partially proved in [7, 8].

2 The proof of existence

By a similar method as in [4], we can prove the following.

Lemma 2.1 *If* $\int_\Omega b_i^{-\frac{1}{p_i-1}}(x)\,dx < \infty$, $v(x,t)$ *is a weak solution of equation* (1.4) *with initial condition* (1.5). *Then, for any given* $t \in [0,T)$,

$$\int_\Omega |v_{x_i}|\,dx \le c, \quad i = 1,2,\ldots,N, \tag{2.1}$$

and the trace of v *on the boundary* $\partial\Omega$ *can be defined in the traditional way.*

We omit the details of the proof here. By this lemma, we know that if $b_i(x)$ *satisfies* (1.8), (1.9) *and if for every* $1 \le r \le l$, $\int_\Omega b_{j_r}^{-\frac{1}{p_{j_r}-1}}(x)\,dx < \infty$, *then* (2.1) *is satisfied. Thus, we can define the trace of* v *on the boundary* $\partial\Omega$.

Consider the regularized equation

$$\begin{aligned} v_t ={}& \sum_{r=1}^{k}\big(b_{i_r}(x)|v_{x_{i_r}}|^{p_{i_r}(x)-2}v_{x_{i_r}}\big)_{x_{i_r}} + \sum_{r=1}^{l}\big(\big(b_{j_r}(x)+\varepsilon\big)|v_{x_{j_r}}|^{p_{j_r}(x)-2}v_{x_{j_r}}\big)_{x_{j_r}} \\ &+ \sum_{i=1}^{N} g^i(x)\frac{\partial a(v)}{\partial x_i}, \quad (x,t) \in Q_T \end{aligned} \tag{2.2}$$

with the initial-boundary condition

$$v(x,0) = v_{0\varepsilon}(x), \quad x \in \Omega, \tag{2.3}$$

$$v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{2.4}$$

Here, $v_{0\varepsilon}(x) \in C_0^\infty(\Omega)$ *and is strongly convergent to* $v_0(x)$ *in* $W_0^{1,p^0}(\Omega)$.

Then, by Wu [9], *we know that problem* (2.2)–(2.4) *has a unique solution* $v_\varepsilon \in L^\infty(Q_T)$, $v_\varepsilon \in W_0^{1,p_0}(\Omega)$.

Proof of Theorem 1.2 Multiplying (2.2) by v_ε and integrating it over Q_T yield

$$\begin{aligned} &\frac{1}{2}\int_\Omega v_\varepsilon^2\,dx + \sum_{i=1}^{N}\iint_{Q_T} b_i(x)|v_{\varepsilon x_i}|^{p_i(x)}\,dx\,dt + \varepsilon\sum_{r=1}^{l}\iint_{Q_T}|v_{\varepsilon x_{j_r}}|^{p_{j_r}(x)}\,dx\,dt \\ &\quad = \frac{1}{2}\int_\Omega v_0^2(x)\,dx - \sum_{i=1}^{N}\iint_{Q_T}\big[a(v_\varepsilon)g^i(x)v_{\varepsilon x_i} + a(v_\varepsilon)g^i_{x_i}v_\varepsilon\big]\,dx\,dt, \end{aligned} \tag{2.5}$$

then

$$\varepsilon \sum_{r=1}^{l} \iint_{Q_T} |v_{\varepsilon x_{j_r}}|^{p_{j_r}(x)}\, dx\, dt \leq c \tag{2.6}$$

and

$$\sum_{i=1}^{N} \iint_{Q_T} b_i(x) |v_{\varepsilon x_i}|^{p_i(x)}\, dx\, dt \leq c. \tag{2.7}$$

Hence, by (2.5), (2.6), and (2.7), there exists a function v and an n-dimensional vector $\overrightarrow{\xi} = (\xi_1, \dots, \xi_n)$ such that

$$v \in L^{\infty}(Q_T), \qquad \frac{\partial v}{\partial t} \in L^2(Q_T), \qquad \xi_i \in L^1\bigl(0, T; L^{\frac{p_i(x)}{p_i(x)-1}}(\Omega)\bigr),$$

and $v_\varepsilon \to v$ *a.e.* $\in Q_T$,

$$\begin{aligned}
&v_\varepsilon \rightharpoonup v, \quad \text{weakly star in } L^{\infty}(Q_T),\\
&v_\varepsilon \to v, \quad \text{in } L^2\bigl(0, T; L^r_{\mathrm{loc}}(\Omega)\bigr),\\
&\frac{\partial v_\varepsilon}{\partial t} \rightharpoonup \frac{\partial v}{\partial t} \quad \text{in } L^2(Q_T),\\
&\varepsilon v_{\varepsilon x_{j_r}} \rightharpoonup 0, \quad \text{in } L^{p_{j_r}(x)}(Q_T),\\
&b_i(x) |v_{\varepsilon x_i}|^{p_i(x)-2} v_{\varepsilon x_i} \rightharpoonup \xi_i \quad \text{in } L^1\bigl(0, T; L^{\frac{p_i(x)}{p_i(x)-1}}(\Omega)\bigr).
\end{aligned}$$

Here, $r < \frac{Np_0}{N-p^0}$.

Now, similar to [1], we can show that

$$v_t \in L^{p^{0\prime}}\bigl(0, T; W^{-1,p^{0\prime}}(\Omega)\bigr),$$

and by Wu [9], by a process of the limit, we are able to prove that

$$\sum_{i=1}^{N} \iint_{Q_T} b_i(x) |v_{x_i}|^{p_i(x)-2} v_{x_i} \varphi_{x_i}\, dx\, dt = \sum_{i=1}^{N} \iint_{Q_T} \xi_i(x) \varphi_{x_i}\, dx\, dt \tag{2.8}$$

for any function $\varphi \in L^2(0, T; W^{1,p^0}(\Omega))$, $\varphi|_{x\in\partial\Omega} = 0$. Thus, $v(x,t)$ satisfies (1.10) and (1.11). Moreover, according to Lemma 2.1, the partial boundary value condition (1.6) is satisfied in the sense of trace.

Now, we can prove the initial value (1.5) in a similar way as that in [10]. In detail, for small given $r > 0$, denote $D_r = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \leq r\}$. For large enough m, n, denoting that $v_m(x,t) = v_{\varepsilon=\frac{1}{m}}(x,t)$, we declare that

$$\int_{D_{2r}} \bigl|v_m(x,t) - v_n(x,t)\bigr|\, dx \leq \int_{D_r} \bigl|v_{0m}(x) - v_{0n}(x)\bigr|\, dx + c_r(t), \tag{2.9}$$

where $c_r(t)$ is independent of m, n, and $\lim_{t\to 0} c_r(t) = 0$. In fact, by (2.2), for any $t \in [0, T)$, we have

$$\begin{aligned}
&\int_0^t \int_{D_r} \varphi(v_{mt} - v_{nt})\,dx\,d\tau \\
&\quad + \sum_{s=1}^{k} \int_0^t \int_{D_r} b_{i_s}(x)\bigl(|v_{mx_{i_s}}|^{p_{i_s}(x)-2} v_{mx_{i_s}} - |v_{nx_{i_s}}|^{p_{i_s}(x)-2} v_{nx_{i_s}}\bigr)\varphi_{x_{i_s}}\,dx\,d\tau \\
&\quad + \sum_{s=1}^{l} \int_0^t \int_{D_r} \biggl[\biggl(b_{j_s}(x) + \frac{1}{m}\biggr)|v_{mx_{j_s}}|^{p_{j_s}(x)-2} v_{mx_{j_s}} \\
&\quad - \biggl(b_{j_s}(x) + \frac{1}{n}\biggr)|v_{nx_{j_s}}|^{p_{j_s}(x)-2} v_{nx_{j_s}}\biggr]\varphi_{x_{j_s}}\,dx\,d\tau \\
&\quad + \sum_{i=1}^{N} \int_0^t \int_{D_r} \bigl\{\bigl[a(v_m) - a(v_n)\bigr]g^i(x)\varphi_{x_i} + \bigl[a(v_m) - a(v_n)\bigr]g^i_{x_i}\varphi\bigr\}\,dx\,d\tau \\
&= 0, \quad \forall \varphi \in L^2\bigl(0, T; W_0^{1,p_0}(\Omega)\bigr).
\end{aligned} \tag{2.10}$$

For small $\eta > 0$, let

$$L_\eta(s) = \int_0^s l_\eta(\tau)\,d\tau, \qquad l_\eta(s) = \frac{2}{\eta}\biggl(1 - \frac{|s|}{\eta}\biggr)_+. \tag{2.11}$$

Obviously, $l_\eta(s) \in C(\mathbb{R})$ and

$$l_\eta(s) \ge 0, \qquad \bigl|s l_\eta(s)\bigr| \le 1, \qquad \bigl|L_\eta(s)\bigr| \le 1. \tag{2.12}$$

Clearly, if we denote $A_\eta(s) = \int_0^s L_\eta(s)\,ds$,

$$\lim_{\eta\to 0} L_\eta(s) = \operatorname{sgn}(s), \qquad \lim_{\eta\to 0} A_\eta(s) = |s|, \quad s \in (-\infty, +\infty), \tag{2.13}$$

and

$$\lim_{\eta\to 0} l_\eta(s)s = 0. \tag{2.14}$$

Suppose that $\xi(x) \in C_0^1(D_r)$ such that

$$0 \le \xi \le 1; \qquad \xi|_{D_{2r}} = 1,$$

and choose $\varphi = \xi L_\eta(v_m - v_n)$ in (2.10), then

$$\begin{aligned}
&\int_0^t \int_{D_r} \xi L_\eta(v_m - v_n)(v_{mt} - v_{nt})\,dx\,d\tau \\
&\quad + \sum_{s=1}^{k} \int_0^t \int_{D_r} b_{i_s}(x)\bigl(|v_{mx_{i_s}}|^{p_{i_s}(x)-2} v_{nx_{i_s}} - |v_{nx_{i_s}}|^{p_{i_s}(x)-2} v_{nx_{i_s}}\bigr) \\
&\quad \times \xi l_\eta(v_m - v_n)(v_m - v_n)_{x_{i_s}}\,dx\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{s=1}^{k}\int_0^t\int_{D_r} b_{i_s}(x)\bigl(|v_{mx_{i_s}}|^{p_{i_s}(x)-2}v_{mx_{i_s}}-|v_{nx_{i_s}}|^{p_{i_s}(x)-2}v_{nx_{i_s}}\bigr)L_\eta(v_m-v_n)\xi_{x_{i_s}}\,dx\,d\tau\\
&+\sum_{s=1}^{l}\int_0^t\int_{D_r}\biggl[\biggl(b_{j_s}(x)+\frac{1}{m}\biggr)|v_{mx_{j_s}}|^{p_{j_s}(x)-2}v_{mx_{j_s}}-\biggl(b_{j_s}(x)+\frac{1}{n}\biggr)|v_{nx_{j_s}}|^{p_{j_s}(x)-2}v_{nx_{j_s}}\biggr]\\
&\times\xi l_\eta(v_m-v_n)(v_m-v_n)_{x_{j_s}}\,dx\,d\tau\\
&+\sum_{s=1}^{l}\int_0^t\int_{D_r}\biggl[\biggl(b_{j_s}(x)+\frac{1}{m}\biggr)|v_{mx_{j_s}}|^{p_{j_s}(x)-2}v_{mx_{j_s}}-\biggl(b_{j_s}(x)+\frac{1}{n}\biggr)|v_{nx_{j_s}}|^{p_{j_s}(x)-2}v_{nx_{j_s}}\biggr]\\
&\times L_\eta(v_m-v_n)\xi_{x_{j_s}}\,dx\,d\tau\\
&+\sum_{i=1}^{N}\int_0^t\int_{D_r}\bigl[a(v_m)-a(v_n)\bigr]g^i(x)\xi l_\eta(v_m-v_n)(v_m-v_n)_{x_i}\,dx\,d\tau\\
&+\sum_{i=1}^{N}\int_0^t\int_{D_r}\bigl[a(v_m)-a(v_n)\bigr]g^i(x)L_\eta(v_m-v_n)\xi_{x_i}\,dx\,d\tau\\
&+\sum_{i=1}^{N}\int_0^t\int_{D_r}\bigl[a(v_m)-a(v_n)\bigr]g^i_{x_i}\xi L_\eta(v_m-v_n)\,dx\,d\tau\\
={}&0. \qquad (2.15)
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
&\int_0^t\int_{D_r} b_{i_s}(x)\bigl(|v_{mx_{i_s}}|^{p_{i_s}(x)-2}v_{mx_{i_s}}-|v_{nx_{i_s}}|^{p_{i_s}(x)-2}v_{nx_{i_s}}\bigr)\xi l_\eta(v_m-v_n)(v_m-v_n)_{x_{i_s}}\,dx\,d\tau\\
&\quad\ge 0 \qquad (2.16)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t\int_{D_r}\biggl[\biggl(b_{j_s}(x)+\frac{1}{m}\biggr)|v_{mx_{j_s}}|^{p_{j_s}(x)-2}v_{mx_{j_s}}-\biggl(b_{j_s}(x)+\frac{1}{n}\biggr)|v_{nx_{j_s}}|^{p_{j_s}(x)-2}v_{nx_{j_s}}\biggr]\\
&\qquad\times\xi l_\eta(v_m-v_n)(v_m-v_n)_{x_{j_s}}\,dx\,d\tau\\
&\quad\ge 0. \qquad (2.17)
\end{aligned}
$$

Noticing that $\xi\in C_0^1(\Omega)$, $a(s)$ is a Lipschitz function, using Hölder's inequality of the variable exponent Sobolev space, by (2.14), we easily deduce that

$$
\lim_{\eta\to 0}\int_{D_r}\bigl[a(v_m)-a(v_n)\bigr]g^i(x)\xi l_\eta(v_m-v_n)(v_m-v_n)_{x_i}\,dx=0. \qquad (2.18)
$$

At the same time,

$$
\begin{aligned}
&\lim_{\eta\to 0}\int_0^t\int_{D_r}\xi L_\eta(v_m-v_n)(v_{mt}-v_{nt})\,dx\,d\tau\\
&\quad=\lim_{\eta\to 0}\int_0^t\int_{D_r}\xi\biggl(\int_0^{v_m-v_n}L_\eta(s)\,ds\biggr)_\tau\,dx\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\eta\to 0} \int_0^t \int_{D_r} \xi \int_0^{v_m - v_n} \operatorname{sgn}_\eta(s)\,ds\Big|_0^t \,dx \\
&= \int_{\Omega_r} \xi |v_m - v_n|\,dx - \int_{D_r} \xi |v_{0m} - v_{0n}|\,dx. \qquad (2.19)
\end{aligned}
$$

Let $\eta \to 0$. By (2.11)–(2.19), we can obtain

$$
\begin{aligned}
&\int_{D_{2r}} \xi |v_m - v_n|\,dx \\
&\quad \leq \int_{D_r} |v_{0m} - v_{0n}|\,dx \\
&\qquad + \frac{c}{r} \sum_{s=1}^{k} \int_0^t \int_{D_r} b_{i_s}(x)\big(|v_{mx_{i_s}}|^{p_{i_s}(x)-1} + |v_{nx_{i_s}}|^{p_{i_s}(x)-1}\big)\,dx\,d\tau \\
&\qquad + \frac{c}{r} \sum_{s=1}^{l} \int_0^t \int_{D_r} \left[\left(b_{j_s}(x) + \frac{1}{m}\right)|v_{mx_{j_s}}|^{p_{j_s}(x)-1} + \left(b_{j_s}(x) + \frac{1}{n}\right)|v_{nx_{j_s}}|^{p_{j_s}(x)-1}\right] dx\,d\tau \\
&\qquad + \frac{c}{r} \sum_{i=1}^{N} \int_0^t \int_{D_r} |v_m - v_n|\,dx\,d\tau \\
&\quad \leq \int_{D_r} |v_{0m} - v_{0n}|\,dx + c_r(t).
\end{aligned}
$$

By $v_\varepsilon \in L^\infty(Q_T)$ and the unform estimates (2.6)–(2.7), we know that $c_r(t)$ is independent of m, n.

Now, for any given small r, if m, n are large enough, by (2.9), we have

$$
\begin{aligned}
&\int_{D_{2r}} \big|v(x,t) - v_0(x)\big|\,dx \\
&\quad \leq \int_{D_{2r}} \big|v(x,t) - v_m(x,t)\big|\,dx + \int_{D_{2r}} \big|v_m(x,t) - v_n(x,t)\big|\,dx \\
&\qquad + \int_{D_{2r}} \big|v_n(x,t) - v_{0n}(x)\big|\,dx + \int_{D_{2r}} \big|v_{0n}(x) - v_0(x)\big|\,dx \\
&\quad \leq \int_{D_{2r}} \big|v(x,t) - v_m(x,t)\big|\,dx + \int_{D_r} \big|v_m(x,0) - v_n(x,0)\big|\,dx + c_r(t) \\
&\qquad + \int_{D_{2r}} \big|v_n(x,t) - v_{0n}(x)\big|\,dx + \int_{D_{2r}} \big|v_{0n}(x) - v_0(x)\big|\,dx. \qquad (2.20)
\end{aligned}
$$

By (2.20), similar to the usual evolutionary p-Laplacian equation (Chap. 2, [9]), we have (1.12). Then Theorem 1.2 is proved.

Certainly, the initial value condition (1.5) can be right in the other sense; for example, in [11], it has the form

$$
\lim_{t\to 0} \int_\Omega v(x,t)\phi(x)\,dx = \int_\Omega v_0(x)\phi(x)\,dx, \quad \forall \phi(x) \in C_0^\infty(\Omega).
$$

Also, the existence of weak solutions can be proved in other ways. Here, we would like to mention some recent related papers [12–14]. □

3 The stability of the initial-boundary value problem

Theorem 3.1 *If $p_0 > 1$, $b_i(x)$ satisfies conditions* (1.8), (1.9), *$g^i(x) \in C^1(\overline{\Omega})$, $a(s)$ is a Lipschitz function and for every $1 \le r \le l$, $\int_\Omega b_{j_r}^{-\frac{1}{p_{j_r}-1}}(x)\,dx < \infty$, $g^i(x)$ satisfies*

$$g^i(x) b_i^{-\frac{1}{p_0}}(x) \le c. \tag{3.1}$$

If $v(x,t)$ and $u(x.t)$ are two solutions of equation (1.4) *with the same homogeneous value*

$$v(x,t) = u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T),$$

and with different initial values $u_0(x)$ and $v_0(x)$, then

$$\int_\Omega \bigl|v(x,t) - u(x,t)\bigr|\,dx \le \int_\Omega \bigl|v_0(x) - u_0(x)\bigr|\,dx.$$

Proof Since $\int_\Omega b_{j_r}^{-\frac{1}{p_{j_r}-1}}(x)\,dx < \infty$, by Lemma 2.1, we can choose $\varphi = \chi_{[\tau,s]} L_\eta(v-u)$ in (1.11), where $\chi_{[\tau,s]}$ is the characteristic function of $[\tau,s] \subset (0,T)$. Then

$$\begin{aligned}
&\int_\tau^s \int_\Omega L_\eta(v-u) \frac{\partial(v-u)}{\partial t}\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_\tau^s \int_\Omega b_i(x)\bigl(|v_{x_i}|^{p_i(x)-2} v_{x_i} - |u_{x_i}|^{p_i(x)-2} u_x x_i\bigr)(v-u)_{x_i} l_\eta(v-u)\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_\tau^s \int_\Omega \bigl[\bigl(a(v) - a(u)\bigr) g^i(x) l_\eta(v-u)(v-u)_{x_i} \\
&\quad + \bigl(a(v) - a(u)\bigr) g^i_{x_i} L_\eta(v-u)\bigr]\,dx\,dt \\
&= 0.
\end{aligned} \tag{3.2}$$

At first, we have

$$\int_\tau^s \int_\Omega b_i(x)\bigl(|v_{x_i}|^{p_i(x)-2} v_{x_i} - |u_{x_i}|^{p_i(x)-2} u_{x_i}\bigr)(v-u)_{x_i} l_\eta(v-u)\,dx\,dt \ge 0. \tag{3.3}$$

By Lemma 3.1 from [11], we have

$$\begin{aligned}
&\lim_{\eta\to 0} \int_\tau^s \int_\Omega L_\eta(v-u) \frac{\partial(v-u)}{\partial t}\,dx\,dt \\
&= \lim_{\eta\to 0} \int_\Omega \bigl[A_\eta(v-u)(x,s) - A_\eta(v-u)(x,\tau)\bigr]\,dx \\
&= \int_\Omega |v-u|(x,s)\,dx - \int_\Omega |v-u|(x,\tau)\,dx.
\end{aligned} \tag{3.4}$$

Moreover, since $g^i(x)$ satisfies condition (3.1)

$$g^i(x) b_i^{-\frac{1}{p_0}} \le c,$$

by (3.4), $a(s)$ is a Lipschitz function, we have

$$\begin{aligned}
&\lim_{\eta\to 0}\sum_{i=1}^{N}\left|\int_{\tau}^{s}\int_{\Omega}\big[a(v)-a(u)\big]g^{i}(x)l_{\eta}(v-u)(v-u)_{x_i}\,dx\,dt\right| \\
&\quad=\lim_{\eta\to 0}\sum_{i=1}^{N}\left|\int_{\tau}^{s}\int_{\Omega}\big[a(v)-a(u)\big]g^{i}(x)b_i^{-\frac{1}{p_0}}l_{\eta}(v-u)b_i^{-\frac{1}{p_0}}(v-u)_{x_i}\,dx\,dt\right| \\
&\quad\leq\sum_{i=1}^{N}\lim_{\eta\to 0}\left(\int_{\tau}^{s}\int_{\Omega}\left|\big[a(v)-a(u)\big]g^{i}(x)b_i^{-\frac{1}{p_0}}l_{\eta}(v-u)\right|^{\frac{p_0}{p_0-1}}dx\,dt\right)^{\frac{p_0-1}{p_0}} \\
&\qquad\times\left(\int_{\tau}^{s}\int_{\Omega}b_i(x)\big(|v_{x_i}|^{p_0}+|u_{x_i}|^{p_0}\big)\,dx\,dt\right)^{\frac{1}{p_0}} \\
&\quad\leq c\sum_{i=1}^{N}\lim_{\eta\to 0}\left(\int_{\tau}^{s}\int_{\Omega}\left|(v-u)l_{\eta}(v-u)\right|^{\frac{p_0}{p_0-1}}dx\,dt\right)^{\frac{p_0-1}{p_0}} \\
&\qquad\times\left(\int_{\tau}^{s}\int_{\Omega}b_i(x)\big(|v_{x_i}|^{p_0}+|u_{x_i}|^{p_0}\big)\,dx\,dt\right)^{\frac{1}{p_0}} \\
&\quad=0. \qquad (3.5)
\end{aligned}$$

$$\begin{aligned}
&\lim_{\eta\to 0}\sum_{i=1}^{N}\left|\int_{\tau}^{s}\int_{\Omega}\big[a(v)-a(u)\big]g^{i}_{x_i}L_{\eta}(v-u)\,dx\,dt\right| \\
&\quad\leq c\int_{\tau}^{s}\int_{\Omega}|v-u|\,dx\,dt. \qquad (3.6)
\end{aligned}$$

Now, let $\eta\to 0$ in (3.2). Then

$$\begin{aligned}
&\int_{\Omega}\big|v(x,s)-u(x,s)\big|\,dx \\
&\quad\leq\int_{\Omega}\big|v(x,\tau)-u(x,\tau)\big|\,dx+c\int_{\tau}^{s}\int_{\Omega}|v-u|\,dx\,dt. \qquad (3.7)
\end{aligned}$$

By Gronwall's inequality, letting $\tau\to 0$, we have

$$\int_{\Omega}\big|v(x,s)-u(x,s)\big|\,dx\leq\int_{\Omega}\big|v_0(x)-u_0(x)\big|\,dx.$$

Theorem 3.1 is proved.

One can see that condition (3.1) is used to prove (3.4). In fact, without this condition, the conclusion of Theorem 3.1 is still true. This is Theorem 1.3. □

Proof of Theorem 1.3 From the above proof of Theorem 3.1, we only need to prove that

$$\begin{aligned}
&\lim_{\eta\to 0}\left|\int_{\Omega}\big[a(v)-a(u)\big]g^{i}(x)(v-u)_{x_i}L'_{\eta}(v-u)\,dx\right| \\
&\quad\leq c\lim_{\eta\to 0}\int_{\Omega}\left|\big[a(v)-a(u)\big](v-u)_{x_i}L'_{\eta}(u-v)\right|\,dx=0, \qquad (3.8)
\end{aligned}$$

without condition (3.1). Let us give an explanation. Noticing

$$\begin{aligned}
&\int_\Omega \big|\big[a(v)-a(u)\big](v-u)_{x_i}L'_\eta(v-u)\big|\,dx \\
&\quad= \int_\Omega \big|\big[a(v)-a(u)\big](v-u)_{x_i}l_\eta(v-u)\big|\,dx \\
&\quad= \int_{\{\Omega:|v-u|<\eta\}} \big|\big[a(v)-a(u)\big](v-u)_{x_i}l_\eta(v-u)\big|\,dx,
\end{aligned}\tag{3.9}$$

if $\{\Omega : |v-u| = 0\}$ is a subset of Ω with a positive measure, then

$$\begin{aligned}
&\lim_{\eta\to 0}\int_{\{\Omega:|v-u|<\eta\}} \big|\big[a(v)-a(u)\big](v-u)_{x_i}l_\eta(v-u)\big|\,dx \\
&\quad\le c\biggl(\int_{\{\Omega:|v-u|=0\}}\bigl(b_i^{\frac{1}{p_i}}|v_{x_i}-u_{x_i}|\bigr)^{p_i}\,dx\biggr)^{\frac{1}{p_i}}\biggl(\int_\Omega b_i^{-\frac{1}{p_i-1}}\,dx\biggr)^{\frac{p_i-1}{p_i}} \\
&\quad= 0.
\end{aligned}$$

At the same time, if $\{\Omega : |v-u| = 0\}$ is a subset of Ω with zero measure, since $b_i(x)$ satisfies (1.8)–(1.9), and for every $1 \le r \le l$, $\int_\Omega b_{j_r}^{-\frac{1}{p_{j_r}-1}}(x)\,dx < \infty$, we have

$$b_i^{-\frac{1}{p_i-1}} \in L^1(\Omega), \quad i = 1,2,\dots,N.$$

Then

$$\begin{aligned}
&\lim_{\eta\to 0}\int_{\{\Omega:|v-u|<\eta\}} \big|\big[a(v)-a(u)\big](v-u)_{x_i}l_\eta(v-u)\big|\,dx \\
&\quad\le c\biggl(\int_\Omega\bigl(b_i^{\frac{1}{p_i}}|v_{x_i}-u_{x_i}|\bigr)^{p_i}\,dx\biggr)^{\frac{1}{p_i}}\biggl(\int_{\{\Omega:|v-u|=0\}} b_i^{-\frac{1}{p_i-1}}\,dx\biggr)^{\frac{p_i-1}{p_i}} \\
&\quad\le c\biggl(\int_\Omega b_i(x)\bigl(|v_{x_i}|^{p_i}+|u_{x_i}|^{p_i}\bigr)\,dx\biggr)^{\frac{1}{p_i}}\biggl(\int_{\{\Omega:|v-u|=0\}} b_i^{-\frac{1}{p_i-1}}\,dx\biggr)^{\frac{p_i-1}{p_i}} \\
&\quad= 0.
\end{aligned}$$

Thus, Theorem 1.3 is true. □

4 The stability based on the partial boundary value condition

Theorem 4.1 *If* $p_0 > 1$, $b_i(x)$ *satisfies conditions* (1.8), (1.9), $g^i(x) \in C^1(\overline{\Omega})$ *satisfies* (3.5), $a(s)$ *is a Lipschitz function. Let* $v(x,t)$ *and* $u(x,t)$ *be two solutions of equation* (1.4). *If the initial values* $u_0(x)$ *and* $v_0(x)$ *are different, while the partial boundary values satisfy*

$$v(x,t) = u(x,t) = 0, \quad (x,t) \in \Sigma_1 \times [0,T),$$

then the stability (1.15) *is true, provided that for every* $1 \le r \le l$, *condition* (1.17) *is true, i.e.,*

$$\frac{1}{\eta}\biggl(\int_{\Omega\setminus\Omega_\eta} b_{j_r}(x)\biggl|\biggl(\sum_{s=1}^{l} b_{j_s}(x)\biggr)_{x_{j_r}}\biggr|^{p_{j_r}}\,dx\biggr)^{\frac{1}{p_{j_r}^+}} \le c.$$

Proof Let $\Omega_\eta = \{x \in \Omega : \sum_{r=1}^{l} b_{j_r}(x) > \eta\}$, and

$$\phi_\eta(x) = \begin{cases} 1 & \text{if } x \in \Omega_\eta, \\ \frac{1}{\eta}\sum_{r=1}^{l} b_{j_r}(x) & \text{if } x \in \Omega \setminus \Omega_\eta. \end{cases} \tag{4.1}$$

Then, if $x \in \Omega \setminus \Omega_\eta$, $\phi_{\eta x_i} = \frac{1}{\eta}(\sum_{r=1}^{l} b_{j_r}(x))_{x_i}$, while $x \in \Omega_\eta$, $\phi_{x_i} = 0$. Let $\varphi = \chi_{[\tau,s]}\phi_\eta L_\eta(v-u)$ be the test function in (1.11). Then

$$\begin{aligned} &\int_\tau^s \int_\Omega \phi_\eta L_\eta(v-u)\frac{\partial(v-u)}{\partial t}\,dx\,dt \\ &\quad+ \sum_{i=1}^{N}\int_\tau^s\int_\Omega b_i(x)\bigl(|v_{x_i}|^{p_i(x)-2}v_{x_i} - |u_{x_i}|^{p_i(x)-2}u_{x_i}\bigr)(u_{x_i}-v_{x_i})l_\eta(u-v)\phi_\eta(x)\,dx\,dt \\ &\quad+ \sum_{r=1}k\int_\tau^s\int_\Omega b_{i_r}(x)\bigl(|v_{x_{i_r}}|^{p_{i_r}(x)-2}v_{x_{i_r}} - |u_{x_{i_r}}|^{p_{i_r}(x)-2}u_{x_{i_r}}\bigr)L_\eta(v-u)\phi_{\eta x_{i_r}}\,dx\,dt \\ &\quad+ \sum_{r=l}^{l}\int_\tau^s\int_\Omega b_{j_r}(x)\bigl(|v_{x_{j_r}}|^{p_{j_r}(x)-2}v_{x_{j_r}} - |u_{x_{j_r}}|^{p_{j_r}(x)-2}u_{x_{j_r}}\bigr)L_\eta(v-u)\phi_{\eta x_{j_r}}\,dx\,dt \\ &\quad+ \sum_{i=1}^{N}\int_\tau^s\int_\Omega\bigl[\bigl(a(v)-a(u)\bigr)g^i(x)l_\eta(v-u)(v-u)_{x_i}\phi_\eta \\ &\quad+ \bigl(a(v)-a(u)\bigr)g^i_{x_i}L_\eta(v-u)\phi_{\eta x_i}\bigr]\,dx\,dt \\ &= 0. \end{aligned} \tag{4.2}$$

At first,

$$\int_\Omega b_i(x)\bigl(|v_{x_i}|^{p_i(x)-2}v_{x_i} - |u_{x_i}|^{p_i(x)-2}v_{x_i}\bigr)(v_{x_i}-u_{x_i})l_\eta(v-u)\phi_\eta(x)\,dx \geq 0, \tag{4.3}$$

and

$$\begin{aligned} &\lim_{\eta\to 0}\int_\tau^s\int_\Omega \phi_\eta(x)L_\eta(v-u)\frac{\partial(v-u)}{\partial t}\,dx\,dt \\ &\quad= \int_\Omega |v-u|(x,s)\,dx - \int_\Omega |v-u|(x,\tau)\,dx. \end{aligned} \tag{4.4}$$

Secondly, by Hölder's inequality of the variable exponent Sobolev space, we have

$$\begin{aligned} &\left|\int_\Omega b_{i_r}(x)\bigl(|v_{x_{i_r}}|^{p_{i_r}(x)-2}v_{x_{i_r}} - |u_{x_{i_r}}|^{p_{i_r}(x)-2}u_{x_{i_r}}\bigr)L_\eta(v-u)\phi_{\eta x_{i_r}}\,dx\right| \\ &\quad= \left|\int_{\Omega\setminus\Omega_\eta} b_{i_r}(x)\bigl(|v_{x_{i_r}}|^{p_{i_r}(x)-2}v_{x_{i_r}} - |u_{x_{i_r}}|^{p_{i_r}(x)-2}u_{x_{i_r}}\bigr)L_\eta(v-u)\phi_{\eta x_{i_r}}\,dx\right| \\ &\quad\leq \int_{\Omega\setminus\Omega_\eta} b_{i_r}(x)\bigl(|v_{x_{i_r}}|^{p_{i_r}(x)-1} + |u_{x_{i_r}}|^{p_{i_r}(x)-1}\bigr)\bigl|L_\eta(v-u)\phi_{\eta x_{i_r}}\bigr|\,dx \\ &\quad\leq \int_{\Omega\setminus\Omega_\eta}\left[\frac{1}{\eta}b_{i_r}(x)\sum_{s=1}^{l}b_{j_s}(x)\right]^{\frac{p_{i_r}(x)-1}{p_{i_r}(x)}}\bigl(|v_{x_{i_r}}|^{p_{i_r}(x)-1} + |u_{x_{i_r}}|^{p_{i_r}(x)-1}\bigr) \end{aligned}$$

$$
\begin{aligned}
&\times \big(b_{i_r}(x)\big)^{\frac{1}{p_{i_r}(x)}} \left| L_\eta(v-u) \frac{(\frac{1}{\eta})^{\frac{1}{p_{i_r}(x)}} [\sum_{s=1}^{l} b_{j_s}(x)]_{x_{i_r}}}{[\sum_{s=1}^{l} b_{j_s}(x)]^{\frac{p_{i_r}(x)-1}{p_{i_r}(x)}}} \right| dx \\
&\leq \left(\int_{\Omega\setminus\Omega_\eta} \frac{1}{\eta} \sum_{s=1}^{l} b_{j_s}(x) b_{i_r}(x) \big(|v_{x_{i_r}}|^{p_{i_r}(x)} + |u_{x_{i_r}}|^{p_{i_r}(x)} \big)\, dx \right)^{\frac{1}{q_{i_r}^1}} \\
&\quad \times \left(\int_{\Omega\setminus\Omega_\eta} \frac{1}{\eta} b_{i_r}(x) \big| L_\eta(v-u) \big|^{p_{i_r}(x)} \frac{|(\sum_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\sum_{s=1}^{l} b_{j_s}(x)]^{p_{i_r}(x)-1}}\, dx \right)^{\frac{1}{p_{i_r}^1}} \\
&\leq c \left(\int_{\Omega\setminus\Omega_\eta} b_{i_r}(x) \big(|v_{x_{i_r}}|^{p_{i_r}(x)} + |u_{x_{i_r}}|^{p_{i_r}(x)} \big)\, dx \right)^{\frac{1}{q_{i_r}^1}} \\
&\quad \times \left(\frac{1}{\eta} \int_{\Omega\setminus\Omega_\eta} \big| L_\eta(v-u) \big|^{p_{i_r}(x)} \frac{|(\sum_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\sum_{s=1}^{l} b_{j_s}(x)]^{p_{i_r}(x)-1}}\, dx \right)^{\frac{1}{p_{i_r}^1}}.
\end{aligned} \tag{4.5}
$$

Here, $p_{i_r}^1 = p_{i_r}^+$ or $p_{i_r}^-$ according to

$$
\int_{\Omega\setminus\Omega_\eta} \frac{1}{\eta} b_{i_r}(x) \big| L_\eta(v-u) \big|^{p_{i_r}(x)} \frac{|(\sum_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\sum_{s=1}^{l} b_{j_s}(x)]^{p_{i_r}(x)-1}}\, dx \geq 1,
$$

or

$$
\int_{\Omega\setminus\Omega_\eta} \frac{1}{\eta} b_{i_r}(x) \big| L_\eta(v-u) \big|^{p_{i_r}(x)} \frac{|(\sum_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\sum_{s=1}^{l} b_{j_s}(x)]^{p_{i_r}(x)-1}}\, dx < 1,
$$

one can refer to Lemma 2.1 of [4]. $q_{j_r}(x) = \frac{p_{j_r}(x)}{p_{j_r}(x)-1}$, $q_{i_r}^1$ has a similar sense.

Let $\Sigma_2 = \partial\Omega \setminus \Sigma_1$, and

$$
\begin{aligned}
\Omega_{\eta 1} &= \big\{ x \in \Omega \setminus \Omega_\eta : \operatorname{dist}(x, \Sigma_2) > \operatorname{dist}(x, \Sigma_1) \big\}, \\
\Omega_{\eta 2} &= \big\{ x \in \Omega \setminus \Omega_\eta : \operatorname{dist}(x, \Sigma_2) \leq \operatorname{dist}(x, \Sigma_1) \big\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{\eta} \int_{\Omega\setminus\Omega_\eta} \big| L_\eta(v-u) \big|^{p_{i_r}(x)} \frac{|(\sum_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\sum_{s=1}^{l} b_{j_s}(x)]^{p_{i_r}(x)-1}}\, dx \\
&\quad \leq \frac{1}{\eta} \int_{\Omega_{\eta 1}} \big| L_\eta(v-u) \big|^{p_{i_r}(x)} \frac{|(\sum_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\sum_{s=1}^{l} b_{j_s}(x)]^{p_{i_r}(x)-1}}\, dx \\
&\qquad + n \int_{\Omega_{\eta 2}} \big| L_\eta(v-u) \big|^{p_{i_r}(x)} \frac{|(\sum_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\sum_{s=1}^{l} b_{j_s}(x)]^{p_{i_r}(x)-1}}\, dx.
\end{aligned} \tag{4.6}
$$

Since

$$
v = u = 0, \qquad x \in \Sigma_1,
$$

by the definition of the trace, we have

$$
\lim_{\eta\to 0} \frac{1}{\eta} \int_{\Omega_{\eta 1}} \big| L_\eta(v-u) \big|^{p_{i_r}(x)} \frac{|(\sum_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\sum_{s=1}^{l} b_{j_s}(x)]^{p_{i_r}(x)-1}}\, dx
$$

$$= \int_{\Sigma_1} \operatorname{sign}(v-u) \frac{|(\sum_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\sum_{s=1}^{l} b_{j_s}(x)]^{p_{i_r}(x)-1}} \, d\Sigma = 0. \tag{4.7}$$

Moreover, since

$$\frac{|(\sum_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\sum_{s=1}^{l} b_{j_s}(x)]^{p_{i_r}(x)-1}} = 0, \quad x \in \Sigma_2,$$

we have

$$\begin{aligned}
&\lim_{\eta\to\infty} \frac{1}{\eta} \int_{\Omega_{\eta 2}} \left|L_\eta(v-u)\right|^{p_{i_r}(x)} \frac{|(\sum_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\sum_{s=1}^{l} b_{j_s}(x)]^{p_{i_r}(x)-1}} \, dx \\
&\quad \le \lim_{\eta\to\infty} \frac{1}{\eta} \int_{\Omega_{\eta 2}} \frac{|(\sum_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\sum_{s=1}^{l} b_{j_s}(x)]^{p_{i_r}(x)-1}} \, dx \\
&\quad = \int_{\Sigma_2} \frac{|(\sum_{s=1}^{l} b_{j_s}(x))_{x_{i_r}}|^{p_{i_r}(x)}}{[\sum_{s=1}^{l} b_{j_s}(x)]^{p_{i_r}(x)-1}} \, d\Sigma = 0.
\end{aligned} \tag{4.8}$$

According to (4.5)–(4.8), we have

$$\lim_{\eta\to 0} \left| \int_\Omega b_{i_r}(x) \big(|u_{x_{i_r}}|^{p_{i_r}(x)-2} u_{x_{i_r}} - |v_{x_{i_r}}|^{p_{i_r}(x)-2} v_{x_{i_r}} \big) L_\eta(u-v) \phi_{\eta x_{i_r}} \, dx \right| = 0. \tag{4.9}$$

Thirdly, for the last term of the left-hand side of (4.2), we have

$$\begin{aligned}
&\left| \int_\Omega b_{j_r}(x) \big(|v_{x_{j_r}}|^{p_{j_r}(x)-2} u_{x_{j_r}} - |u_{x_{j_r}}|^{p_{j_r}(x)-2} v_{x_{j_r}} \big) L_\eta(v-u) \phi_{\eta x_{j_r}} \, dx \right| \\
&\quad = \left| \int_{\Omega\setminus\Omega_\eta} b_{j_r}(x) \big(|v_{x_{j_r}}|^{p_{j_r}(x)-2} v_{x_{j_r}} - |u_{x_{j_r}}|^{p_{j_r}(x)-2} u_{x_{j_r}} \big) L_\eta(v-u) \phi_{\eta x_{j_r}} \, dx \right| \\
&\quad \le \int_{\Omega\setminus\Omega_\eta} b_{j_r}(x) \big(|v_{x_{j_r}}|^{p_{j_r}(x)-1} + |u_{x_{j_r}}|^{p_{j_r}(x)-1} \big) \left| \left(\sum_{s=1}^{l} b_{j_s}(x) \right)_{x_{j_r}} L_\eta(v-u) \right| dx \\
&\quad \le c \left(\int_{\Omega\setminus\Omega_\eta} b_{j_r}(x) \big(|v_{x_{j_r}}|^{p_{j_r}(x)} + |u_{x_{j_r}}|^{p_{j_r}(x)} \big) \, dx \right)^{\frac{1}{q_{j_r}^+}} \\
&\qquad \times \frac{1}{\eta} \left(\int_{\Omega\setminus\Omega_\eta} b_{j_r}(x) \left| \left(\sum_{s=1}^{l} b_{j_s}(x) \right)_{x_i} \right|^{p_{j_r}(x)} dx \right)^{\frac{1}{p_{j_r}^+}}.
\end{aligned} \tag{4.10}$$

By condition (1.17),

$$\frac{1}{\eta} \left(\int_{\Omega\setminus\Omega_\eta} b_{j_r}(x) \left| \left(\sum_{s=1}^{l} b_{j_s}(x) \right)_{x_{j_r}} \right|^{p_{j_r}(x)} dx \right)^{\frac{1}{p_{j_r}^+}} \le c.$$

Then

$$\lim_{\eta\to 0} \left| \int_\Omega b_{j_r}(x) \big(|v_{x_{j_r}}|^{p_{j_r}(x)-2} v_{x_{j_r}} - |u_{x_{j_r}}|^{p_{j_r}(x)-2} u_{x_{j_r}} \big) L_\eta(v-u) \phi_{\eta x_{j_r}} \, dx \right| = 0. \tag{4.11}$$

Fourthly, since $g^i(x)$ satisfies condition (3.5), we have

$$\begin{aligned}
&\lim_{\eta\to 0}\sum_{i=1}^{N}\left|\int_\tau^s\int_\Omega\big[a(v)-a(u)\big]g^i(x)l_\eta(v-u)(v-u)_{x_i}\phi_\eta(x)\,dx\,dt\right|\\
&\quad=\lim_{\eta\to 0}\sum_{i=1}^{N}\left|\int_\tau^s\int_\Omega\big[a(v)-a(u)\big]g^i(x)a^{-\frac{1}{p_0}}l_\eta(v-u)b_i^{-\frac{1}{p_0}}(v-u)_{x_i}\phi_\eta(x)\,dx\,dt\right|\\
&\quad\le\sum_{i=1}^{N}\lim_{\eta\to 0}\left(\int_\tau^s\int_\Omega\left|\big[a(v)-a(u)\big]g^i(x)b_i^{-\frac{1}{p_0}}l_\eta(v-u)\right|^{\frac{p_0}{p_0-1}}dx\,dt\right)^{\frac{p_0-1}{p_0}}\\
&\quad\quad\times\left(\int_\tau^s\int_\Omega b_i(x)\big(|v_{x_i}|^{p_0}+|u_{x_i}|^{p_0}\big)\,dx\,dt\right)^{\frac{1}{p_0}}\\
&\quad=0.
\end{aligned}\tag{4.12}$$

At last, we have

$$\lim_{\eta\to 0}\sum_{i=1}^{N}\left|\int_\tau^s\int_\Omega\big[a(v)-a(u)\big]g^i_{x_i}L_\eta(u-v)\phi_\eta(x)\,dx\,dt\right|\le c\int_\tau^s\int_\Omega|v-u|\,dx\,dt.\tag{4.13}$$

Now, let $\eta\to 0$ in (4.2). By (4.3), (4.4), (4.9), (4.11), (4.12), and (4.13), we have

$$\int_\Omega\big|v(x,s)-u(x,s)\big|\,dx\le\int_\Omega\big|v(x,\tau)-u(x,\tau)\big|\,dx+c\int_\tau^s\big|v(x,t)-u(x,t)\big|\,dx\,dt.\tag{4.14}$$

By Gronwall's inequality, we have

$$\int_\Omega\big|v(x,s)-u(x,s)\big|\,dx\le\int_\Omega\big|v(x,\tau)-u(x,\tau)\big|\,dx.\tag{4.15}$$

Let $\tau\to 0$. Then

$$\int_\Omega\big|v(x,s)-u(x,s)\big|\,dx\le\int_\Omega\big|v_0(x)-u_0(x)\big|\,dx.$$ □

Proof of Theorem 1.4 If $|g^i(x)|\le c$ and $a(s)$ is a Lipschitz function, for every r satisfies (1.17), similar to the proof of Theorem 1.3 in Sect. 3, combining with Theorem 4.1, we know Theorem 1.4 is true. □

5 The stability without boundary value condition

Theorem 5.1 *Let $v(x,t)$ and $u(x,t)$ be two solutions of equation* (1.4) *with different initial values $v_0(x)$ and $u_0(x)$, respectively. If $p_0>1$, $b_i(x)$ satisfies conditions* (1.8), (1.9), *$g^i(x)\in C^1(\overline{\Omega})$ satisfies* (3.5), *$a(s)$ is a Lipschitz function, conditions* (1.17) *and* (1.23) *are true, then the stability* (1.15) *is true.*

Proof As in the proof of Theorem 4.1, let $\varphi=\chi_{[\tau,s]}\phi_\eta L_\eta(v-u)$ be the test function and obtain (4.2)–(4.4).

Since

$$\left|\int_\Omega b_{i_r}(x)\big(|v_{x_{i_r}}|^{p_{i_r}(x)-2}v_{x_{i_r}}-|u_{x_{i_r}}|^{p_{i_r}(x)-2}u_{x_{i_r}}\big)L_\eta(v-u)\phi_{\eta x_{i_r}}\,dx\right|$$

$$
\begin{aligned}
&= \left| \int_{\Omega\setminus\Omega_\eta} b_{i_r}(x)\bigl(|v_{x_{i_r}}|^{p_{i_r}(x)-2}v_{x_{i_r}} - |u_{x_{i_r}}|^{p_{i_r}(x)-2}u_{x_{i_r}}\bigr)L_\eta(v-u)\phi_{\eta x_{i_r}}\,dx \right| \\
&\leq \int_{\Omega\setminus\Omega_\eta} b_{i_r}(x)\bigl(|V_{x_{i_r}}|^{p_{i_r}(x)-1} + |U_{x_{i_r}}|^{p_{i_r}(x)-1}\bigr)\bigl|L_\eta(v-u)\phi_{\eta x_{i_r}}\bigr|\,dx \\
&\leq \left(\int_{\Omega\setminus\Omega_\eta} \sum_{s=1}^{l} b_{j_s}(x)b_{i_r}(x)\bigl(|v_{x_{i_r}}|^{p_{i_r}(x)} + |u_{x_{i_r}}|^{p_{i_r}(x)}\bigr)\,dx\right)^{\frac{1}{q_{i_r}^1}} \\
&\quad \times \frac{c}{\eta}\left(\int_{\Omega\setminus\Omega_\eta} \left|\sum_{s=1}^{l} b_{j_s}(x)_{x_{i_r}}\right|^{p_{i_r}(x)} dx\right)^{\frac{1}{p_{i_r}^1}},
\end{aligned}
\tag{5.1}
$$

by condition (1.23), we have (4.9). At the same time, we have (4.10)–(4.15). The proof is complete. □

Proof of Theorem 1.5 If $|g^i(x)| \leq c$ and $a(s)$ is a Lipschitz function, condition (1.23) is true. Similar to the proof of Theorem 1.3 in Sect. 3, combining with Theorem 5.1, we know Theorem 1.5 is true. □

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