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Energy decay of solutions of nonlinear viscoelastic problem with the dynamic and acoustic boundary conditions

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Abstract
In this paper, we are concerned with the energy decay rate of the nonlinear viscoelastic problem with dynamic and acoustic boundary conditions.

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1 Introduction

In this paper, we are concerned with the energy decay rate of the following nonlinear viscoelastic problem with a time-varying delay in the boundary feedback and acoustic boundary conditions:

$$\begin{aligned} & u_{tt} - \delta_0 \Delta u + \int_0^t g(t-s)\operatorname{div}\bigl(a(x)\nabla u(s)\bigr)\,ds + \bigl(\delta_1 + b(x)\bigl|u_t(t)\bigr|^{m-2}\bigr)u_t \\ & \quad = |u|^{p-2}u \quad \text{in } \Omega \times (0,+\infty), \end{aligned} \tag{1.1}$$

$$u = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \tag{1.2}$$

$$\begin{aligned} & u_{tt} + \delta_0 \frac{\partial u(t)}{\partial \nu} - \int_0^t g(t-s)\bigl(a(x)\nabla u(s)\bigr)\cdot \nu\,ds + \mu_1 k_1\bigl(u_t(t)\bigr) + \mu_2 k_2\bigl(u_t\bigl(t-\tau(t)\bigr)\bigr) \\ & \quad = h(x)y_t \quad \text{on } \Gamma_1 \times (0,\infty), \end{aligned} \tag{1.3}$$

$$u_t + f(x)y_t + m(x)y = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.4}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{1.5}$$

$$y(x,0) = y_0(x) \quad \text{on } \Gamma_1, \tag{1.6}$$

$$u_t\bigl(x,t-\tau(t)\bigr) = j_0\bigl(x,t-\tau(0)\bigr) \quad \text{on } \Gamma_1 \times \bigl(0,\tau(0)\bigr), \tag{1.7}$$

where Ω regular and is a bounded domain of R^n, $n \geq 1$, $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Here Γ_0, Γ_1 are closed and disjoint with $\operatorname{meas}(\Gamma_0) > 0$ and ν is the unit outward normal to $\partial\Omega$, $\delta_0 > 0$, $\delta_1 \geq 0$, $m \geq 2$, $p > 2$, g denotes the memory kernel and a, b are real valued functions which satisfy appropriate conditions. The functions $f, m, h : \Gamma_1 \to R^+$ are essentially bounded,

$k_1, k_2 : R \to R$ are given functions, $\tau(t) > 0$ represents the time-varying delay, μ_1, μ_2 are real numbers with $\mu_1 > 0$, $\mu_2 \neq 0$ and the initial data (u_0, u_1, y_0) belongs to a suitable space. This type of equation usually arises in the theory of viscoelasticity. It is well known that viscoelastic materials have memory effects, which is due to the mechanical response influenced by the history of the materials themselves. From the mathematical point of view, their memory effects are modeled by integrodifferential equations. Hence, equations related to the behavior of the solutions for the PDE system have attracted considerable attention in recent years. We can refer to recent work in [1–7].

The dynamic boundary conditions are not only important from the theoretical point of view but also arise in numerous practical problems. Among the early results dealing with this type of boundary conditions are those of [8, 9] in which the author has made contributions to this field. Recently, some authors have studied the existence and decay of solutions for a wave equation with dynamic boundary conditions [10–13].

Moreover, the acoustic boundary conditions was introduced by Morse and Ingard in [14] and developed by Beale and Rosencrans in [15], where the authors proved the global existence and regularity of the linear problem. Recently, some authors have studied the existence and decay of solutions for a viscoelastic wave equation with acoustic boundary conditions (see [16–19]). Time delay so often arises in many physical, chemical, biological and economical phenomena because these phenomena depend not only on the present state but also on the history of the system in a more complicated way.

In recent years, differential equations with time delay effects have become an active area of research; see for example [20–26] and the references therein. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary. For instance in [22], the authors are proved the boundary stabilization of a nonlinear viscoelastic equation with interior time-varying delay and nonlinear dissipative boundary feedback. In particular, Wu and Chen [27] consider the nonlinear viscoelastic wave equation with boundary dissipation

$$
\begin{aligned}
&u_{tt}(t) - K_0\Delta u(t) + \int_0^t g(t-s)\,\operatorname{div}\bigl(a(x)\nabla u(s)\bigr)\,ds + b(x)u_t = f(u) \quad \text{in } \Omega \times (0,\infty),\\
&u = 0 \quad \text{on } \Gamma_0 \times (0,\infty),\\
&K_0\frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\bigl(a(x)\nabla u(s)\bigr)\cdot \nu\,ds + h(u_t) = 0 \quad \text{on } \Gamma_1 \times (0,\infty),\\
&u(0) = 0, \qquad u_t(0) = u_1, \quad x \in \Omega,
\end{aligned}
$$

where $K_0 > 0$ and Ω is a bounded domain in R^n $(n \geq 1)$ with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. The authors studied the uniform decay of solutions for a nonlinear viscoelastic wave equation with boundary dissipation. In [28], Boukhatem and Benabderrahmane have proved the existence and decay of solutions for a viscoelastic wave equation with acoustic boundary conditions as follows:

$$
\begin{aligned}
&u_{tt} + Lu - \int_0^t g(t-s)Lu(s)\,ds = |u|^{p-2}u \quad \text{in } \Omega \times (0,\infty),\\
&u = 0 \quad \text{on } \Gamma_0 \times (0,\infty),\\
&\frac{\partial u}{\partial \nu_L} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu_L}(s)\,ds = h(x)z_t \quad \text{on } \Gamma_1 \times (0,\infty),
\end{aligned}
$$

$$u_t + f(x)z_t + m(x)z = 0 \quad \text{on } \Gamma_1 \times (0,\infty),$$
$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{in } \Omega,$$
$$z(x,0) = z_0(x) \quad \text{on } \Gamma_1,$$

where $Lu = -\operatorname{div}(A\nabla u) = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j})$ and $\frac{\partial u}{\partial \nu_L} = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_j}\nu_i$.

Liu [29] investigated the following viscoelastic wave equation with an interval time-varying delay term:

$$u_{tt}(x,t) - \Delta u(x,t) + \alpha(t)\int_0^t g(t-s)\Delta u(x,s)\,ds + a_0 u_t(x,t) + a_1\bigl(x,t-\tau(t)\bigr) = 0$$
$$\text{in } \Omega \times (0,\infty),$$
$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,\infty),$$
$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{in } \Omega,$$
$$u_t\bigl(x,t-\tau(0)\bigr) = f_0\bigl(x,t-\tau(0)\bigr) \quad \text{on } \Omega \times \bigl(0,\tau(0)\bigr),$$

where Ω is a bounded domain R^n $(n \geq 2)$ with a boundary $\partial\Omega$ of class C^2, α and g are positive non-increasing functions defined on R^+, a_0 and a_1 are real number with $a_0 > 0$, $\tau(t) > 0$ represents the time-varying delay. He also proved the general decay rate for the energy of a weak viscoelastic wave equation with an interval time-varying delay term.

Recently, Li and Chai [30] have investigated the energy decay for a nonlinear wave equation of variable coefficients with acoustic boundary conditions and a time-varying delay in the boundary feedback form:

$$u_{tt} - \operatorname{div}\bigl(A(x)\nabla u\bigr) + \varphi(u_t) = 0 \quad \text{in } \Omega \times (0,\infty),$$
$$u = 0 \quad \text{on } \Gamma_1 \times (0,\infty),$$
$$u_t + f(x)z_t + k(x)z = 0 \quad \text{on } \Gamma_1 \times (0,\infty),$$
$$\frac{\partial u}{\partial \nu_A} - h(x)z_t + \mu_1\beta\bigl(u_t(x,t)\bigr) + \mu_2 u_t\bigl(x,t-\tau(t)\bigr) = 0 \quad \text{on } \Gamma_1 \times (0,\infty),$$
$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{in } \Omega,$$
$$z(x,0) = z_0(x) \quad \text{on } \Gamma_1,$$
$$u_t\bigl(x,t-\tau(0)\bigr) = j_0\bigl(x,t-\tau(0)\bigr) \quad \text{on } \Gamma_1 \times \bigl(0,\tau(0)\bigr),$$

where $\operatorname{div} X$ denotes the divergence of the vector field X in the Euclidean metric, $A(x) = (a_{ij}(x))$ are symmetric and positive definite metrics for all $x \in R^n$ and $a_{ij}(x)$ are smooth functions on R^n, $\frac{\partial u}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\nu_i$, where $\nu = (\nu_1,\nu_2,\dots,\nu_n)^T$ denotes the outward unit normal vector of the boundary and $\nu + A = A\mu$.

Motivated by previous work, in this paper, we study the energy decay rate of the nonlinear viscoelastic problem with a time-varying delay in the boundary conditions. Previously many authors have considered the uniform decay of solutions for a nonlinear viscoelastic wave equations with boundary dissipations. However, to our knowledge, there is no energy decay result of the nonlinear viscoelastic problem with a the dynamic, time-varying delay and acoustic boundary conditions. Thus this work is significant. The outline of the paper is the following. In Section 2, we give some notation and hypotheses for our result. In Section 3, we prove our main result.

2 Preliminary

In this section, we present some material that we shall use in order to present our results. We denote by $(u,v) = \int_\Omega u(x)v(x)\,dx$ the scalar product in $L^2(\Omega)$. We denote by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \le q \le \infty$ and $\|\cdot\|_{q,\Gamma_1}$ for $L^q(\Gamma_1)$. We introduce by

$$V = \{u \in H^1(\Omega); u = 0 \text{ on } \Gamma_0\}$$

the closed subspace of $H^1(\Omega)$ equipped with the norm equivalent to the usual norm in $H^1(\Omega)$. The Poincaré inequality holds in V, *i.e.*, there exists a constant C_* such that

$$\forall u \in V, \quad \|u\|_p \le C_* \|\nabla u\|, \quad 2 < p \le \bar{p}, \tag{2.1}$$

where

$$\bar{p} = \begin{cases} \frac{2(n-2)}{n-2}, & \text{if } n \ge 3, \\ \infty, & \text{if } n = 1, 2, \end{cases}$$

and there exists a constant $\tilde{C}_* > 0$ such that

$$\|u\|_{\Gamma_1} \le \tilde{C}_* \|\nabla u\|, \quad \forall u \in V. \tag{2.2}$$

For studying the problem (1.1)-(1.7) we will need the following assumptions:

($H1$) The kernel function $g : R^+ \to R^+$ is a bounded C^1 function satisfying

$$g(0) > 0, \quad \delta_0 - \|a\|_\infty \int_0^\infty g(s)\,ds := l > 0, \tag{2.3}$$

and there exists a non-increasing C^1 positive differentiable function $\zeta : R^+ \to R^+$ satisfying

$$g'(t) \le -\zeta(t)g(t), \qquad \int_0^\infty \zeta(s)\,ds = \infty, \quad \forall t \ge 0. \tag{2.4}$$

($H2$) $a : \Omega \to R$ is a nonnegative function and $a \in C^1(\bar{\Omega})$ such that

$$a(x) \ge a_0 > 0, \tag{2.5}$$

$$\|\nabla a(x)\|^2 \le \alpha_1^2 \|a\|_\infty^2 \tag{2.6}$$

for some positive constant α_1, $b : \Omega \to R$ is a nonnegative functions and $b \in C^1(\bar{\Omega})$ such that $b(x) \ge b_0 > 0$.

($H3$) Similarly to [31] $k_1 : R \to R$ is a nondecreasing C^1 function such that there exist $\varepsilon_1, C_1, C_2 > 0$ and a convex, increasing function $K_1 : R^+ \to R^+$, of the class $C^1(R^+) \cap C^2(R^+)$ satisfying $K_1(0) = 0$, K_1 is linear (or ($K_1'(0) = 0$) and $K_1'' > 0$ on $[0, \varepsilon_1]$) such that

$$C_1|s| \le |k_1(s)| \le C_2|s| \quad \text{for all } |s| \ge \varepsilon_1, \tag{2.7}$$

$$s^2 + k_1^2(s) \le K_1^{-1}\big(sk_1(s)\big) \quad \text{for all } |s| \le \varepsilon_1, \tag{2.8}$$

$k_2 : R \to R$ is an odd nondecreasing C^1 function such that there exist $C_3, C_4, C_5 > 0$,

$$\big|k_2'(s)\big| \le C_3, \tag{2.9}$$

$$C_4 sk_2(s) \le K_2(s) \le C_5 sk_1(s) \quad \text{for } s \in R, \tag{2.10}$$

where

$$K_2(s) = \int_0^s k_2(r)\,dr.$$

($H4$) For the time-varying delay, we assume as in [22] that $\tau \in W^{2,\infty}([0,T])$ for $T > 0$ and there exist positive constants τ_0, τ_1 such that

$$0 < \tau_0 \le \tau(t) \le \tau_1, \quad \forall t > 0. \tag{2.11}$$

Moreover, we assume that there exists $d > 0$ such that

$$\tau'(t) \le d < 1 \quad \text{for } t > 0, \tag{2.12}$$

and that μ_1, μ_2 satisfy

$$|\mu_2| < \frac{C_4(1-d)}{C_5(1-C_4 d)}\mu_1. \tag{2.13}$$

($H5$) The functions $f, m, h > 0$ are essentially bounded such that $f(x), m(x), h(x) > 0$. Furthermore, there exist positive constants f_0, m_0 and h_0 such that

$$f(x) \ge f_0, \qquad m(x) \ge m_0, \qquad h(x) \ge h_0 \quad \text{for all a.e. } x \in \Gamma_1.$$

Remark 2.1 By the mean value theorem for integrals and the monotonicity of k_2, we find that

$$K_2(s) = \int_0^s k_2(r)\,dr \le sk_2(s).$$

Then from (2.10), we obtain $C_4 \le 1$.

For studying problem (1.1)-(1.7), we introduce a new variable z as in [22],

$$z(x,\rho,t) = u_t\big(x, t-\rho\tau(t)\big), \quad x \in \Gamma_1, \rho \in (0,1), t > 0. \tag{2.14}$$

Then problem (1.1)-(1.7) is equivalent to

$$\begin{aligned} &u_{tt} - \delta_0 \Delta u + \int_0^t g(t-s)\operatorname{div}\big(a(x)\nabla u(s)\big)\,ds + \big(\delta_1 + b(x)\big|u_t(t)\big|^{m-2}\big)u_t(t) \\ &\quad = |u|^{p-2}u \quad \text{in } \Omega \times (0,\infty), \end{aligned} \tag{2.15}$$

$$u = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \tag{2.16}$$

$$u_{tt} + \delta_0 \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\big(a(x)\nabla u(s)\big) \cdot \nu \, ds + \mu_1 k_1\big(u_t(x,t)\big) + \mu_2 k_2\big(z(x,1,t)\big)$$
$$= h(x)y_t \quad \text{on } \Gamma_1 \times (0,\infty), \tag{2.17}$$

$$u_t + f(x)y_t + m(x)y = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{2.18}$$

$$\tau(t)z_t(x,\rho,t) + \big(1 - \rho\tau'(t)\big)z_\rho(x,\rho,t) = 0 \quad \text{on } \Gamma_1 \times (0,1) \times (0,\infty), \tag{2.19}$$

$$z(x,0,t) = u_t(x,t) \quad \text{on } \Gamma_1 \times (0,\infty), \tag{2.20}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{2.21}$$

$$y(x,0) = y_0(x) \quad \text{on } \Gamma_1, \tag{2.22}$$

$$z_t(x,\rho,0) = j_0\big(x, -\rho\tau(0)\big) \quad \text{on } \Gamma_1 \times \big(0,\tau(0)\big). \tag{2.23}$$

Now we are in a position to state the local existence result of problem (2.15)-(2.23) which can be established by combining with the argument of [15, 32].

Theorem 2.1 *Assume that* $(H1)$-$(H5)$ *hold. Then given* $(u_0,u_1) \in V \times L^2(\Omega)$, $y_0 \in L^2(\Gamma_1)$ *and* $j_0 \in L^2(\Gamma_1 \times (0,1))$, *there exist* $T > 0$ *and a unique weak solution* (u,y,z) *of problem* (2.15)-(2.23) *such that*

$$u \in C(0,T;V), \qquad u_t \in C\big(0,T;L^2(\Omega)\big) \cap L^2\big(\Gamma_1 \times (0,1)\big),$$
$$h^{1/2}y \in L^2\big(0,T;L^2(\Gamma_1)\big), \qquad h^{1/2}y_t \in L^2\big(0,T;L^2(\Gamma_1)\big).$$

3 Global existence and asymptotic behavior

In order to study the global existence of solution for problem (2.15)-(2.23) given by Theorem 2.1, we define the functions

$$J(t) = \frac{1}{2}\left(\delta_0 - a(x)\int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2 + \frac{1}{2}(g \circ \nabla u)(t)$$
$$- \frac{1}{p}\big\|u(t)\big\|_p^p + \frac{1}{2}\int_{\Gamma_1} m(x)h(x)y^2(t)\,d\Gamma + \frac{\xi\tau(t)}{2}\int_{\Gamma_1}\int_0^1 K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma \tag{3.1}$$

and

$$I(t) = \left(\delta_0 - a(x)\int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2 + (g \circ \nabla u)(t)$$
$$- \big\|u(t)\big\|_p^p + \int_{\Gamma_1} m(x)h(x)y^2(t)\,d\Gamma + \xi\tau(t)\int_{\Gamma_1}\int_0^1 K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma, \tag{3.2}$$

where $(g \circ \nabla u)(t) = \int_\Omega \int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|^2\,ds\,dx$. Adopting the proof of [33], we still have the following results.

Lemma 3.1 *For any $u \in C^1(0,t;H^1(\Omega))$, we have*

$$\begin{aligned}
&\int_\Omega a(x)\int_0^t g(t-s)\nabla u(s)\nabla u_t(t)\,ds\,dx \\
&\quad = -\frac{1}{2}\int_\Omega a(x)g(t)\big|\nabla u(t)\big|^2\,dx + \frac{1}{2}\big(g'\circ\nabla u\big)(t) \\
&\qquad - \frac{1}{2}\frac{d}{dt}\left[(g\circ\nabla u)(t) - \int_\Omega a(x)\int_0^t g(s)\,ds\big|\nabla u(t)\big|^2\,dx\right].
\end{aligned} \tag{3.3}$$

We define the modified energy functional $E(t)$ associated with problem (2.15)-(2.23) by

$$\begin{aligned}
E(t) &= \frac{1}{2}\big\|u_t(t)\big\|^2 + \frac{1}{2}\left(\delta_0 - a(x)\int_0^t g(s)\,ds\right)\big\|\nabla u(t)\big\|^2 + \frac{1}{2}(g\circ\nabla u)(t) \\
&\quad - \frac{1}{p}\big\|u(t)\big\|_p^p + \frac{1}{2}\int_{\Gamma_1} m(x)h(x)y^2(t)\,d\Gamma + \frac{\xi\tau(t)}{2}\int_{\Gamma_1}\int_0^1 K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma \\
&\quad + \frac{1}{2}\big\|u_t(t)\big\|_{\Gamma_1}^2 \\
&= \frac{1}{2}\big\|u_t(t)\big\|^2 + J(t) + \frac{1}{2}\big\|u_t(t)\big\|_{\Gamma_1}^2,
\end{aligned} \tag{3.4}$$

where ξ is a positive constant such that

$$\frac{|\mu_2|(1-C_4)}{C_4(1-d)} \le \xi \le \frac{\mu_1 - |\mu_2|C_5}{C_5}. \tag{3.5}$$

Lemma 3.2 *Let (u,y,z) be the solution of* (2.15)-(2.23). *Then the energy functional defined by* (3.4) *is a non-increasing function and for all $t > 0$, we have*

$$\begin{aligned}
\frac{d}{dt}E(t) &\le -\left(\mu_1 - |\mu_2|C_5 - \frac{\xi C_5}{2}\right)\int_{\Gamma_1} k_1\big(u_t(t)\big)u_t(t)\,d\Gamma \\
&\qquad - \left[\frac{\xi C_4}{2}\big(1-\tau'(t)\big) - |\mu_2|(1-C_4)\right]\int_{\Gamma_1} z(x,1,t)k_2\big(z(x,1,t)\big)\,d\Gamma \\
&\qquad - \int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma + \frac{1}{2}\big(g'\circ\nabla u\big)(t) \\
&\qquad - \frac{a_0}{2}g(t)\big\|\nabla u(t)\big\|^2 - \delta_1\big\|u_t(t)\big\|^2 - b_0\big\|u_t(t)\big\|_m^m \\
&\le 0.
\end{aligned} \tag{3.6}$$

Proof Multiplying in (2.15) by u_t, integrating over Ω, using Green's formula and exploiting the conditions (2.16) and (2.17), we have

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\big\|u_t(t)\big\|^2 + \frac{1}{2}\delta_0\big\|\nabla u(t)\big\|^2 - \frac{1}{p}\big\|u(t)\big\|_p^p + \frac{1}{2}\big\|u_t(t)\big\|_{\Gamma_1}^2\right] - \int_{\Gamma_1} h(x)y_t(t)u_t(t)\,d\Gamma \\
&\quad = -\int_\Omega\int_0^t g(t-s)\big(a(x)\nabla u(s)\big)\nabla u_t(t)\,ds\,dx - \int_\Omega\big(\delta_1 + b(x)\big|u_t(t)\big|^{m-2}\big)\big|u_t(t)\big|^2\,dx \\
&\qquad - \mu_1\int_{\Gamma_1} k_1\big(u_t(t)\big)u_t(t)\,d\Gamma - \mu_2\int_{\Gamma_1} k_2\big(z(x,1,t)\big)u_t(t)\,d\Gamma.
\end{aligned} \tag{3.7}$$

On the other hand, from (2.18), we see that

$$-\int_{\Gamma_1} h(x)y_t(t)u_t(t)\,d\Gamma = \int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma + \int_{\Gamma_1} h(x)m(x)y(t)y_t(t)\,d\Gamma. \tag{3.8}$$

We also multiply the equation in (2.19) by $\xi k_2(z(x,\rho,t))$ and integrate over $\Gamma_1 \times (0,1)$ to obtain

$$\begin{aligned}
&\frac{\xi\tau(t)}{2}\int_{\Gamma_1}\int_0^1 z_t(x,\rho,t)k_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma \\
&\quad = -\frac{\xi}{2}\int_{\Gamma_1}\int_0^1 \big(1-\rho\tau'(t)\big)\frac{\partial}{\partial\rho}K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma,
\end{aligned} \tag{3.9}$$

and it follows that

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\xi\tau(t)}{2}\int_{\Gamma_1}\int_0^1 K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma\right) \\
&\quad = -\frac{\xi}{2}\int_{\Gamma_1}\int_0^1 \frac{\partial}{\partial\rho}\big[\big(1-\rho\tau'(t)\big)K_2\big(z(x,\rho,t)\big)\big]\,d\rho\,d\Gamma \\
&\quad = \frac{\xi}{2}\int_{\Gamma_1}\big(K_2\big(z(x,0,t)\big)-K_2\big(z(x,1,t)\big)\big)\,d\Gamma + \frac{\xi}{2}\tau'(t)\int_{\Gamma_1}K_2\big(z(x,1,t)\big)\,d\Gamma.
\end{aligned} \tag{3.10}$$

Thus from (3.7), (3.8), (3.10) and using (2.10) and (3.3), we deduce

$$\begin{aligned}
\frac{d}{dt}E(t) \le{}& -\left(\mu_1 - \frac{\xi C_5}{2}\right)\int_{\Gamma_1} k_1\big(u_t(t)\big)u_t(t)\,d\Gamma + |\mu_2|\int_{\Gamma_1} k_2\big(z(x,1,t)\big)u_t(t)\,d\Gamma \\
&- \frac{\xi}{2}\big(1-\tau'(t)\big)\int_{\Gamma_1} K_2\big(z(x,1,t)\big)\,d\Gamma - \int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma \\
&+ \frac{1}{2}\big(g'\circ\nabla u\big)(t) - \frac{1}{2}g(t)\int_\Omega a(x)\big|\nabla u(t)\big|^2\,dx \\
&- \int_\Omega \big(\delta_1 + b(x)\big|u_t(t)\big|^{m-2}\big)\big|u_t(t)\big|^2\,dx.
\end{aligned} \tag{3.11}$$

Let us denote by K_2^* the conjugate of the convex function K_2, *i.e.*,

$$K_2^*(s) = \sup_{t\in R^+}\big(st - K_2(t)\big),$$

then

$$st \le K_2^*(s) + K_2(t), \quad \forall s,t \ge 0. \tag{3.12}$$

Moreover, K_2 is the Legendre transform of K_2 (thanks to the argument given in [34])

$$K_2^*(s) = s\big(K_2'\big)^{-1}(s) - K_2\big(\big(K_2'\big)^{-1}(s)\big), \quad \forall s \ge 0. \tag{3.13}$$

Then from the definition of K_2 and (3.13), we get

$$K_2^*(s) = s(k_2)^{-1}(s) - K_2\big((k_2)^{-1}(s)\big), \quad \forall s \ge 0. \tag{3.14}$$

Let us recall the following relations (Eq. (3.7) in [35]) derived from (3.14):

$$\begin{aligned} &|\mu_2|\int_{\Gamma_1} k_2\big(z(x,1,t)\big)u_t(t)\,d\Gamma \\ &\quad\le |\mu_2|\int_{\Gamma_1}\big(k_2\big(z(x,1,t)\big)z(x,1,t) - K_2\big(z(x,1,t)\big) + K_2\big(u_t(t)\big)\big)\,d\Gamma, \end{aligned} \tag{3.15}$$

$$\begin{aligned} K_2^*\big(k_2\big(z(x,1,t)\big)\big) &= z(x,1,t)k_2\big(z(x,1,t)\big) - K_2\big(z(x,1,t)\big) \\ &\le (1-C_4)z(x,1,t)k_2\big(z(x,1,t)\big). \end{aligned} \tag{3.16}$$

Using (3.11), (3.15) and (3.16), we obtain

$$\begin{aligned} \frac{d}{dt}E(t) \le{}& -\left(\mu_1 - \frac{\xi C_5}{2}\right)\int_{\Gamma_1} k_1\big(u_t(t)\big)u_t(t)\,d\Gamma \\ &+ |\mu_2|\int_{\Gamma_1}\big[K_2\big(u_t(t)\big) + K_2^*\big(k_2\big(z(x,1,t)\big)\big)\big]\,d\Gamma \\ &- \frac{\xi}{2}\big(1-\tau'(t)\big)\int_{\Gamma_1} K_2\big(z(x,1,t)\big)\,d\Gamma - \int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma + \frac{1}{2}\big(g'\circ\nabla u\big)(t) \\ &- \frac{1}{2}g(t)\int_\Omega a(x)\big|\nabla u(t)\big|^2\,dx - \int_\Omega\big(\delta_1 + b(x)\big|u_t(t)\big|^{m-2}\big)\big|u_t(t)\big|^2\,dx. \end{aligned}$$

Also using (2.10), (2.19), (3.16), this estimate becomes

$$\begin{aligned} \frac{d}{dt}E(t) \le{}& -\left(\mu_1 - |\mu_2|C_5 - \frac{\xi C_5}{2}\right)\int_{\Gamma_1} k_1\big(u_t(t)\big)u_t(t)\,d\Gamma \\ &- \frac{\xi C_4}{2}\big(1-\tau'(t)\big)\int_{\Gamma_1} z(x,1,t)k_2(x,1,t)\,d\Gamma \\ &+ |\mu_2|(1-C_4)\int_{\Gamma_1} z(x,1,t)k_2\big(z(x,1,t)\big)\,d\Gamma \\ &- \int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma + \frac{1}{2}\big(g'\circ\nabla u\big)(t) \\ &- \frac{1}{2}g(t)\int_\Omega a(x)\big|\nabla u(t)\big|^2\,dx - \int_\Omega\big(\delta_1 + b(x)|u_t(t)|^{m-2}\big)\big|u_t(t)\big|^2\,dx. \end{aligned}$$

Consequently, using (3.5), estimate (3.6) follows. Thus the proof of Lemma 3.2 is complete. □

Lemma 3.3 *Let* (u,y,z) *be the solution of* (2.15)-(2.23). *Assume that* $I(0) > 0$ *and*

$$\gamma = C_*^p\left(\frac{2p}{l(p-2)}E(0)\right)^{(p-2)/2} < 1, \tag{3.17}$$

then $I(t) > 0$ *for all* $t \ge 0$.

Proof Since $I(0) > 0$, by continuity of $u(t)$ there exists $T_* < T$ such that $I(t) > 0$ for all $t \in [0, T_*]$. From (3.1), (3.2)

$$\begin{aligned} J(t) &= \frac{p-2}{2p}\bigg[\left(\delta_0 - a(x)\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + (g\circ\nabla u)(t) \\ &\quad + \int_{\Gamma_1} m(x)h(x)y^2(t)\,d\Gamma + \xi\tau(t)\int_{\Gamma_1}\int_0^1 K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma\bigg] + \frac{1}{p}I(t) \\ &> \frac{p-2}{2p}\bigg[\left(\delta_0 - a(x)\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + (g\circ\nabla u)(t) \\ &\quad + \int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma + \xi\tau(t)\int_{\Gamma_1}\int_0^1 K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma\bigg]. \end{aligned}$$

Hence from (2.3), (2.10) and the fact that $(g\circ\nabla u)(t) > 0$, $\forall t \ge 0$, we can deduce

$$\begin{aligned} & l\|\nabla u(t)\|^2 + \int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma + \xi\tau(t)\int_{\Gamma_1}\int_0^1 K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma \\ &\quad \le \left(\delta_0 - a(x)\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + \int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma \\ &\quad\quad + \xi\tau(t)\int_{\Gamma_1}\int_0^1 K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma \\ &\quad < \frac{2p}{p-2}J(t), \quad \forall t\in[0,T_*), \end{aligned} \tag{3.18}$$

it follows that

$$\|\nabla u(t)\|^2 \le \frac{2p}{l(p-2)}J(t) \le \frac{2p}{l(p-2)}E(t) \le \frac{2p}{l(p-2)}E(0), \quad \forall t\in[0,T_*). \tag{3.19}$$

Thus from (2.1), (3.17) and (3.19), we arrive at

$$\begin{aligned} \|u(t)\|_p^p &\le C_*^p\|\nabla u(t)\|^p \le C_*^p\|\nabla u(t)\|^{p-2}\|\nabla u(t)\|^2 \\ &\le C_*^p\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{2}}\|\nabla u(t)\|^2 \le \|\nabla u(t)\|^2, \quad \forall t\in[0,T_*). \end{aligned} \tag{3.20}$$

Hence $\|u(t)\|_p^p \le C\|\nabla u(t)\|^2$, $\forall t\in[0,T_*)$, which implies that $I(t) > 0$, $\forall \in [0,T_*)$. Note that

$$C_*^p\left(\frac{2p}{l(p-2)}E(T_*)\right)^{\frac{p-2}{p}} \le C_*^p\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{p}} < 1.$$

We repeat the procedure with T_* extended to T. □

Theorem 3.1 *Let* (u,y,z) *be the solutions of problem* (2.15)-(2.23). *Suppose that* (3.17) *holds and* $I(0) > 0$, *then the solution* (u,y,z) *is a global time*.

Proof It suffices to show that

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \xi\tau(t)\int_{\Gamma_1}\int_0^1 K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma + \int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma$$

is bounded independent of t. Under the hypotheses in Theorem 3.1, we see from Lemma 3.3 that $I(t) > 0$ on $[0, T]$. Using Lemma 3.2, from (3.18) it follows that

$$\begin{aligned}
&\frac{1}{2}\|u_t(t)\|^2 + \frac{p-2}{2p}\left(l\|\nabla u(t)\|^2 + \xi\tau(t)\int_{\Gamma_1}\int_0^1 K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma\right.\\
&\qquad\left.+\int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma\right)\\
&\qquad\le \frac{1}{2}\|u_t(t)\|^2 + J(t) + \frac{1}{2}\|u_t(t)\|^2_{\Gamma_1} = E(t) < E(0).
\end{aligned}$$

Thus, there exists a constant $C > 0$ depending p and l such that

$$\begin{aligned}
&\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \xi\tau(t)\int_{\Gamma_1}\int_0^1 K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma + \int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma\\
&\qquad\le CE(0) < +\infty.
\end{aligned}$$

Thus the proof of Theorem 3.1 is finished. □

4 General energy decay rate

In this section, we shall investigate the asymptotic behavior of the energy function $E(t)$. For this purpose we construct a Lyapunov function $\mathcal{L}(t)$ equivalent to $E(t)$, which we can show to lead to the desired result. First, we define some functional and establish several lemmas. Let

$$\mathcal{L}(t) = ME(t) + \varepsilon\Psi(t) + \Phi(t) + \varepsilon\Lambda(t), \tag{4.1}$$

where M and ε are positive constants to be chosen later and

$$\begin{aligned}
\Psi(t) &= \int_\Omega u_t(t)u(t)\,dx + \int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma + \frac{1}{2}\int_{\Gamma_1} h(x)f(x)y^2(t)\,d\Gamma\\
&\quad + \int_{\Gamma_1} u(t)u_t(t)\,d\Gamma,
\end{aligned}\tag{4.2}$$

$$\begin{aligned}
\Phi(t) &= -\int_\Omega a(x)u_t(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx\\
&\quad - \int_{\Gamma_1} a(x)u_t(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,d\Gamma,
\end{aligned}\tag{4.3}$$

and

$$\Lambda(t) = \tau(t)\int_{\Gamma_1}\int_0^1 e^{-\rho\tau(t)}K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma. \tag{4.4}$$

The functional $\mathcal{L}(t)$ is equivalent to the energy function $E(t)$ by the following lemma.

Lemma 4.1 *For $\varepsilon > 0$ small enough while M is large enough, there exist two positive constants β_1 and β_2 such that*

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_2 E(t), \quad \forall t \ge 0. \tag{4.5}$$

Proof From Hölder's and Young's inequality, (2.1), (2.2) and (4.2)-(4.4) we have

$$\begin{aligned}
\left|\Psi(t)\right| &\le \left|\int_\Omega u(t)u_t(t)\,dx\right| + \left|\int_{\Gamma_1} h(x)u(t)y(t)\,d\Gamma\right| + \frac{1}{2}\left|\int_{\Gamma_1} h(x)f(x)y^2(t)\,d\Gamma\right| \\
&\quad + \left|\int_{\Gamma_1} u(t)u_t(t)\,d\Gamma\right| \\
&\le \frac{1}{2}\left\|u_t(t)\right\|^2 + \frac{C_*^2}{2}\left\|\nabla u(t)\right\|^2 \\
&\quad + \frac{\|h\|_\infty^{1/2}\|m\|_\infty^{1/2}}{m_0}\left(\int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma\right)^{1/2}\left(\int_{\Gamma_1}\left|u(t)\right|^2 d\Gamma\right)^{1/2} \\
&\quad + \frac{1}{2}\int_{\Gamma_1} h(x)f(x)y^2(y)\,d\Gamma + \frac{1}{2}\left\|u_t(t)\right\|_{\Gamma_1}^2 + \frac{\tilde{C}_*^2}{2}\left\|\nabla u(t)\right\|^2 \\
&\le \frac{1}{2}\left\|u_t(t)\right\|^2 + \frac{C_*^2}{2}\left\|\nabla u(t)\right\|^2 + \frac{\|h\|_\infty\|m\|_\infty}{2m_0^2}\int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma \\
&\quad + \frac{\tilde{C}_*^2}{2}\left\|\nabla u(t)\right\|^2 + \frac{\|f\|_\infty}{2m_0}\int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma + \frac{1}{2}\left\|u_t(t)\right\|_{\Gamma_1}^2 + \frac{\tilde{C}_*^2}{2}\left\|\nabla u(t)\right\|^2 \\
&= \frac{1}{2}\left\|u_t(t)\right\|^2 + \left(\frac{C_*^2}{2} + \frac{\tilde{C}_*^2}{2}\right)\left\|\nabla u(t)\right\|^2 \\
&\quad + \left(\frac{\|f\|_\infty}{2m_0} + \frac{\|h\|_\infty\|m\|_\infty}{2m_0^2}\right)\int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma + \frac{1}{2}\left\|u_t(t)\right\|_{\Gamma_1}^2, \qquad (4.6)
\end{aligned}$$

$$\begin{aligned}
\left|\Phi(t)\right| &\le \left|-\int_\Omega a(x)u_t(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx\right| \\
&\quad + \left|-\int_{\Gamma_1} a(x)u_t(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,d\Gamma\right| \\
&\le \frac{1}{2}\left\|u_t(t)\right\|^2 \\
&\quad + \frac{1}{2}\int_\Omega\left(a(x)u_t(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)^2 dx \\
&\quad + \frac{1}{2}\left\|u_t(t)\right\|_{\Gamma_1}^2 + \frac{1}{2}\int_{\Gamma_1}\left(a(x)u_t(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)^2 d\Gamma \\
&\le \frac{1}{2}\left\|u_t(t)\right\|^2 \\
&\quad + \frac{\|a\|_\infty}{2}\int_0^t g(s)\,ds\int_\Omega\int_0^t g(t-s)\left|u(t)-u(s)\right|^2 ds\,dx + \frac{1}{2}\left\|u_t(t)\right\|_{\Gamma_1}^2 \\
&\quad + \frac{\|a\|_\infty}{2}\int_0^t g(s)\,ds\int_{\Gamma_1}\int_0^t g(t-s)\left|u(t)-u(s)\right|^2 ds\,d\Gamma \\
&\le \frac{1}{2}\left\|u_t(t)\right\|^2 + \frac{1}{2}\left\|u_t(t)\right\|_{\Gamma_1}^2 + \frac{(\delta_0 - l)}{2}\big(C_*^2 + \tilde{C}_*^2\big)(g\circ\nabla u)(t), \qquad (4.7)
\end{aligned}$$

where we used

$$\delta_0 - \|a\|_\infty\int_0^\infty g(s)\,ds = l > 0$$

and

$$\begin{aligned}\left|\Lambda(t)\right| &= \left|\tau(t)\int_{\Gamma_1}\int_0^1 e^{-\rho\tau(t)}K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma\right| \\ &\le C\tau(t)\int_{\Gamma_1}\int_0^1 K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma, \end{aligned} \tag{4.8}$$

where C is a positive constant. Combining (4.1) and (4.6)-(4.8), then we arrive at

$$\begin{aligned}\left|\mathcal{L}(t)-ME(t)\right| &\le \frac{1}{2}(\varepsilon+1)\left\|u_t(t)\right\|^2+\frac{\varepsilon}{2}\big(C_*^2+\tilde{C}_*^2\big)\left\|\nabla u(t)\right\|^2 \\ &\quad+\frac{(\delta_0-l)}{2}\big(C_*^2+\tilde{C}_*^2\big)(g\circ\nabla u)(t) \\ &\quad+\frac{\varepsilon}{2m_0}\left(\|f\|_\infty+\frac{\|h\|_\infty\|m\|_\infty}{m_0}\right)\int_{\Gamma_1}h(x)m(x)y^2(t)\,d\Gamma \\ &\quad+\frac{1}{2}(\varepsilon+1)\left\|u_t(t)\right\|_{\Gamma_1}^2+\varepsilon C\tau(t)\int_{\Gamma_1}\int_0^1 K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma \\ &\le CE(t),\end{aligned}$$

where C is a positive constant. Choosing $M > 0$ large, we complete the proof of Lemma 4.1. □

Lemma 4.2 *Let* (u,y,z) *be the solution of* (2.15)-(2.23). *Then the functional* Ψ *defined in* (4.2) *satisfies*

$$\begin{aligned}\frac{d}{dt}\Psi(t) &\le \left(1+\frac{\delta_1}{4\eta}\right)\left\|u_t(t)\right\|^2 \\ &\quad-\left[\frac{\delta_0}{2}-\delta_1C_*^2\eta-\eta\tilde{C}_*^2\big(1+\mu_1+|\mu_2|\big)-\frac{1}{2\delta_0}(1+\eta)(\delta_0-l)^2\right. \\ &\quad\left.-\frac{\|b\|_\infty\eta^{-m}C_*^m}{m}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{m-2}{2}}\right]\left\|\nabla u(t)\right\|^2 \\ &\quad+\frac{1}{2\delta_0}\left(1+\frac{1}{\eta}\right)(\delta_0-l)(g\circ\nabla u)(t)+\left\|u(t)\right\|_p^p+\left\|u_t(t)\right\|_{\Gamma_1}^2 \\ &\quad+\frac{\|b\|_\infty(m-1)}{m}\eta^{\frac{m}{m-1}}\left\|u_t(t)\right\|_m^m-\int_{\Gamma_1}h(x)m(x)y^2(t)\,d\Gamma \\ &\quad+\frac{\|h\|_\infty\|f\|_\infty}{\eta f_0^2}\int_{\Gamma_1}h(x)f(x)y_t^2(t)\,d\Gamma \\ &\quad+\frac{\mu_1}{4\eta}\int_{\Gamma_1}k_1^2\big(u_t(t)\big)\,d\Gamma+\frac{|\mu_2|}{4\eta}\int_{\Gamma_1}k_2^2\big(z(x,1,t)\big)\,d\Gamma.\end{aligned} \tag{4.9}$$

Proof Taking the derivatives of $\Psi(t)$ defined in (4.2) and using (2.15)-(2.18) we have

$$\begin{aligned}\frac{d}{dt}\Psi(t) &= \int_\Omega u(t)u_{tt}(t)\,dx+\int_{\Gamma_1}u(t)u_{tt}(t)\,d\Gamma+\int_\Omega\left|u_t(t)\right|^2dx+\int_{\Gamma_1}\left|u_t(t)\right|^2d\Gamma \\ &\quad+\int_{\Gamma_1}h(x)u_t(t)y(t)\,d\Gamma+\int_{\Gamma_1}h(x)u(t)y_t(t)\,d\Gamma+\int_{\Gamma_1}h(x)f(x)y(t)y_t(t)\,d\Gamma\end{aligned}$$

$$
\begin{aligned}
&= -\delta_0 \int_\Omega \left|\nabla u(t)\right|^2 dx + \int_\Omega \nabla u(t) \int_0^t g(t-s)\big(a(x)\nabla u(s)\big)\, ds\, dx \\
&\quad - \delta_1 \int_\Omega u(t)u_t(t)\, dx - \int_\Omega b(x)\left|u_t(t)\right|^{m-2} u_t(t)u(t)\, dx \\
&\quad + \int_\Omega \left|u(t)\right|^p dx + \int_\Omega \left|u_t(t)\right|^2 dx + \int_{\Gamma_1} \left|u_t(t)\right|^2 d\Gamma \\
&\quad + 2\int_{\Gamma_1} h(x)u(t)y_t(t)\, d\Gamma - \int_{\Gamma_1} h(x)m(x)y^2(t)\, d\Gamma \\
&\quad - \mu_1 \int_{\Gamma_1} k_1\big(u_t(t)\big)u(t)\, d\Gamma - \mu_2 \int_{\Gamma_1} k_2\big(z(x,1,t)\big)u(t)\, d\Gamma.
\end{aligned} \tag{4.10}
$$

Now, by using Hölder's and Young's inequality, $(H1)$, (2.1) and (2.2) we estimate the right hand side of (2.10) as follows, for any $\eta > 0$:

$$
\begin{aligned}
&\left| \int_\Omega \nabla u(t) \int_0^t g(t-s)\big(a(x)\nabla u(s)\big)\, ds\, dx \right| \\
&\quad \le \left[\frac{\delta_0}{2} + \frac{1}{2\delta_0}(1+\eta)(\delta_0 - l)^2 \right] \left\|\nabla u(t)\right\|^2 \\
&\qquad + \frac{1}{2\delta_0}\left(1 + \frac{1}{\eta}\right)(\delta_0 - l)(g \circ \nabla u)(t),
\end{aligned} \tag{4.11}
$$

$$
\left| -\delta_1 \int_\Omega u(t)u_t(t)\, dx \right| \le \delta_1 \eta C_*^2 \left\|\nabla u(t)\right\|^2 + \frac{\delta_1}{4\eta}\left\|u_t(t)\right\|^2, \tag{4.12}
$$

$$
\begin{aligned}
&\left| -\int_\Omega b(x)\left|u_t(t)\right|^{m-2} u_t(t)u(t)\, dx \right| \\
&\quad \le \frac{\|b\|_\infty \eta^{-m}}{m} C_*^m \left(\frac{2p}{l(p-2)}E(0) \right)^{\frac{m-2}{2}} \left\|u(t)\right\|_m^m + \|b\|_\infty \frac{m-1}{m} \eta^{m/m-1} \left\|u_t(t)\right\|_m^m,
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
2\left| \int_{\Gamma_1} h(x)u(t)y_t(t)\, d\Gamma \right| &= 2\left| \int_{\Gamma_1} \frac{h(x)f(x)u(t)y_t(t)}{f(x)}\, d\Gamma \right| \\
&\le \eta \tilde{C}_*^2 \left\|\nabla u(t)\right\|^2 + \frac{\|h\|_\infty \|f\|_\infty}{\eta f_0^2} \int_{\Gamma_1} h(x)f(x)y_t^2(t)\, d\Gamma,
\end{aligned} \tag{4.14}
$$

$$
\left| \mu_1 \int_{\Gamma_1} k_1\big(u_t(x,t)\big)u(t)\, d\Gamma \right| \le \mu_1 \eta \tilde{C}_*^2 \left\|\nabla u(t)\right\|^2 + \frac{\mu_1}{4\eta} \int_{\Gamma_1} k_1^2\big(u_t(t)\big)\, d\Gamma, \tag{4.15}
$$

and

$$
\left| \mu_2 \int_{\Gamma_1} k_2\big(z(x,1,t)\big)u(t)\, d\Gamma \right| \le |\mu_2| \eta \tilde{C}_*^2 \left\|\nabla u(t)\right\|^2 + \frac{|\mu_2|}{4\eta} \int_{\Gamma_1} k_2^2\big(z(x,1,t)\big)\, d\Gamma. \tag{4.16}
$$

Thus from (4.10)-(4.16) we conclude that

$$
\begin{aligned}
\frac{d}{dt}\Psi(t) &\le \left(1 + \frac{\delta_1}{4\eta}\right)\left\|u_t(t)\right\|^2 \\
&\quad - \left[\frac{\delta_0}{2} - \delta_1 C_*^2 \eta - \eta \tilde{C}_*^2\big(1 + \mu_1 + |\mu_2|\big) - \frac{1}{2\delta_0}(1+\eta)(\delta_0 - l)^2 \right. \\
&\quad \left. - \frac{\|b\|_\infty \eta^{-m} C_*^m}{m} \left(\frac{2p}{l(p-2)}E(0) \right)^{\frac{m-2}{2}} \right] \left\|\nabla u(t)\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2\delta_0}\left(1+\frac{1}{\eta}\right)(\delta_0-l)(g\circ\nabla u)(t)+\left\|u(t)\right\|_p^p+\left\|u_t(t)\right\|_\Gamma^2\\
&+\frac{\|b\|_\infty(m-1)\eta^{m/m-1}}{m}\left\|u_t(t)\right\|_m^m\\
&-\int_{\Gamma_1}h(x)m(x)y^2(t)\,d\Gamma+\frac{\|h\|_\infty\|f\|_\infty}{\eta f_0^2}\int_{\Gamma_1}h(x)f(x)y_t^2(t)\,d\Gamma\\
&+\frac{\mu_1}{4\eta}\int_{\Gamma_1}k_1^2\big(u_t(t)\big)\,d\Gamma+\frac{|\mu_2|}{4\eta}\int_{\Gamma_1}k_2^2\big(z(x,1,t)\big)\,d\Gamma.
\end{aligned}
$$

Thus we finished the proof of Lemma 4.2. □

Lemma 4.3 *Let* (u,y,z) *be the solution of problem* (2.15)-(2.23). *Then the functional* $\Phi(t)$ *defined in* (4.3) *satisfies*

$$
\begin{aligned}
\frac{d}{dt}\Phi(t)\le{}&-\left(a_0\int_0^t g(s)\,ds-\delta_1\eta-\eta\right)\left\|u_t(t)\right\|^2\\
&+\eta\left[\delta_0^2+\delta_0^2\alpha_1^2+2\alpha_1^2(\delta_0-l)^2+2(\delta_0-l)^2+C_*^{2(p-1)}\left(\frac{2pE(0)}{l(p-2)}\right)^{p-2}\right]\left\|\nabla u(t)\right\|^2\\
&+\left[\frac{\delta_0-l}{4\eta}\big(1+2C_*^2+8\eta+\|a\|_\infty+2\|a\|_\infty C_*^2\right.\\
&+\mu_1\|a\|_\infty\tilde{C}_*^2+|\mu_2|\|a\|_\infty\tilde{C}_*^2+\|a\|_\infty\tilde{C}_*^2\big)\\
&\left.+\frac{2\eta^m\|b\|_\infty}{m}(\delta_0-l)^{m-1}C_*^m\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{m-2}{2}}\right](g\circ\nabla u)(t)\\
&-\frac{g(0)\|a\|_\infty}{4\eta}\big(C_*^2+\tilde{C}_*^2\big)\big(g'\circ\nabla u\big)(t)-\left(a_0\int_0^t g(s)\,ds-\eta\right)\left\|u_t(t)\right\|_{\Gamma_1}^2\\
&+\mu_1\eta\int_{\Gamma_1}k_1^2\big(u_t(t)\big)\,d\Gamma+|\mu_2|\eta\int_{\Gamma_1}k_2^2\big(z(x,1,t)\big)\,d\Gamma\\
&+\frac{\eta\|h\|_\infty}{f_0}\int_{\Gamma_1}h(x)f(x)y_t^2(t)\,d\Gamma.
\end{aligned}
\tag{4.17}
$$

Proof Taking the derivative of $\Phi(t)$ defined in (4.3) and using (2.15)-(2.18), we have

$$
\begin{aligned}
\frac{d}{dt}\Phi(t)={}&-\int_\Omega a(x)u_{tt}(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx\\
&-\int_\Omega a(x)u_t(t)\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx\\
&-\left(\int_0^t g(s)\,ds\right)\int_\Omega a(x)u_t^2(t)\,dx\\
&-\int_{\Gamma_1} a(x)u_{tt}(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx\\
&-\int_{\Gamma_1} a(x)u_t(t)\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx\\
&-\left(\int_0^t g(s)\,ds\right)\int_{\Gamma_1} a(x)u_t^2(t)\,d\Gamma
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
&= \int_\Omega \delta_0 a(x)\nabla u(t)\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx \\
&\quad + \int_\Omega \delta_0 \nabla u(t)\cdot\nabla a(x)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad - \int_\Omega \left(\int_0^t g(t-s)a(x)\nabla u(s)\cdot\nabla a(x)\,ds\right)\left(\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)dx \\
&\quad - \int_\Omega a(x)\left(\int_0^t g(t-s)a(x)\nabla u(s)\,ds\right)\left(\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right)dx \\
&\quad + \int_\Omega a(x)\big(\delta_1 + b(x)|u_t(t)|^{m-2}\big)u_t(t)\left(\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)dx \\
&\quad - \int_\Omega a(x)|u(t)|^{p-2}u(t)\left(\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)dx \\
&\quad - \int_\Omega a(x)u_t(t)\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad - \left(\int_0^t g(s)\,ds\right)\int_\Omega a(x)u_t^2(t)\,dx \\
&\quad + \int_{\Gamma_1} a(x)\big[\mu_1 k_1\big(u_t(t)\big) + \mu_2 k_2\big(z(x,1,t)\big) - h(x)y_t(t)\big] \\
&\quad \times \left(\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)d\Gamma \\
&\quad - \int_{\Gamma_1} a(x)u_t(t)\left(\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\right) \\
&\quad - \left(\int_0^t g(s)\,ds\right)\int_{\Gamma_1} a(x)u_t^2(t)\,d\Gamma. \qquad (4.19)
\end{aligned}
$$

Similarly to (4.9), we estimate each terms in the right hand side of (4.19). Using Hölder's and Young's inequality, $(H1)$, $(H2)$, (2.1), (2.2), (2.3), (2.5), (2.6) and (3.19), for any $\eta > 0$, we have

$$
\begin{aligned}
&\left|\int_\Omega \delta_0 a(x)\nabla u(t)\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx\right| \\
&\quad \le \delta_0^2\eta\|\nabla u(t)\|^2 + \frac{1}{4\eta}\int_\Omega\left(a(x)\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right)^2dx \\
&\quad \le \delta_0^2\eta\|\nabla u(t)\|^2 + \frac{\delta_0 - l}{4\eta}(g\circ\nabla u)(t), \qquad (4.20) \\
&\left|\int_\Omega \delta_0\nabla u(t)\cdot\nabla a(x)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx\right| \\
&\quad \le \delta_0\alpha_1\int_\Omega|\nabla u(t)|\sqrt{a(x)}\left(\int_0^t g(s)\,ds\right)^{1/2}\left(\int_0^t g(t-s)\big(u(t)-u(s)\big)^2\,ds\right)^{1/2}dx \\
&\quad \le \delta_0^2\alpha_1^2\eta\|\nabla u(t)\|^2 + \frac{(\delta_0 - l)C_*^2}{4\eta}(g\circ\nabla u)(t), \qquad (4.21) \\
&\left|-\int_\Omega\left(\int_0^t g(t-s)a(x)\nabla u(s)\cdot\nabla a(x)\,ds\right)\left(\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)dx\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha_1^2\eta\int_\Omega a^2(x)\left(\int_0^t g(t-s)\left|\nabla u(s)\right|ds\right)^2 dx \\
&\quad + \frac{1}{4\eta}\int_\Omega a(x)\left(\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)^2 dx \\
&\leq 2\alpha_1^2\eta(\delta_0-l)^2\left\|\nabla u(t)\right\|^2 + \left(2\alpha_1^2\eta(\delta_0-l)+\frac{(\delta_0-l)C_*^2}{4\eta}\right)(g\circ\nabla u)(t), \qquad (4.22)
\end{aligned}
$$

$$
\begin{aligned}
&\left|-\int_\Omega a(x)\left(\int_0^t g(t-s)a(x)\nabla u(s)\,ds\right)\left(\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right)dx\right| \\
&\leq \eta\int_\Omega a^2(x)\left(\int_0^t g(t-s)\left|\nabla u(s)\right|ds\right)^2 dx \\
&\quad + \frac{1}{4\eta}\int_\Omega a^2(x)\left(\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right)^2 dx \\
&\leq 2\eta(\delta_0-l)^2\left\|\nabla u(t)\right\|^2 + \left(2\eta+\frac{\|a\|_\infty}{4\eta}\right)(\delta_0-l)(g\circ\nabla u)(t), \qquad (4.23)
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int_\Omega a(x)\big(\delta_1+b(x)\left|u_t(t)\right|^{m-2}\big)u_t(t)\left(\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)dx\right| \\
&\leq \delta_1\left|\int_\Omega a(x)u_t(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx\right| \\
&\quad + \left|\int_\Omega a(x)b(x)\left|u_t(t)\right|^{m-1}\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx\right| \\
&\leq \delta_1\eta\left\|u_t(t)\right\|^2 + \frac{(\delta_0-l)\|a\|_\infty C_*^2}{4\eta}(g\circ\nabla u)(t) + \frac{m-1}{m}\|b\|_\infty\eta^{-\frac{m}{m-1}}\left\|u_t(t)\right\|_m^m \\
&\quad + \frac{\eta^m}{m}\|b\|_\infty(\delta_0-l)^{m-1}C_*^m\int_\Omega\int_0^t g(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2\left|\nabla u(t)-\nabla u(s)\right|^{m-2}\,ds\,dx \\
&= \delta_1\eta\left\|u_t(t)\right\|^2 + \frac{m-1}{m}\|b\|_\infty\eta^{-\frac{m}{m-1}}\left\|u_t(t)\right\|_m^m + \left[\frac{(\delta_0-l)\|a\|_\infty C_*^2}{4\eta}\right. \\
&\quad \left. + \frac{2\eta^m}{m}\|b\|_\infty(\delta_0-l)^{m-1}C_*^m\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{m-2}{2}}\right](g\circ\nabla u)(t), \qquad (4.24)
\end{aligned}
$$

$$
\begin{aligned}
&\left|-\int_\Omega a(x)\left|u(t)\right|^{p-2}u(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx\right| \\
&\leq \eta\left\|u(t)\right\|_{2(p-1)}^{2(p-1)} + \frac{(\delta_0-l)\|a\|_\infty C_*^2}{4\eta}(g\circ\nabla u)(t) \\
&\leq \eta C_*^{2(p-1)}\left(\frac{2pE(0)}{l(p-2)}\right)^{p-2}\left\|\nabla u(t)\right\|^2 + \frac{(\delta_0-l)\|a\|_\infty C_*^2}{4\eta}(g\circ\nabla u)(t), \qquad (4.25)
\end{aligned}
$$

$$
\begin{aligned}
&\left|-\int_\Omega a(x)u_t(t)\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx\right| \\
&\leq \eta\left\|u_t(t)\right\|^2 - \frac{g(0)\|a\|_\infty C_*^2}{4\eta}(g'\circ\nabla u)(t), \qquad (4.26)
\end{aligned}
$$

$$
\begin{aligned}
&\left|\mu_1\int_{\Gamma_1} a(x)k_1\big(u_t(t)\big)\left(\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)dx\right| \\
&\leq \mu_1\eta\int_{\Gamma_1} k_1^2\big(u_t(t)\big)\,d\Gamma + \frac{\mu_1(\delta_0-l)\|a\|_\infty\tilde{C}_*^2}{4\eta}(g\circ\nabla u)(t), \qquad (4.27)
\end{aligned}
$$

$$\begin{aligned}&\left|\mu_2\int_{\Gamma_1} a(x)k_2\big(z(x,1,t)\big)\left(\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)dx\right|\\&\quad\le|\mu_2|\eta\int_{\Gamma_1} k_2^2\big(z(x,1,t)\big)\,d\Gamma+\frac{|\mu_2|(\delta_0-l)\|a\|_\infty\tilde{C}_*^2}{4\eta}(g\circ\nabla u)(t),\end{aligned}\tag{4.28}$$

$$\begin{aligned}&\left|-\int_{\Gamma_1} a(x)h(x)y_t(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx\right|\\&\quad\le\frac{\eta\|h\|_\infty}{f_0}\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma+\frac{(\delta_0-l)\|a\|_\infty\tilde{C}_*^2}{4\eta}(g\circ\nabla u)(t),\end{aligned}\tag{4.29}$$

$$\begin{aligned}&\left|-\int_{\Gamma_1} a(x)u_t(t)\left(\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\right)d\Gamma\right|\\&\quad\le\eta\big\|u_t(t)\big\|_{\Gamma_1}^2-\frac{g(0)\|a\|_\infty\tilde{C}_*^2}{4\eta}\big(g'\circ\nabla u\big)(t).\end{aligned}\tag{4.30}$$

Combining the estimates (4.20)-(4.30), then (4.19) becomes

$$\begin{aligned}\frac{d}{dt}\Phi(t)\le&-\left(a_0\int_0^t g(s)\,ds-\delta_1\eta-\eta\right)\big\|u_t(t)\big\|^2\\&+\eta\left[\delta_0^2+\delta_0^2\alpha_1^2+2\alpha_1^2(\delta_0-l)^2+2(\delta_0-l)^2+C_*^{2(p-1)}\left(\frac{2pE(0)}{l(p-2)}\right)^{p-2}\right]\big\|\nabla u(t)\big\|^2\\&+\left[\frac{\delta_0-l}{4\eta}\big(1+2C_*^2+8\eta+\|a\|_\infty+2\|a\|_\infty C_*^2\right.\\&+\mu_1\|a\|_\infty\tilde{C}_*^2+|\mu_2|\|a\|_\infty\tilde{C}_*^2+\|a\|_\infty\tilde{C}_*^2\big)\\&\left.+\frac{2\eta^m\|b\|_\infty}{m}(\delta_0-l)^{m-1}C_*^m\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{m-2}{2}}\right](g\circ\nabla u)(t)\\&-\frac{g(0)\|a\|_\infty}{4\eta}\big(C_*^2+\tilde{C}_*^2\big)\big(g'\circ\nabla u\big)(t)-\left(a_0\int_0^t g(s)\,ds-\eta\right)\big\|u_t(t)\big\|_{\Gamma_1}^2\\&+\mu_1\eta\int_{\Gamma_1} k_1^2\big(u_t(t)\big)\,d\Gamma+|\mu_2|\eta\int_{\Gamma_1} k_2^2\big(z(x,1,t)\big)\,d\Gamma\\&+\frac{\eta\|h\|_\infty}{f_0}\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma.\end{aligned}$$

□

Lemma 4.4 *Let* (u,y,z) *be the solution of problem* (2.15)-(2.23). *Then the functional* $\Lambda(t)$ *defined in* (4.4) *satisfies*

$$\begin{aligned}\frac{d}{dt}\Lambda(t)\le&-\rho\Lambda(t)+C_5\int_{\Gamma_1} k_1\big(u_t(t)\big)u_t(t)\,d\Gamma\\&-C_4(1-d)e^{-\tau(t)}\int_{\Gamma_1} k_2\big(z(x,1,t)\big)z(x,1,t)\,d\Gamma.\end{aligned}\tag{4.31}$$

Proof Multiplying (2.19) by $e^{-\rho\tau(t)}k_2(z(x,\rho,t))$ and integrating over $\Gamma_1\times(0,1)$, we obtain

$$\begin{aligned}&\tau(t)\int_{\Gamma_1} e^{-\rho\tau(t)}\int_0^1 z_t(x,\rho,t)k_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma\\&\quad=-\int_{\Gamma_1}\int_0^1\big(1-\rho\tau'(t)\big)e^{-\rho\tau(t)}\frac{\partial}{\partial\rho}K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma.\end{aligned}\tag{4.32}$$

Differentiating (4.4) with respect to t and using (4.32), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\tau(t)\int_{\Gamma_1}\int_0^1 e^{-\rho\tau(t)}K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma\right)\\
&\quad=\tau'(t)\big(1-\tau(t)\big)\int_{\Gamma_1}\int_0^1 e^{-\rho\tau(t)}\frac{\partial}{\partial\rho}K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma\\
&\qquad-\int_{\Gamma_1}\int_0^1\big(1-\rho\tau'(t)\big)e^{-\rho\tau(t)}\frac{\partial}{\partial\rho}K_2\big(z(x,\rho,t)\big)\,d\rho\,d\Gamma. \qquad (4.33)
\end{aligned}$$

Then, by integration by parts and using (2.12), (4.33) lead to

$$\begin{aligned}
\frac{d}{dt}\Lambda(t)&\le-\big(1-\rho\tau'(t)+\tau'(t)\big)\Lambda(t)+\int_{\Gamma_1}\big[K_2\big(z(x,0,t)\big)-e^{-\tau(t)}K_2\big(z(x,1,t)\big)\big]\,d\Gamma\\
&\quad+\tau'(t)e^{-\tau(t)}\int_{\Gamma_1}K_2\big(z(x,1,t)\big)\,d\Gamma\\
&=-\rho\Lambda(t)+\int_{\Gamma_1}K_2\big(z(x,0,t)\big)\,d\Gamma+(d-1)e^{-\tau(t)}\int_{\Gamma_1}K_2\big(z(x,1,t)\big)\,d\Gamma.
\end{aligned}$$

Using (2.10), then (4.31) holds. □

Now we are in a position to state our main result.

Theorem 4.1 *Assume that* $(H1)$*-*$(H5)$ *and* (3.5) *hold. Then, for each* $t_0 > 0$, *there exist positive constants* θ, θ_1, θ_2 *and* ε_0 *such that the solution energy of* (2.15)-(2.23) *satisfies*

$$E(t)\le\theta O^{-1}\left(\theta_1\int_{t_0}^t\zeta(s)\,ds+\theta_2\right)\quad\textit{for } t\ge t_0,$$

where

$$O(t)=\int_t^1\frac{1}{O_1(s)}\,ds$$

and

$$O_1(t)=\begin{cases} t & \textit{if } K_1 \textit{ is linear on } [0,\varepsilon_1],\\ tK_1'(\varepsilon_0 t) & \textit{if } K_1'(0)=0 \textit{ and } K_1''(t)>0 \textit{ on } (0,\varepsilon_1].\end{cases}$$

Proof Since the function $g(t)$ is positive, there exists $t_0 > 0$ such that

$$\int_0^t g(s)\,ds\ge\int_0^{t_0}g(s)\,ds:=g_0\quad\text{for } t\ge t_0.$$

Using (3.6), (4.9), (4.17) and (4.31), we arrive at

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t)&\le-M\left(\mu_1-|\mu_2|C_5-\frac{\xi C_5}{2}\right)\int_{\Gamma_1}k_1\big(u_t(t)\big)u_t(t)\,d\Gamma\\
&\quad-M\left[\frac{\xi C_4}{2}\big(1-\tau'(t)\big)-|\mu_2|(1-C_4)\right]\int_{\Gamma_1}k_2\big(z(x,1,t)\big)z(x,1,t)\,d\Gamma
\end{aligned}$$

$$
\begin{aligned}
&- M\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma + \frac{M}{2}\big(g'\circ\nabla u\big)(t) - \frac{a_0M}{2}g(t)\big\|\nabla u(t)\big\|^2 \\
&- \delta_1 M\big\|u_t(t)\big\|^2 - b_0M\big\|u_t(t)\big\|_m^m + \varepsilon\left(1+\frac{\delta_1}{4\eta}\right)\big\|u_t(t)\big\|^2 \\
&- \varepsilon\left[\frac{\delta_0}{2} - \delta_1 C_*^2\eta - \eta\tilde{C}_*^2\big(1+\mu_1+|\mu_2|\big) - \frac{1}{2\delta_0}(1+\eta)(\delta_0-l)^2\right. \\
&\left.- \frac{\|b\|_\infty\eta^{-m}C_*^m}{m}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{m-2}{2}}\right]\big\|\nabla u(t)\big\|^2 \\
&+ \frac{\varepsilon}{2\delta_0}\left(1+\frac{1}{\eta}\right)(\delta_0-l)(g\circ\nabla u)(t) + \varepsilon\big\|u(t)\big\|_p^p + \varepsilon\big\|u_t(t)\big\|_{\Gamma_1}^2 \\
&+ \frac{\varepsilon\|b\|_\infty(m-1)}{m}\eta^{\frac{m}{m-1}}\big\|u_t(t)\big\|_m^m - \varepsilon\int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma \\
&+ \frac{\varepsilon\|h\|_\infty\|f\|_\infty}{\eta f_0^2}\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma + \frac{\mu_1\varepsilon}{4\eta}\int_{\Gamma_1} k_1^2\big(u_t(t)\big)\,d\Gamma \\
&+ \frac{|\mu_2|\varepsilon}{4\eta}\int_{\Gamma_1} k_2^2\big(z(x,1,t)\big)\,d\Gamma - (a_0g_0 - \delta_1\eta - \eta)\big\|u_t(t)\big\|^2 \\
&+ \eta\left[\delta_0^2 + \delta_0^2\alpha_1^2 + 2\alpha_1^2(\delta_0-l)^2 + 2(\delta_0-l)^2 + C_*^{2(p-1)}\left(\frac{2pE(0)}{l(p-2)}\right)^{p-2}\right]\big\|\nabla u(t)\big\|^2 \\
&+ \left[\frac{(\delta_0-l)}{4\eta}\big(1 + 2C_*^2 + 8\eta + \|a\|_\infty + 2\|a\|_\infty C_*^2\right. \\
&+ \mu_1\|a\|_\infty\tilde{C}_*^2 + |\mu_2|\|a\|_\infty\tilde{C}_*^2 + \|a\|_\infty\tilde{C}_*^2\big) \\
&\left.+ \frac{2\eta^m\|b\|_\infty}{m}(\delta_0-l)^{m-1}C_*^m\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{m-2}{2}}\right](g\circ\nabla u)(t) \\
&- \eta C_*^{2(p-2)}\big(C_*^2+\tilde{C}_*^2\big)\big(g'\circ\nabla u\big)(t) - (a_0g_0-\eta)\big\|u_t(t)\big\|_{\Gamma_1}^2 \\
&+ \mu_1\eta\int_{\Gamma_1} k_1^2\big(u_t(t)\big)\,d\Gamma + |\mu_2|\eta\int_{\Gamma_1} k_2^2\big(z(x,1,t)\big)\,d\Gamma \\
&+ \frac{\eta\|h\|_\infty}{f_0}\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma - \varepsilon\rho\Lambda(t) + \varepsilon C_5\int_{\Gamma_1} k_1\big(u_t(t)\big)u_t(t)\,d\Gamma \\
&- \varepsilon C_4(1-d)e^{-\tau(t)}\int_{\Gamma_1} k_2\big(z(x,1,t)\big)z(x,1,t)\,d\Gamma \\
= {}&-\left[\delta_1M - \varepsilon\left(1+\frac{\delta_1}{4\eta}\right) + a_0g_0 - \eta(\delta_1+1)\right]\big\|u_t(t)\big\|^2 \\
&- \left[\frac{M}{2} + \eta C_*^{2(p-2)}\big(C_*^2+\tilde{C}_*^2\big)\right]\big(g'\circ\nabla u\big)(t) + \varepsilon\big\|u(t)\big\|_p^p \\
&- \left(M - \frac{\varepsilon\|h\|_\infty\|f\|_\infty}{\eta f_0^2} - \frac{\eta\|h\|_\infty}{f_0}\right)\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma \\
&- \varepsilon\int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma \\
&- \left\{\frac{a_0M}{2}g(t) + \varepsilon\left[\frac{\delta_0}{2} - \delta_1C_*^2\eta - \eta\tilde{C}_*^2\big(1+\mu_1+|\mu_2|\big) - \frac{(1+\eta)(\delta_0-l)^2}{2\delta_0}\right.\right. \\
&\left.- \frac{\|b\|_\infty\eta^{-m}C_*^m}{m}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{m-2}{2}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \eta\left[\delta_0^2 + \delta_0^2\alpha_1^2 + 2\alpha_1^2(\delta_0 - l)^2 + 2(\delta_0 - l)^2 + C_*^{2(p-1)}\left(\frac{2pE(0)}{l(p-2)}\right)^{p-2}\right]\biggr\}\bigl\|\nabla u(t)\bigr\|^2 \\
&+ \left[\frac{\varepsilon}{2\delta_0}\left(1 + \frac{1}{\eta}\right)(\delta_0 - l) + \frac{(\delta_0 - l)}{4\eta}\bigl(1 + 2C_*^2 + 8\eta + \|a\|_\infty + 2\|a\|_\infty C_*^2 \right. \\
&+ \mu_1\|a\|_\infty\tilde{C}_*^2 + |\mu_2|\|a\|_\infty\tilde{C}_*^2 + \|a\|_\infty\tilde{C}_*^2\bigr) \\
&\left. + \frac{2\eta^m\|b\|_\infty}{m}(\delta_0 - l)^{m-1}C_*^m\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{m-2}{2}}\right](g \circ \nabla u)(t) \\
&- \left(b_0M - \frac{\varepsilon\|b\|_\infty(m-1)}{m}\eta^{\frac{m}{m-1}}\right)\bigl\|u_t(t)\bigr\|_m^m - (a_0g_0 - \eta - \varepsilon)\bigl\|u_t(t)\bigr\|_{\Gamma_1}^2 - \varepsilon\rho\Lambda(t) \\
&- \left[M\left(\mu_1 - |\mu_2|C_5 - \frac{\xi C_5}{2}\right) - \varepsilon C_5\right]\int_{\Gamma_1} k_1\bigl(u_t(t)\bigr)u_t(t)\,d\Gamma \\
&+ \mu_1\left(\frac{\varepsilon}{4\eta} + \eta\right)\int_{\Gamma_1} k_1^2\bigl(u_t(t)\bigr)\,d\Gamma \\
&- \left\{M\left[\frac{\xi C_4}{2}\bigl(1 - \tau'(t)\bigr) - |\mu_2|(1 - C_4)\right] + \varepsilon C_4(1-d)e^{-\tau(t)}\right\} \\
&\times \int_{\Gamma_1} k_2\bigl(z(x,1,t)\bigr)z(x,1,t)\,d\Gamma \\
&+ |\mu_2|\left(\frac{\varepsilon}{4\eta} + \eta\right)\int_{\Gamma_1} k_2^2\bigl(z(x,1,t)\bigr)\,d\Gamma.
\end{aligned}
$$

At this point, we choose $\varepsilon > 0$ small enough and we pick $\eta > 0$ sufficiently small and M is so large such that

$$
\begin{aligned}
&M_1 = \delta_1M - \varepsilon\left(1 + \frac{\delta_1}{4\eta}\right) + a_0g_0 - \eta(\delta_1 + 1) > 0, \\
&M_2 = \frac{M}{2} + \eta C_*^{2(p-2)}\bigl(C_*^2 + \tilde{C}_*^2\bigr) > 0, \\
&M_3 = M - \frac{\varepsilon\|h\|_\infty\|f\|_\infty}{\eta f_0^2} - \frac{\eta\|h\|_\infty}{f_0} > 0, \\
&M_4 = \frac{a_0M}{2}g(t) \\
&\qquad + \varepsilon\left[\frac{\delta_0}{2} - \delta_1C_*^2\eta - \eta\tilde{C}_*^2\bigl(1 + \mu_1 + |\mu_2|\bigr) - \frac{(1+\eta)(\delta_0 - l)^2}{2\delta_0}\right. \\
&\qquad \left. + \frac{\|b\|_\infty\eta^{-m}C_*^m}{m}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{m-2}{2}}\right] \\
&\qquad - \eta\left[\delta_0^2 + \delta_0^2\alpha_1^2 + 2\alpha_1^2(\delta_0 - l)^2 + 2(\delta_0 - l)^2 + C_*^{2(p-1)}\left(\frac{2pE(0)}{l(p-2)}\right)^{p-2}\right] \\
&\quad > 0, \\
&M_5 = b_0M - \frac{\varepsilon\|b\|_\infty(m-1)}{m}\eta^{\frac{m}{m-1}} > 0, \\
&M_6 = a_0g_0 - \eta - \varepsilon > 0, \\
&M_7 = M\left(\mu_1 - |\mu_2|C_5 - \frac{\xi C_5}{2}\right) - \varepsilon C_5 > 0.
\end{aligned}
$$

Therefore, for all $t \geq t_0$, we deduce

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leq{}& -M_1\|u_t(t)\|^2 - M_2(g\circ\nabla u)(t) \\
& - M_3\int_{\Gamma_1} h(x)f(x)y_t^2(t)\,d\Gamma - \varepsilon\int_{\Gamma_1} h(x)m(x)y^2(t)\,d\Gamma \\
& - M_4\|\nabla u(t)\|^2 - M_5\|u_t(t)\|_m^m - M_6\|u_t(t)\|_{\Gamma_1}^2 - \varepsilon\rho\Lambda(t) \\
& - M_7\int_{\Gamma_1} k_1\big(u_t(t)\big)u_t(t)\,d\Gamma + \mu_1\left(\frac{\varepsilon}{4\eta}+\eta\right)\int_{\Gamma_1} k_1^2\big(u_t(t)\big)\,d\Gamma \\
& - M_8\int_{\Gamma_1} k_2\big(z(x,1,t)\big)z(x,1,t)\,d\Gamma + |\mu_2|\left(\frac{\varepsilon}{4\eta}+\eta\right)\int_{\Gamma_1} k_2^2\big(z(x,1,t)\big)\,d\Gamma \\
& + M_9(g\circ\nabla u)(t),
\end{aligned}$$

where

$$\begin{aligned}
M_8 ={}& M\left[\frac{\xi C_4}{2}\big(1-\tau'(t)\big) - |\mu_2|(1-C_4)\right] + \varepsilon C_4(1-d)e^{-\tau(t)} > 0, \\
M_9 ={}& \frac{\varepsilon}{2\delta_0}\left(1+\frac{1}{\eta}\right)(\delta_0 - l) + \frac{(\delta_0 - l)}{4\eta}\big(1 + 2C_*^2 + 8\eta + \|a\|_\infty + 2\|a\|_\infty C_*^2 \\
& + \mu_1\|a\|_\infty\tilde{C}_*^2 + |\mu_2|\|a\|_\infty\tilde{C}_*^2 + \|a\|_\infty\tilde{C}_*^2\big) \\
& + \frac{2\eta^m\|b\|_\infty}{m}(\delta_0 - l)^{m-1}C_*^m\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{m-2}{2}} \\
& > 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leq{}& -M_{10}E(t) + M_{11}(g\circ\nabla u)(t) + \mu_1\left(\frac{\varepsilon}{4\eta}+\eta\right)\int_{\Gamma_1} k_1^2\big(u_t(t)\big)\,d\Gamma \\
& + |\mu_2|\left(\frac{\varepsilon}{4\eta}+\eta\right)\int_{\Gamma_1} k_2^2\big(z(x,1,t)\big)\,d\Gamma,
\end{aligned}$$

where M_{10} and M_{11} are some positive constants.

Multiplying the above inequality by $\zeta(t)$, we obtain, for any $t \geq t_0$,

$$\begin{aligned}
\zeta(t)\mathcal{L}'(t) \leq{}& -M_{10}\zeta(t)E(t) + M_{11}\zeta(t)(g\circ\nabla u)(t) + \mu_1\left(\frac{\varepsilon}{4\eta}+\eta\right)\zeta(t)\int_{\Gamma_1} k_1^2\big(u_t(t)\big)\,d\Gamma \\
& + |\mu_2|\left(\frac{\varepsilon}{4\eta}+\eta\right)\zeta(t)\int_{\Gamma_1} k_2^2\big(z(x,1,t)\big)\,d\Gamma.
\end{aligned}$$

From (2.9), we obtain

$$k_2^2(s) \leq csk_2(s) \quad \text{for all } s \in R,$$

where c is some positive constant. Recalling (2.4) and (3.6), we get for any $t \geq t_0$

$$\zeta(t)\mathcal{L}'(t) \leq -M_{10}\zeta(t)E(t) + M_{12}\zeta(t)\int_{\Gamma_1} k_1^2\big(u_t(t)\big)\,d\Gamma - M_{13}E'(t). \tag{4.34}$$

Now, we define

$$G(t) = \zeta(t)\mathcal{L}(t) + M_{13}E(t).$$

As ζ is a non-increasing positive function, by using Lemma 4.1, the function $G(t)$ is equivalent to $E(t)$. Using the fact that $\zeta'(t) \le 0$, (4.34) implies that

$$G'(t) \le -M_{10}\zeta(t)E(t) + M_{12}\zeta(t)\int_{\Gamma_1} k_1^2\big(u_t(t)\big)\,d\Gamma \quad \text{for } t \ge t_0. \tag{4.35}$$

In the following, we shall estimate the term $\int_{\Gamma_1} k_1^2(u_t(t))\,d\Gamma$ in (4.35). To do this, let $\Gamma_{11} = \{x \in \Gamma_1 : |u_t| > \varepsilon_1\}$, $\Gamma_{12} = \{x \in \Gamma_1 : |u_t| \le \varepsilon_1\}$.

Case (I): K_1 is linear on $[0, \varepsilon_1]$.

There exist positive constants C_1 and C_2 such that

$$C_1|s| \le \big|k_1(s)\big| \le C_2|s| \quad \text{for } s \in R.$$

This and (4.35) yield

$$G'(t) \le -M_{10}\zeta(t)E(t) - M_{13}E'(t) \quad \text{for } t \ge t_0,$$

where M_{13} is a positive constant. This gives

$$\big(G(t) + M_{13}E(t)\big)' \le -M_{10}\zeta(t)E(t) \quad \text{for } t \ge t_0.$$

Employing that G is equivalent to E, we get

$$E(t) \le Ce^{-\nu\int_{t_0}^t \zeta(s)\,ds} \quad \text{for } t \ge t_0,$$

where C and ν are positive constants. Owing to $K_1(s) = \sqrt{s}k_1(\sqrt{s}) = cs$,

$$E(t) \le Ce^{-\nu\int_{t_0}^t \zeta(s)\,ds} = K_1^{-1}\left(\int_{t_0}^t \zeta(s)\,ds\right) \quad \text{for } t \ge t_0.$$

Case (II): $K_1'(0) = 0$ and $K_1''(t) > 0$ on $[0, \varepsilon_1]$.

Since K_1 is convex and increasing, K_1^{-1} is concave and increasing. By (2.7), (2.8), (3.6) and the reversed Jensen inequality

$$\begin{aligned}
\zeta(t)\int_{\Gamma_1} k_1^2\big(u_t(t)\big)\,d\Gamma &= \zeta(t)\int_{\Gamma_{11}} k_1^2\big(u_t(t)\big)\,d\Gamma + \zeta(t)\int_{\Gamma_{12}} k_1^2\big(u_t(t)\big)\,d\Gamma \\
&\le C_2\zeta(0)\int_{\Gamma_{11}} u_t(t)k_1\big(u_t(t)\big)\,d\Gamma + \zeta(t)\int_{\Gamma_{12}} K_1^{-1}\big(u_tk_1(u_t)\big)\,d\Gamma \\
&\le -\mu_3E'(t) + \zeta(t)|\Gamma_{12}|K_1^{-1}\left(\frac{1}{|\Gamma_{12}|}\int_{\Gamma_{12}} u_tk_1(u_t)\,d\Gamma\right) \\
&\le -\mu_3E'(t) + \zeta(t)|\Gamma_{12}|K_1^{-1}\big(-\mu_4E'(t)\big),
\end{aligned}$$

where μ_3 and μ_4 are positive constants. Thus, we get from (4.35)

$$G_1'(t) \le -M_{10}\zeta(t)E(t) + |\Gamma_{12}|M_{12}\zeta(t)K_1^{-1}\bigl(-\mu_4 E'(t)\bigr),$$

where $G_1(t) = G(t) + \mu_3 M_{12}E(t)$. Now, for $0 < \varepsilon_0 < \varepsilon_1$ and $\mu > 0$, we define

$$H(t) = K_1'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)G_1(t) + \mu E(t), \quad \forall t \ge t_0. \tag{4.36}$$

It is easily noted that

$$\gamma_1 H(t) \le E(t) \le \gamma_2 H(t), \tag{4.37}$$

where γ_1, γ_2 are positive constant. Thanks to the similar argument in (3.13) and (3.12), we have

$$\begin{aligned}
H'(t) &= \varepsilon_0\frac{E'(t)}{E(0)}K_1''\left(\varepsilon_0\frac{E(t)}{E(0)}\right)G_1(t) + K_1'\left(\varepsilon_0\frac{E'(t)}{E(0)}\right)G_1'(t) + \mu E'(t)\\
&\le -M_{10}\zeta(t)E(t)K_1'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\\
&\quad + M_{12}|\Gamma_{12}|\zeta(t)K_1'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)K_1^{-1}\bigl(-\mu_4 E'(t)\bigr) + \mu E'(t)\\
&\le -M_{10}\zeta(t)E(t)K_1'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + \mu E'(t)\\
&\quad + M_{12}|\Gamma_{12}|\zeta(t)K_1^*\left\{K_1'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right\} - \mu_4|\Gamma_{12}|M_{12}\zeta(t)E'(t)\\
&\le -M_{10}\zeta(t)E(t)K_1'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + (\mu - M_{13})E'(t)\\
&\quad + M_{14}\zeta(t)K_1'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\varepsilon_0\frac{E(t)}{E(0)} - M_{14}\zeta(t)K_1\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\\
&\le -\bigl(M_{10}E(0) - \varepsilon_0 M_{14}\bigr)\zeta(t)\frac{E(t)}{E(0)}K_1'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + (\mu - M_{13})E'(t),
\end{aligned}$$

where $M_{13} = |\Gamma_{12}|M_{12}\mu_4\zeta(0)$ and $M_{14} = |\Gamma_{12}|M_{12}$. Taking ε_0 sufficiently small such that $M_{10}E(0) - \varepsilon_0 M_{14} > 0$ and $\mu > 0$ suitably so that $\mu - M_{13} > 0$, we arrive at

$$H'(t) \le -\mu_5\zeta(t)\frac{E(t)}{E(0)}K_1'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) = -\mu_5\zeta(t)O_1\left(\frac{E(t)}{E(0)}\right) \quad \text{for } t \ge t_0, \tag{4.38}$$

where $O_1(t) = tK_1'(\varepsilon_0 t)$ and μ_5 is a positive constant. Finally, we define

$$\mathcal{E}(t) = \gamma_1\frac{H(t)}{E(0)}.$$

Using (4.37), we see that $\mathcal{E}$ is equivalent to E. Therefore,

$$\mathcal{E}'(t) \le -\theta_1\zeta(t)O_1\bigl(\mathcal{E}(t)\bigr) \quad \text{for } t \ge t_0,$$

for some $\theta_1 > 0$, and

$$\mathcal{E}(t) \le O^{-1}\left(\theta_1 \int_{t_0}^{t} \zeta(s)\,ds + \theta_2\right) \quad \text{for } t \ge t_0.$$

Thus the proof is completed. □

5 Conclusion

We have investigated the energy decay rate of the nonlinear viscoelastic problem with dynamic and acoustic boundary conditions.

It is well known that viscoelastic materials have memory effects, which is due to the mechanical response influenced by the history of the materials themselves. As these materials have a wide application in the natural sciences, their dynamics is interesting and of great importance. Also, the dynamic boundary conditions are not only important from the theoretical point of view but also arise in numerous practical problems and the acoustic boundary conditions are related to noise control and suppression in practical applications. Moreover, time delay so often arises in many physical, chemical, biological and economical phenomena because these phenomena depend not only on the present state but also on the history of the system in a more complicated way. We established a decay rate estimate for the energy by introducing suitable Lyapunov functionals.

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Competing interests
The authors declare that they have no competing interests

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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