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Existence of weak solutions for a boundary value problem of a second order ordinary differential equation

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Abstract

The aim of this paper is to investigate the existence of weak solutions for a boundary value problem of a second order differential equation. As the main tool, we apply a Krasnosel'skii type fixed point theorem in conjunction with the technique of measures of weak noncompactness in Banach spaces. Finally, two examples are given to illustrate our abstract results.

MSC: 35G30; 47H08; 47H09

Keywords: boundary value problem; measure of weak noncompactness; quadratic integral equation

1 Introduction

In this paper, we investigate the existence of weak solutions for the boundary value problem of second order differential equations of the form

$$\left(\frac{x(t)-g(t,x(t))}{h(t,x(t))}\right)'' + f\bigl(t,x(t)\bigr) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$x(0) = g\bigl(0,x(0)\bigr), \qquad x(1) = g\bigl(1,x(1)\bigr), \tag{1.2}$$

where the functions $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $h : [0,1] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ satisfy some special assumptions that will be given in detail in Section 3.

Boundary value problems arise in a variety of applied mathematics and physics areas (refer to [1, 2] etc.). Some boundary value problems of ordinary equations may be turned into (1.1)-(1.2). For example, in the design of bridge, denote $u(t)$ by the displacement of the bridge from the unloaded position. Small size bridges are often designed with two supported points, which leads to a two-point boundary value problem (cf. [3]):

$$u''(t) + \upsilon(t) + \varphi\bigl(t,u(t)\bigr) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \qquad u(1) = 0.$$

If we define $x(t) = g(t,x(t)) + h(t,x(t))u(t)$ for the known functions g and h, and $f(t,x(t)) = \upsilon(t) + \varphi(t,u(t))$, then the above problem is turned into (1.1)-(1.2).

As another example, for appropriate functions $p(t)$ and $\psi(t,x)$, if we take the functions involved in Eq. (1.1) in the form

$$g\big(t,x(t)\big) = e^{-\int_0^t p(s)\,ds} \int_0^t x(s)p(s)e^{\int_0^s p(\tau)\,d\tau}\,ds,$$
$$h\big(t,x(t)\big) = e^{-\int_0^t p(s)\,ds} \quad \text{and} \quad f\big(t,x(t)\big) = e^{\int_0^t p(s)\,ds}\psi\big(t,x(t)\big),$$

then Eq. (1.1) can be easily transformed into the following equation:

$$x''(t) + p(t)x'(t) + \psi\big(t,x(t)\big) = 0, \quad 0 < t < 1.$$

Unlike initial value problems, which are normally uniquely solvable, boundary value problems can have no solution or several solutions. However, we will adopt a strategy in the present paper to cope with the weak solutions of this problem. To consider the existence of weak solutions for problem (1.1)-(1.2), we will turn it into the following perturbed quadratic integral equation:

$$x(t) = g\big(t,x(t)\big) + h\big(t,x(t)\big)\int_0^1 G(t,s)f\big(s,x(s)\big)\,ds, \quad 0 \le t \le 1, \tag{1.3}$$

where G is the Green's function associated with problem (1.1)-(1.2). Our considerations are put in $L^1(I)$, the Banach space consisting of all real functions defined and Lebesgue integrable on the interval $I := [0,1]$. More specifically, by using the techniques of fixed point associated with measures of weak noncompactness, we will establish the existence of integrable solutions to Eq. (1.3) in a certain ball of $L^1(I)$. For the existence of integrable solutions of some nonlinear integral equations, we refer the reader to the literature [4–10].

2 Preliminaries

Let E be a Banach space. From now on we denote by $\mathcal{B}(E)$ the collection of all nonempty bounded subsets of E, and $\mathcal{W}(E)$ is a sub-collection of $\mathcal{B}(E)$ consisting of all weakly compact subsets of E. Denote by B_r the closed ball in E centered at zero with radius r. In what follows we accept the following definition (cf. [11]).

Definition 2.1 A mapping $\omega : \mathcal{B}(E) \to \mathbb{R}^+$ is said to be a (regular) measure of weak noncompactness if, for all $M,N \in \mathcal{B}(E)$, the following conditions are satisfied:

(1) The family $\ker(\omega) := \{M \in \mathcal{B}(E) : \omega(M) = 0\}$ is nonempty, and $M \in \ker(\omega)$ if and only if M is relatively weakly compact;
(2) $N \subseteq M \Rightarrow \omega(N) \le \omega(M)$;
(3) $\omega(\overline{M}^{\omega}) = \omega(M)$, where $\overline{M}^{\omega}$ is the weak closure of M;
(4) $\omega(M \cup N) = \max\{\omega(M), \omega(N)\}$;
(5) $\omega(\lambda M) = |\lambda|\omega(M)$ for all $\lambda \in \mathbb{R}$;
(6) $\omega(\mathrm{co}(M)) = \omega(M)$, where $\mathrm{co}(M)$ is the convex hull of M;
(7) $\omega(M + N) \le \omega(M) + \omega(N)$;
(8) If $(M_n)_{n=1}^{\infty}$ is a decreasing sequence of nonempty, bounded and weakly closed subsets of E with $\lim_{n\to\infty} \omega(M_n) = 0$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.

The family $\ker(\omega)$ described in (1) is called the kernel of the measure of weak noncompactness ω. Note that the intersection set M_∞ from (8) belongs to $\ker(\omega)$ since $\omega(M_\infty) \leq \omega(M_n)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \omega(M_n) = 0$.

The first important example of a measure of weak noncompactness was defined by De Blasi [12] as follows:

$$\omega(M) = \inf\bigl\{r > 0 : \exists W \in \mathcal{W}(E) \text{ such that } M \subseteq W + \mathrm{B}_r\bigr\}.$$

The De Blasi measure of weak noncompactness has some interesting properties. It plays a significant role in nonlinear analysis and has some applications.

Nevertheless, it is rather difficult to express the above De Blasi measure of weak noncompactness with the help of a convenient formula in a concrete Banach space. Such a formula is known in the case of the space of $L^1(I)$. In [13], Appell and De Pascale showed that the measure of noncompactness $\omega(\cdot)$ in $L^1(I)$ possesses the following simple form:

$$\omega(M) = \limsup_{\varepsilon\to 0}\left\{\sup_{x\in M}\left[\int_D \bigl|x(t)\bigr|\,dt : D \subseteq I, \operatorname{meas}(D) \leq \varepsilon\right]\right\} \tag{2.1}$$

for any nonempty bounded subset M of $L^1(I)$, where $\operatorname{meas}(\cdot)$ denotes the Lebesgue measure.

Recall that a useful characterization of relatively weakly compact sets in $L^1(I)$ is provided by the following Dunford-Pettis theorem (cf. [14, p. 115]).

Theorem 2.2 *A bounded set N is relatively weakly compact in $L^1(I)$ if and only if N is equi-integrable, that is,*

$$\forall \varepsilon > 0, \exists \delta > 0 \textit{ such that } \int_D \bigl|x(t)\bigr|\,dt \leq \varepsilon, \quad \forall x \in N,$$

for any measurable set $D \subseteq I$ with $\operatorname{meas}(D) \leq \delta$.

Definition 2.3 (see [15, 16]) Let E_1 and E_2 be two Banach spaces, and let $\mathcal{D}$ be a subset of E_1. A continuous operator $T : \mathcal{D} \to E_2$ is said to be

(1) ws-compact if $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ is a weakly convergent sequence in E_1, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence;

(2) ww-compact if $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ is a weakly convergent sequence in E_1, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence.

Remark 2.4 A continuous operator is ws-compact if and only if it maps relatively weakly compact sets into relatively strongly compact ones; and it is ww-compact if and only if it maps relatively weakly compact sets into relatively weakly compact ones, since the weak compactness of a set in Banach spaces is equivalent to its weakly sequential compactness by the Eberlein-Šmulian theorem (cf. [17, p. 430]).

A function $f : I \times \mathbb{R} \to \mathbb{R}$ is said to satisfy the Carathéodory conditions if it is measurable in t for each x in $\mathbb{R}$ and continuous in x for almost every (or a.e. for short) $t \in I$.

Let $\mathbf{m}(I)$ denote the collection of all measurable functions $x : I \to \mathbb{R}$. If a function f : $I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition, then f defines a mapping $\mathcal{N}_f : \mathbf{m}(I) \to$

$\mathbf{m}(I)$ by $\mathcal{N}_f x(t) := f(t, x(t))$. This mapping is called the superposition operator (Nemytskii operator) associated with f. Regarding its continuity, we have the following theorem (cf. [18, p. 93]).

Theorem 2.5 *The superposition operator $\mathcal{N}_f$ maps $L^1(I)$ into $L^1(I)$ if and only if there exist a function $a \in L^1_+(I)$ and a constant $b > 0$ such that*

$$\left|f(t,x)\right| \leq a(t) + b|x|,$$

where $L^1_+(I)$ denotes the positive cone of $L^1(I)$. In this case, $\mathcal{N}_f$ is continuous and bounded in the sense that it maps bounded sets into bounded ones.

Definition 2.6 Let $\mathcal{D}$ be a nonempty subset of the Banach space E. An operator $T : \mathcal{D} \to E$ is said to be

(1) contractive with ℓ if there exists $\ell \in [0,1)$ such that $\|Tx_1 - Tx_2\| \leq \ell\|x_1 - x_2\|$ for all $x_1, x_2 \in \mathcal{D}$;

(2) ω-contractive with ℓ if it maps bounded sets into bounded sets, and there exists $\ell \in [0,1)$ such that $\omega(T(M)) \leq \ell\omega(M)$ for all bounded sets M in $\mathcal{D}$.

We end these preliminaries with the following Krasnosel'skii type fixed point result (cf. [19, Corollary 3.4]). It plays an important role in the proof of our main result.

Theorem 2.7 *Let M be a nonempty, bounded, closed and convex subset of a Banach space E. Suppose that the operators $A : M \to E$ and $B : E \to E$ satisfy*

(i) *A is ω-contractive with α, and A is ws-compact*;

(ii) *B is contractive with β, and B is ww-compact*;

(iii) *the equality $y = By + Ax$ with $x \in M$ implies $y \in M$.*

Then there exists $x \in M$ such that $x = Ax + Bx$ provided $\alpha + \beta < 1$.

3 Main results

Throughout this paper, $L^1(I)$ will denote the Banach space consisting of all real functions defined and Lebesgue integrable on $I := [0,1]$, with the standard norm $\|\cdot\|$; and $L^\infty(I)$ will denote the Banach space consisting of all real functions defined and essentially bounded on I, with the standard norm $\|\cdot\|_\infty$.

Lemma 3.1 *A function $x = x(t)$ is a solution of the boundary value problem* (1)-(2) *if and only if x is a solution of the following integral equation*:

$$x(t) = g\bigl(t, x(t)\bigr) + h\bigl(t, x(t)\bigr) \int_0^1 G(t,s) f\bigl(s, x(s)\bigr)\, ds, \tag{3.1}$$

where the Green's function associated with (1.1)-(1.2) *is defined by*

$$G(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.2}$$

Proof $x : I \to \mathbb{R}$ is a solution of Eq. (1.1) if and only if it satisfies

$$\frac{d}{dt}\left(\frac{x(t)-g(t,x(t))}{h(t,x(t))}\right) = -\int_0^t f\bigl(s,x(s)\bigr)\,ds + c_1,$$

it follows that

$$\frac{x(t)-g(t,x(t))}{h(t,x(t))} = -\int_0^t (t-s)f\bigl(s,x(s)\bigr)\,ds + c_1 t + c_0. \tag{3.3}$$

By choosing $t = 0$ and $t = 1$ in (3.3) respectively and applying the boundary condition (1.2), we get

$$c_0 = \frac{x(0)-g(0,x(0))}{h(0,x(0))} = 0, \qquad c_1 = \int_0^1 (1-s)f\bigl(s,x(s)\bigr)\,ds. \tag{3.4}$$

From (3.3)-(3.4) we deduce that

$$\begin{aligned}\frac{x(t)-g(t,x(t))}{h(t,x(t))} &= \int_0^t s(1-t)f\bigl(s,x(s)\bigr)\,ds + \int_t^1 t(1-s)f\bigl(s,x(s)\bigr)\,ds \\ &= \int_0^1 G(t,s)f\bigl(s,x(s)\bigr)\,ds.\end{aligned}$$

Thus, problem (1.1)-(1.2) has been transformed into the perturbed quadratic integral equations (3.1). □

Remark 3.2 Since the maximum value of the Green's function G will be gotten as $s = t$, we have $\max_{(t,s)\in I^2} G(t,s) = \max_{t\in I} t(1-t) = 1/4$. Thus, the linear operator $\mathbb{G}$ defined by

$$\mathbb{G}x(t) := \int_0^1 G(t,s)x(s)\,ds, \quad \forall x \in L^1(I),$$

is bounded from $L^1(I)$ into $L^\infty(I)$. In fact, we have

$$\|\mathbb{G}x\|_\infty \le \frac{1}{4}\int_0^1 \bigl|x(s)\bigr|\,ds \le \frac{1}{4}\|x\|. \tag{3.5}$$

Definition 3.3 A function $x \in L^1(I)$ is said to be a weak solution of problem (1.1)-(1.2) if x satisfies Eq. (3.1) on the interval I.

We will consider (1.1)-(1.2) under the following assumptions.

($\mathcal{H}1$) The functions $f : I \times \mathbb{R} \to \mathbb{R}$ and $h : I \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ satisfy the Carathéodory conditions. Moreover, there exist functions $f_0, h_0 \in L^1_+(I)$ and positive numbers η and γ, respectively, such that

$$\bigl|f(t,x)\bigr| \le f_0(t) + \eta|x|, \qquad \bigl|h(t,x)\bigr| \le h_0(t) + \gamma|x|, \quad \forall x \in \mathbb{R} \text{ and a.e. } t \in I.$$

($\mathcal{H}2$) The function $g : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions, and there exists a positive number β such that

$$\left|g(t,x) - g(t,y)\right| \le \beta|x - y|, \quad \forall x, y \in \mathbb{R} \text{ and a.e. } t \in I.$$

Moreover, $g_0(t) := |g(t,0)|$ belongs to $L^1_+(I)$.

($\mathcal{H}3$) The following inequality holds:

$$4\beta + \gamma\|f_0\| + \eta\|h_0\| + 2\sqrt{\gamma\eta\big(4\|g_0\| + \|f_0\|\|h_0\|\big)} < 4. \tag{3.6}$$

Remark 3.4 (1) Note that from ($\mathcal{H}2$) we deduce that $|g(t,x)| \le g_0(t) + \beta|x|$ for all $x \in \mathbb{R}$ and a.e. $t \in I$. Thus, by Theorem 2.5, ($\mathcal{H}1$) and ($\mathcal{H}2$) imply that the superposition operators $\mathcal{N}_f$, $\mathcal{N}_g$ and $\mathcal{N}_h$, respectively, map $L^1(I)$ into itself continuously. Further, according to [7, Lemma 3.2], $\mathcal{N}_f$, $\mathcal{N}_g$ and $\mathcal{N}_h$ are ww-compact.

(2) It is easily deduced from inequality (3.6) of ($\mathcal{H}3$) that the following quadratic inequality about $\mathbf{r}$

$$\gamma\eta\mathbf{r}^2 - \big(4 - 4\beta - \gamma\|f_0\| - \eta\|h_0\|\big)\mathbf{r} + 4\|g_0\| + \|f_0\|\|h_0\| \le 0 \tag{3.7}$$

has positive solutions, since inequality (3.6) implies that

$$\begin{aligned}
&\big(4 - 4\beta - \gamma\|f_0\| - \eta\|h_0\|\big)^2 - 4\gamma\eta\big(4\|g_0\| + \|f_0\|\|h_0\|\big) > 0,\\
&\frac{4 - 4\beta - \gamma\|f_0\| - \eta\|h_0\|}{2\gamma\eta} > 0.
\end{aligned}$$

Further, there is a certain solution $\mathbf{r}_0$ of (3.7) such that

$$0 < \mathbf{r}_0 \le \frac{4 - 4\beta - \gamma\|f_0\| - \eta\|h_0\|}{2\gamma\eta}. \tag{3.8}$$

Theorem 3.5 *Under assumptions* ($\mathcal{H}1$)-($\mathcal{H}3$), *problem* (1.1)-(1.2) *has at least one weak solution $x \in M$, where $M := \{x : \|x\| \le \mathbf{r}_0\}$ is a closed ball of $L^1(I)$, and $\mathbf{r}_0$ is a solution of* (3.7) *and satisfies* (3.8) .

Proof Let $\mathcal{B} := \mathcal{N}_g$. Define $\mathcal{A}$ by $\mathcal{A}x(t) := \mathcal{N}_h x(t) \cdot \mathbb{G}\mathcal{N}_f x(t)$ for $x \in M$. For proving the operator equation $x = \mathcal{A}x + \mathcal{B}x$ has a unique solution in $L^1(I)$, our processes are divided into several steps.

(1). $\mathcal{A}$ is ws-compact.

For all $x_1, x_2 \in M$, from ($\mathcal{H}1$)-($\mathcal{H}2$), Remark 3.4 and Remark 3.2, we deduce that

$$\begin{aligned}
&\|\mathcal{A}x_1 - \mathcal{A}x_2\|\\
&\quad\le \big\|\mathbb{G}(\mathcal{N}_f x_1 - \mathcal{N}_f x_2)\big\|_\infty \cdot \|\mathcal{N}_h x_1\| + \|\mathbb{G}\mathcal{N}_f x_2\|_\infty \cdot \|\mathcal{N}_h x_1 - \mathcal{N}_h x_2\|\\
&\quad\le \frac{1}{4}\big(\|h_0\| + \gamma\mathbf{r}_0\big) \cdot \|\mathcal{N}_f x_1 - \mathcal{N}_f x_2\| + \frac{1}{4}\big(\|f_0\| + \eta\mathbf{r}_0\big) \cdot \|\mathcal{N}_h x_1 - \mathcal{N}_h x_2\|.
\end{aligned}$$

Thus, according to the continuity of the operators $\mathcal{N}_f$ and $\mathcal{N}_h$, we obtain that $\mathcal{A}$ is continuous on M.

Further, let us take a weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ from M. Then, for any measurable subset D of the interval I, from (3.5) we obtain the following estimate:

$$\begin{aligned}\int_D \bigl|\mathcal{A}x_n(t)\bigr|\,dt &\le \|\mathbb{G}\mathcal{N}_f x_n\|_\infty \int_D \bigl|\mathcal{N}_h x_n(t)\bigr|\,dt \\ &\le \frac{1}{4}\bigl(\|f_0\| + \eta \mathbf{r}_0\bigr)\int_D \bigl|\mathcal{N}_h x_n(t)\bigr|\,dt \quad (n = 1,2,\ldots). \end{aligned} \tag{3.9}$$

Since the sequence $(\mathcal{N}_h x_n)_{n\in\mathbb{N}}$ is relatively weakly compact (cf. Remark 3.4(1)), then applying formula (2.1) we deduce from (3.9) that $(\mathcal{A}x_n)_{n\in\mathbb{N}}$ is relatively weakly compact as well.

In what follows, let us fix a number $\varepsilon > 0$. According to Theorem 2.2, we can choose a number $\delta > 0$ such that, for any measurable subset D_δ of the interval I with $\operatorname{meas}(D_\delta) \le \delta$, we have

$$\int_{D_\delta} \bigl|\mathcal{A}x_n(s)\bigr|\,ds \le \frac{\varepsilon}{3}, \quad m = 1,2,\ldots. \tag{3.10}$$

According to Remark 3.4(1), $(\mathcal{N}_f x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence, say $(\mathcal{N}_f x_{n_k})_{k\in\mathbb{N}}$. Since the continuity of the linear operator $\mathbb{G}$ implies its weak continuity on $L^1(I)$ for a.e. $t \in I$, then $(\mathbb{G}\mathcal{N}_f x_{n_k})_{k\in\mathbb{N}}$ converges pointwise for a.e. $t \in I$. Now, applying Egoroff's theorem, there exists a measurable subset $I_0 \subseteq I$ with $\operatorname{meas}(I\backslash I_0) \le \delta$ such that $(\mathbb{G}\mathcal{N}_f x_{n_k})_{k\in\mathbb{N}}$ is uniformly convergent on I_0.

On the other hand, $(\mathcal{N}_h x_{n_k})_{k\in\mathbb{N}}$ has a weakly convergent subsequence as well according to Remark 3.4(1). Without loss of generality, we can assume that it is still $(\mathcal{N}_h x_{n_k})_{k\in\mathbb{N}}$. Thus, the sequence $(\mathcal{N}_h x_{n_k} \cdot \mathbb{G}\mathcal{N}_f x_{n_k})_{k\in\mathbb{N}}$, that is, $(\mathcal{A}x_{n_k})_{k\in\mathbb{N}}$ is strongly convergent in $L^1(I_0)$ (cf. [14, Proposition 3.5, p. 58]). Therefore, $(\mathcal{A}x_{n_k})_{k\in\mathbb{N}}$ satisfies the Cauchy criterion on I_0, i.e., there exists $k_0 \in \mathbb{N}$ such that for arbitrary natural numbers $j,k \ge k_0$ the following inequality holds:

$$\int_{I_0} \bigl|\mathcal{A}x_{n_j}(t) - \mathcal{A}x_{n_k}(t)\bigr|\,dt \le \frac{\varepsilon}{3}. \tag{3.11}$$

Thus, we deduce from (3.10) and (3.11) that

$$\begin{aligned}\|\mathcal{A}x_{n_j} - \mathcal{A}x_{n_k}\| &\le \int_{I_0} \bigl|\mathcal{A}x_{n_j}(t) - \mathcal{A}x_{n_k}(t)\bigr|\,dt \\ &\quad + \int_{I\backslash I_0} \bigl|\mathcal{A}x_{n_j}(t)\bigr|\,dt + \int_{I\backslash I_0} \bigl|\mathcal{A}x_{n_k}(t)\bigr|\,dt \\ &\le \varepsilon, \end{aligned}$$

which implies that the sequence $(\mathcal{A}x_{n_k})_{k\in\mathbb{N}}$ is convergent in $L^1(I)$, and then $\mathcal{A}$ maps relatively weakly compact subsets of M into relatively strongly compact ones.

Thus, we complete the proof that $\mathcal{A}$ is ws-compact.

(2). $\mathcal{A}$ is ω-contractive.

Let D be a measurable subset of the interval I, and let S be a nonempty subset of M. From Remark (3.4) and Remark (3.2), for all $x \in S$, we deduce that

$$\int_D \bigl|\mathcal{A}x(t)\bigr|\,dt \le \|\mathbb{G}\mathcal{N}_f x\|_\infty \int_D \bigl|\mathcal{N}_h x(t)\bigr|\,dt \le \frac{1}{4}\bigl(\|f_0\| + \eta\|x\|\bigr)\int_D \bigl(h_0(t) + \gamma\bigl|x(t)\bigr|\bigr)\,dt.$$

Taking into account the facts that the set of a single element is weakly compact and $\|x\| \leq \mathbf{r}_0$, the use of formula (2.1) leads to

$$\omega\big(\mathcal{A}(S)\big) \leq \frac{(\|f_0\| + \eta\mathbf{r_0})\gamma}{4}\omega(S) = \alpha\omega(S), \tag{3.12}$$

where $\alpha := (\|f_0\| + \eta\mathbf{r_0})\gamma/4$. Applying inequality (3.8), we have

$$\alpha < \frac{\gamma}{4}\left(\|f_0\| + \frac{4 - 4\beta - \gamma\|f_0\| - \eta\|h_0\|}{\gamma}\right) < 1 - \beta. \tag{3.13}$$

Thus, (3.12) implies that $\mathcal{A}$ is ω-contraction with α.

(3). If $y = \mathcal{B}y + \mathcal{A}x$ for $x \in M$, then $y \in M$.

If $y \in L^1(I)$ satisfies $y = \mathcal{B}y + \mathcal{A}x$ for $x \in M$, then we have

$$\begin{aligned}\big|y(t)\big| &\leq \big|\mathcal{N}_g y(t)\big| + \big|\mathcal{N}_h x(t)\big| \cdot \big|\mathbb{G}\mathcal{N}_f x(t)\big| \\ &\leq g_0(t) + \beta\big|y(t)\big| + \big(h_0(t) + \gamma\big|x(t)\big|\big) \cdot \frac{1}{4}\big(\|f_0\| + \eta\|x\|\big),\end{aligned}$$

for a.e. $t \in I$, it follows that

$$4\|y\| \leq 4\|g_0\| + 4\beta\|y\| + \big(\|f_0\| + \eta\|x\|\big)\big(\|h_0\| + \gamma\|x\|\big).$$

Noting that $\|x\| \leq \mathbf{r_0}$ and applying inequality (3.7), we deduce from the above that

$$\|y\| \leq \frac{4\|g_0\| + \|f_0\|\|h_0\| + (\gamma\|f_0\| + \eta\|h_0\|)\mathbf{r}_0 + \gamma\eta\mathbf{r}_0^2}{4(1-\beta)} \leq \mathbf{r}_0,$$

which implies $y \in M$.

(4). Conclusion.

The condition (i) of Theorem 2.7 is verified in (1)-(2), and the condition (iii) of Theorem 2.7 is verified in (3). Moreover, $\mathcal{B} = \mathcal{N}_g$ is contractive with β by ($\mathcal{H}2$), and $\mathcal{B}$ is ww-compact by Remark 3.4(1). Then the condition (ii) of Theorem 2.7 is satisfied. The estimate $\alpha + \beta < 1$ is from (3.13).

Now, according to Theorem 2.7, we obtain that Eq. (3.1) has at least one solution in M, and then the existence of weak solutions in $L^1(I)$ for problem (1.1)-(1.2) is proved. □

4 Examples

In this section we give two examples to illustrate the existence result involved in Theorem 3.5.

Example 4.1 Consider the following boundary problem of second order ordinary differential equations:

$$\left(\frac{2x(t) - (t - t^2)\sin x(t)}{\sqrt{e^t + [x(t)]^2}}\right)'' + \frac{\ln(1+t) + x(t)}{1 + [x(t)]^2} = 0, \quad 0 < t < 1, \tag{4.1}$$

$$x(0) = 0, \qquad x(1) = 0. \tag{4.2}$$

In order to show that problem (4.1)-(4.2) admits at least one weak solution in a certain ball of $L^1(I)$, we are going to check that the conditions of Theorem 3.5 are satisfied. To this end, define the functions as follows:

$$g(t,x) := \frac{1}{2}\left(t - t^2\right)\sin x, \qquad h(t,x) := \frac{1}{2}\sqrt{e^t + x^2}, \qquad f(t,x) := \frac{\ln(1+t) + x}{1 + x^2}.$$

For all $x \in \mathbb{R}$ and $t \in [0,1]$, we have

$$\left|f(t,x)\right| \le \ln(1+t) + |x|, \qquad \left|h(t,x)\right| \le \frac{1}{2}\sqrt{e^t} + \frac{1}{2}|x|.$$

Thus, $(\mathcal{H}1)$ is satisfied with $f_0(t) = \ln(1+t)$, $\eta = 1$, $h_0(t) = \sqrt{e^t}/2$ and $\gamma = 1/2$.

For any $x_1, x_2 \in \mathbb{R}$ and $t \in I$, we have

$$\left|g(t,x_1) - g(t,x_2)\right| = \frac{t - t^2}{2}|\sin x_1 - \sin x_2| \le \frac{1}{8}|x_1 - x_2|.$$

Thus, $(\mathcal{H}2)$ is satisfied with $\beta = 1/8$ and $g_0(t) = 0$.

Moreover, a simple calculation yields

$$\|f_0\| = 2\ln 2 - 1, \qquad \|g_0\| = 0, \qquad \|h_0\| = \sqrt{e} - 1.$$

It is easy to infer that inequality (3.6) holds, and then $(\mathcal{H}3)$ is satisfied.

Now, based on Theorem 3.5, we infer that there exists $x \in L^1(I)$ such that it is a weak solution of problem (4.1)-(4.2) in the set $M := \{x : \|x\| \le \mathbf{r}_0\}$, where $\mathbf{r}_0$ satisfies

$$0 < \mathbf{r}_0 \le \frac{4 - 4\beta - \gamma\|f_0\| - \eta\|h_0\|}{2\gamma\eta} = 5 - \ln 2 - \sqrt{e} = 2.8581\ldots.$$

Example 4.2 Consider the following boundary problem of second order ordinary differential equations:

$$x''(t) + tx'(t) + \ln\left(1 + \left|x(t)\right|\right) + 2 - t + 2t^2 - \ln\left(1 + t - t^2\right) = 0, \quad t \in [0,1], \tag{4.3}$$

$$x(0) = 0, \qquad x(1) = 0. \tag{4.4}$$

According to the introduction in Section 1, we can take the functions as follows:

$$g\left(t,x(t)\right) = e^{-\frac{1}{2}t^2}\int_0^t x(s)se^{\frac{1}{2}s^2}\,ds, \qquad h\left(t,x(t)\right) = e^{-\frac{1}{2}t^2},$$

$$f\left(t,x(t)\right) = e^{\frac{1}{2}t^2}\left[\ln\left(1 + \left|x(t)\right|\right) + 2 - t + 2t^2 - \ln\left(1 + t - t^2\right)\right].$$

For all $x \in \mathbb{R}$ and $t \in [0,1]$, we have

$$\left|f(t,x)\right| \le e^{\frac{1}{2}t^2}\left[2 - t + 2t^2 - \ln\left(1 + t - t^2\right)\right] + \sqrt{e}|x|,$$

$$\left|h(t,x)\right| \le \int_0^1 e^{-\frac{1}{2}t^2}\,dt + \gamma|x|.$$

Thus, $(\mathcal{H}1)$ is satisfied with $f_0(t) = e^{\frac{1}{2}t^2}[2 - t + 2t^2 - \ln(1 + t - t^2)]$, $\eta = \sqrt{e}$, $h_0(t) = e^{-\frac{1}{2}t^2}$ and γ being chosen as enough small positive number.

Further, for all $x_1, x_2 \in \mathbb{R}$ and $t \in [0,1]$, we have

$$\bigl|g(t,x_1) - g(t,x_2)\bigr| = e^{-\frac{1}{2}t^2} \int_0^t |x_1 - x_2| s e^{\frac{1}{2}s^2}\, ds \le \left(1 - \frac{1}{\sqrt{e}}\right)|x_1 - x_2|,$$

and $g_0(t,0) = 0$. Thus, $(\mathcal{H}2)$ is satisfied with $\beta = 1 - 1/\sqrt{e}$. Note that a simple calculation yields

$$\|h_0\| = \int_0^1 e^{-\frac{1}{2}t^2}\, dt \le 0.9, \qquad \|g_0\| = 0, \qquad \|f_0\| \le \int_0^1 e^{-\frac{1}{2}t^2}\bigl(2 - t + 2t^2\bigr)\, dt \le 1.69.$$

Thus, if we take $0 < \gamma \le 0.02$, then we have

$$\begin{aligned}
&4\beta + \gamma\|f_0\| + \eta\|h_0\| + 2\sqrt{\gamma\eta\bigl(4\|g_0\| + \|f_0\|\|h_0\|\bigr)} \\
&\quad\le 4\left(1 - \frac{1}{\sqrt{e}}\right) + 0.02 \times 1.69 + \sqrt{e} \times 0.9 + 2\sqrt{0.02 \times \sqrt{e} \times 1.69 \times 0.9} \\
&\quad= 3.5394\ldots < 4,
\end{aligned}$$

which implies that $(\mathcal{H}3)$ is satisfied.

Finally, based on Theorem 3.5, we infer that there exists $x \in L^1(I)$ such that it is a weak solution of problem (4.3)-(4.4). Moreover, it is easy to see that the exact solution of (4.3)-(4.4) is $x(t) = t - t^2$.

5 Conclusions

In this work, we have established an existence result for weak solutions of the boundary value problem of nonlinear differential equations of second order. Our main assumptions about the functions being involved in the equation are the Carathéodory conditions, and the main tool is a Krasnosel'skii type fixed point theorem in conjunction with the technique of measures of weak noncompactness.

In the proof of Theorem 3.5, we avoided using the Scorza-Dragoni theorem [20], the Arzelà-Ascoli theorem, the modulus of continuity of the function G, etc. By using Egoroff's theorem, we have replaced this method so that the proof process is simplified. The reader can compare it with [5, 8, 10, 21, 22].

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