Teng et al. *Boundary Value Problems* (2017) 2017:179
DOI 10.1186/s13661-017-0910-x

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A natural boundary element method for the Sobolev equation in the 2D unbounded domain

Fei Teng[1], Zhendong Luo[2*] and Jing Yang[1]

*Correspondence: zhdluo@163.com
[2]School of Mathematics and Physics, North China Electric Power University, No. 2, Bei Nong Road, Beijing, Changping District 102206, China
Full list of author information is available at the end of the article

Abstract
In this article, we devote ourselves to establishing a natural boundary element (NBE) method for the Sobolev equation in the 2D unbounded domain. To this end, we first constitute the time semi-discretized super-convergence format for the Sobolev equation by means of the Newmark method. Then, using the principle of natural boundary reduction, we establish a fully discretized NBE format based on the natural integral equation and the Poisson integral formula of this problem and analyze the errors between the exact solution and the fully discretized NBE solutions. Finally, we use some numerical experiments to verify that the NBE method is effective and feasible for solving the Sobolev equation in the 2D unbounded domain.

MSC: 65N30; 35Q10

Keywords: natural boundary element method; Sobolev equation; error estimate; numerical experiment

1 Introduction

Let $\Theta \subset R^2$ be a connected and bounded region with smooth boundary $\Gamma := \partial\Theta$, $\Theta^c := R^2 \backslash \overline{\Theta}$, $\boldsymbol{x} = (x_1, x_2)$, $|\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2}$. For given time upper bound T, we consider the following Sobolev equation in the 2D unbounded domain:

Problem I Find ϖ that satisfies

$$\begin{cases} \varpi_t - \varepsilon\Delta\varpi_t - \gamma\Delta\varpi = f(\boldsymbol{x},t), & (\boldsymbol{x},t) \in \Theta^c \times (0,T), \\ \frac{\partial\varpi}{\partial\boldsymbol{n}} = g(\boldsymbol{x},t), & (\boldsymbol{x},t) \in \Gamma \times (0,T), \\ \varpi(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Theta^c, \end{cases} \tag{1}$$

where $\varpi_t = \frac{\partial\varpi}{\partial t}$ represents the partial derivative for the unknown function $\varpi(\boldsymbol{x},t)$ about time, ε and γ are positive constants, $f(\boldsymbol{x},t)$, $g(\boldsymbol{x},t)$, and $u_0(\boldsymbol{x})$ are three known functions satisfying the appropriate conditions, $\frac{\partial}{\partial\boldsymbol{n}}$ denotes the external normal derivative operator, and $\boldsymbol{n}$ represents a normal vector onto boundary Γ of domain Θ^c toward the interior of domain Θ. In addition, we also assume the function $\varpi(\boldsymbol{x},t)$ is bounded at infinity point (see [1, 2]).

Just like the heat equation (see [3]) and the reaction diffusion equation (see [4]), Problem I is an important system of equations with real-life application background. It has been widely applied to many practical engineering fields (see [5, 6]), for example, it is used to describe the procedure of fluid flow permeating rocks, the soils moisture migration, and the different media heat transfer. It usually includes complex computing domain, initial and boundary value functions, or source term for the Sobolev equation in the 2D unbounded domain in the real world so that it has no analytic solution. Hence, one has to depend on the numerical methods.

The natural boundary element (NBE), which was proposed by Feng and Yu at the end of 1970s (see [7–11]), is a novel type of boundary element method (BEM) and suitable for solving the unbound regional problem and is an attractive and promising numerical method. It has been effectively applied to complicated boundary and infinity regional problem. It has more specific advantages than the usual BEM, for instance, it has a unique form of boundary integral equation, remains the unchanged energy functional, holds very good numerical stability, adopts the same variation principle as the finite element method (FEM), can couple with FEM and we need not calculate a great many singular integrations in practice. The main idea of the NBE method consists in introducing an appropriate artificial boundary and then restricting the computation to an appropriate large finite spatial domain. It divides the domain into two subregions, a bounded inner region Θ_1 (bounded annular region by Γ_0 and Γ_R) and a regular unbounded region Θ_2 (unbounded domain outside circle Γ_R) (see [1, 2]). One can obtain the natural integral equation on boundary Γ_R and corresponding Poisson integral formula of the subproblem over unbounded domain Θ_2 by the natural boundary reduction. Only the function itself, not its normal derivative at the common boundary Γ_R, appears in the variational. It has been used to solve the second-order elliptic equations, standard parabolic equations, and hyperbolic equations (see [1, 2, 7, 9, 10]).

However, for all we know, up to now, the NBE method has yet not been used to solve the Sobolev equation. Especially, the Sobolev equation not only includes a first-order derivative term of time and two second-order derivative terms of spatial variables but also does two mixed derivative terms of spatial variables (second-order) and time (first-order) so that either the establishment of NBE format or the theoretical analysis needs more skills and is confronted with more difficulties than the second-order elliptic equations, standard parabolic equations, and hyperbolic equations as mentioned above, but it has certain special applications as mentioned above. Therefore, it is worth to study the NBE method for the Sobolev equation in the 2D unbounded domain.

The forthcoming contents are scheduled as follows. In Section 2, we establish a semi-discretized super-convergence format about time for the Sobolev equation in the 2D unbounded domain and deduce the super-convergence error estimates of the semi-discretized solutions about time. In the next Section 3, by the principle of natural boundary reduction, we establish a fully discretized NBE format based on the natural integral equation and the Poisson integral formula of this problem and provide the error estimates between the exact solution and the fully discretized NBE solutions. In Section 4, we supply some numerical experimentations to validate that the numerical computational consequences are concordant with the theoretical ones. Section 5 summarizes main conclusions.

2 Semi-discretized formulation by the Newmark method and error estimate about time for the Sobolev equation in the 2D unbounded domain

By using Green's formula, we can obtain the following variational form for the Sobolev equation.

Problem II For $t \in (0,T)$, find $\varpi(t) \in H^1(\Theta^c)$ that satisfies

$$(I - \varepsilon\Delta)(\varpi_t, \upsilon) + \gamma(\nabla\varpi, \nabla\upsilon) = (f, \upsilon) + \langle g, \upsilon\rangle, \quad \forall \upsilon \in H^1(\Theta^c), \tag{2}$$

where $(\cdot,\cdot)$ represents the inner product in $L^2(\Theta^c)$, but $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\Gamma)$.

The existence and uniqueness of the solution for the variational form, Problem II, are well known (see [5, 6, 12]).

Set τ is the time-step, $t_k = k \cdot \tau$, $\varpi^k = \varpi(\boldsymbol{x}, t_k)$, and $\dot{\varpi}(\boldsymbol{x}, t_k) = \frac{\partial\varpi}{\partial t}(\boldsymbol{x}, t_k)$. By the Newmark method (see, e.g., [13]), we can establish the following iterative scheme:

$$\dot{\varpi}^k(\boldsymbol{x}) = \gamma(I - \varepsilon\Delta)^{-1}\Delta\varpi^k(\boldsymbol{x}) + (I - \varepsilon\Delta)^{-1}f^k(\boldsymbol{x}), \quad k = 1, 2, \ldots, N, \tag{3}$$

$$\varpi^k(\boldsymbol{x}) = \varpi^{k-1}(\boldsymbol{x}) + \tau\big[(1-\beta)\dot{\varpi}^{k-1}(\boldsymbol{x}) + \beta\dot{\varpi}^k(\boldsymbol{x})\big], \quad k = 1, 2, \ldots, N, \tag{4}$$

where $\beta \in (0,1]$ and $N = [T/\tau]$ ($[T/\tau]$ denotes the integer part of T/τ).

By using $[(\varpi^k - \varpi^{k-1})/\tau - (1-\beta)\dot{\varpi}^{k-1}]/\beta$ to approximate to ϖ_t^k, we obtain the following semi-discretized format about time for the Sobolev equation.

Problem III Find $\varpi^k \in H^1(\Theta^c)$ such that

$$\begin{aligned}
&\big(\varpi^k - \varpi^{k-1}, \upsilon\big) + (\varepsilon + \beta\tau\gamma)\big(\nabla\varpi^k, \nabla\upsilon\big) + \big(\tau\gamma(1-\beta) - \varepsilon\big)\big(\nabla\varpi^{k-1}, \nabla\upsilon\big) \\
&\quad = \big(\beta\tau f^k + \tau(1-\beta)f^{k-1}, \upsilon\big) + \big\langle(\beta\tau\gamma + \varepsilon)g^k + \big(\tau\gamma(1-\beta) - \varepsilon\big)g^{k-1}, \upsilon\big\rangle, \\
&\quad \forall\upsilon \in H^1\big(\Theta^c\big), k = 1, 2, \ldots, N,
\end{aligned} \tag{5}$$

where $\varpi^0 = u_0(\boldsymbol{x})$, $f^k = f(\boldsymbol{x}, t_k)$, and $g^k = g(\boldsymbol{x}, t_k)$.

The solutions to Problem III possess the following conclusions.

Theorem 1 *If $f \in L^2(0,T;L^2(\Theta^c))$, $g \in L^2(0,T;L^2(\Gamma))$, and $u_0 \in H^1(\Theta^c)$, then Problem* III *has a unique set of solutions $\{\varpi^k\}_{k=1}^N \subset H^1(\Theta^c)$ satisfying*

$$\begin{aligned}
\big\|\varpi^k\big\|_0 \le \Bigg(&2\|u_0\|_0^2 + 2(\varepsilon + \tau\beta\gamma)\|\nabla u_0\|_0^2 \\
&+ 16\sum_{i=1}^k\big(C_0\big(\beta\gamma + \varepsilon\tau^{-1}\big)\big\|g^i\big\|_{0,\Gamma}^2 + \beta^2\tau\big\|f^i\big\|_{0,\Theta^c}^2\big)\Bigg)^{\frac{1}{2}}\exp(T/2),
\end{aligned} \tag{6}$$

where C_0 is the nonnegative constant in the trace theorem and $k = 1, 2, \ldots, N$. This signifies that the solutions of Problem III *are stable and consecutively rely on the source function f,* *boundary value function g, and initial value function u_0. In addition, when ϖ is sufficiently smooth about t, we have the following error estimates*:

$$\left\|\nabla\left(\varpi(t_k)-\varpi^k\right)\right\|_0=\left\|\nabla e^k\right\|_0\le C\tau,\quad k=1,2,\ldots,N, \tag{7}$$

where $C^2=2\tau\varepsilon^{-1}\sum_{i=1}^{k}(4\|(I-\varepsilon\Delta)\varpi_{tt}(\xi_2^i)\|_0^2+\|(I-\varepsilon\Delta)\varpi_{tt}(\xi_1^i)\|_0^2)\cdot\exp(T)$, $t_{k-1}\le\xi_1^k,\xi_2^k\le t_{k+1}$.

Proof Because Problem III is a linear equation about unknown function ϖ, to prove the existence and uniqueness of solutions of Problem III is equivalent to demonstrating that it has only a zero solution when $g=f=u_0=0$.

Choosing $\upsilon=\varpi^k$ in Problem III together with utilizing the Hölder and Cauchy-Schwarz inequalities and the trace theorem (see [14]), we have

$$\begin{aligned}&\left\|\varpi^k\right\|_0^2+(\varepsilon+\beta\tau\gamma)\left\|\nabla\varpi^k\right\|_0^2\\&\quad\le 4\left(\beta^2\tau\left\|f^k\right\|_0^2+C_0\tau^{-1}(\varepsilon+\beta\tau\gamma)\left\|g^k\right\|_{0,\Gamma}^2\right)+\frac{1}{2}\left(\left\|\varpi^k\right\|_0^2+\left\|\varpi^{k-1}\right\|_0^2\right)\\&\qquad+\frac{\tau}{4}\left(\left\|\varpi^k\right\|_0^2+(\varepsilon+\beta\tau\gamma)\left\|\nabla\varpi^k\right\|_0^2\right)+\frac{(\varepsilon+\beta\tau\gamma)}{2}\left(\left\|\nabla\varpi^{k-1}\right\|_0^2+\left\|\nabla\varpi^k\right\|_0^2\right),\end{aligned} \tag{8}$$

where C_0 is the nonnegative constant in the trace theorem (see [14]). By summing (8) from 1 to k, we have

$$\begin{aligned}&\left\|\varpi^k\right\|_0^2+(\varepsilon+\beta\tau\gamma)\left\|\nabla\varpi^k\right\|_0^2\\&\quad\le\frac{\tau}{2}\sum_{i=1}^{k}\left(\left\|u^i\right\|_0^2+(\varepsilon+\beta\tau\gamma)\left\|\nabla u^i\right\|_0^2\right)+\|u_0\|_0^2+(\varepsilon+\beta\tau\gamma)\|\nabla u_0\|_0^2\\&\qquad+8\sum_{i=1}^{k}\left(C_0\tau^{-1}(\varepsilon+\beta\tau\gamma)\left\|g^i\right\|_{0,\Gamma}^2+\beta^2\tau\left\|f^i\right\|_0^2\right),\quad 1\le k\le N.\end{aligned} \tag{9}$$

When τ is adequately small such that $\tau\le 1$, by Gronwall's lemma (see [15, 16]), we have

$$\begin{aligned}&\left\|\varpi^k\right\|_0\\&\quad\le\Bigg[2\|u_0\|_0^2+2(\varepsilon+\tau\beta\gamma)\|\nabla u_0\|_0^2\\&\qquad+16\sum_{i=1}^{k}\left(C_0\left(\beta\gamma+\varepsilon\tau^{-1}\right)\left\|g^i\right\|_{0,\Gamma}^2+\beta^2\tau\left\|f^i\right\|_{0,\Theta^c}^2\right)\Bigg]^{\frac{1}{2}}\cdot\exp(T/2),\quad 1\le k\le N.\end{aligned} \tag{10}$$

Thus, when $f=g=u_0=0$, from (10), we obtain $\|\varpi^k\|_0=0$, implying $\varpi^k=0$ ($k=1,2,\ldots,N$). Therefore, Problem III has a unique set of solutions.

With Taylor's expanding formula, we acquire

$$\begin{aligned}\varpi_t(t_k)&=\frac{1}{\beta}\left\{\frac{1}{\tau}\left[\varpi(t_k)-\varpi(t_{k-1})\right]-(1-\beta)\varpi_t(t_{k-1})\right\}\\&\quad+\frac{\tau}{2\beta}\varpi_{tt}\left(\xi_1^k\right)-\frac{(1-\beta)\tau}{\beta}\varpi_{tt}\left(\xi_2^k\right),\end{aligned} \tag{11}$$

where $t_{k-1} \leq \xi_1^k, \xi_2^k \leq t_{k+1}$. From Problem II, we obtain

$$\begin{aligned}
&(\varpi(t_k) - \varpi(t_{k-1}, \upsilon) + (\varepsilon + \beta\tau\gamma)\big(\nabla\varpi(t_k), \nabla\upsilon\big) + \big(\tau\gamma(1-\beta) - \varepsilon\big)\big(\nabla\varpi(t_{k-1}), \nabla\upsilon\big) \\
&\quad = \big(\beta\tau f(t_k) + \tau(1-\beta)f(t_{k-1}), \upsilon\big) + \tau^2(1-\beta)\big((I - \varepsilon\Delta)\varpi_{tt}\big(\xi_2^k\big), \upsilon\big) \\
&\qquad + \big\langle(\beta\tau\gamma + \varepsilon)g(t_k) + \big(\tau\gamma(1-\beta) - \varepsilon\big)g(t_{k-1}), \upsilon\big\rangle - \frac{\tau^2}{2}\big((I - \varepsilon\Delta)\varpi_{tt}\big(\xi_1^k\big), \upsilon\big), \\
&\quad \forall\upsilon \in H^1\big(\Theta^c\big).
\end{aligned} \tag{12}$$

Let $e^k = \varpi(t_k) - \varpi^k$. By subtracting (5) from (12) taking $t = t_k$, we obtain

$$\begin{aligned}
&\big(e^k - e^{k-1}, \upsilon\big) + (\varepsilon + \beta\tau)\big(\nabla e^k, \nabla\upsilon\big) + \big(\tau\gamma(1-\beta) - \varepsilon\big)\big(\nabla e^{k-1}, \nabla\upsilon\big) \\
&\quad = \tau^2(1-\beta)\big((I - \varepsilon\Delta)\varpi_{tt}\big(\xi_2^k\big), \upsilon\big) - \frac{\tau^2}{2}\big((I - \varepsilon\Delta)\varpi_{tt}\big(\xi_1^k\big), \upsilon\big).
\end{aligned} \tag{13}$$

By taking $\upsilon = e^k$ in (13) and using the Cauchy and Hölder-Schwarz inequalities, we have

$$\begin{aligned}
&\|e^k\|_0^2 + (\varepsilon + \beta\tau\gamma)\|\nabla e^k\|_0^2 \\
&\quad \leq \frac{1}{2}\big(\|e^k\|_0^2 + \|e^{k-1}\|_0^2\big) + \tau^3\|(I - \varepsilon\Delta)\varpi_{tt}\big(\xi_1^k\big)\|_0^2 \\
&\qquad + \frac{\tau}{4}\|e^k\|_0^2 + 2\tau^3\beta^2\|(I - \varepsilon\Delta)\varpi_{tt}\big(\xi_2^k\big)\|_0^2 + \frac{(\varepsilon + \beta\tau\gamma)}{2}\big(\|\nabla e^{k-1}\|_0^2 + \|\nabla e^k\|_0^2\big).
\end{aligned} \tag{14}$$

By summing (14) from 1 to k, when τ is adequately small such that $\tau/2 \leq 1/2$, we have

$$\begin{aligned}
\|e^k\|_0^2 + 2(\varepsilon + \beta\tau\gamma)\|\nabla e^k\|_0^2 &\leq \tau\sum_{i=1}^{k-1}\|e^i\|_0^2 + 2\tau^3\sum_{i=1}^{k}\big(\|(I - \varepsilon\Delta)\varpi_{tt}\big(\xi_1^i\big)\|_0^2 \\
&\quad + 4(1-\beta)^2\|(I - \varepsilon\Delta)\varpi_{tt}\big(\xi_2^i\big)\|_0^2\big).
\end{aligned} \tag{15}$$

By Gronwall's lemma (see [15, 16]), we have

$$\begin{aligned}
\|e^k\|_0^2 + 2(\varepsilon + \beta\tau\gamma)\|\nabla e^k\|_0^2 &\leq 2\tau^3\sum_{i=1}^{k}\big(4(1-\beta)^2\|(I - \varepsilon\Delta)\varpi_{tt}\big(\xi_2^i\big)\|_0^2 \\
&\quad + \|(I - \varepsilon\Delta)\varpi_{tt}\big(\xi_1^i\big)\|_0^2\big) \cdot \exp(T).
\end{aligned} \tag{16}$$

From (16), we obtain

$$\|\nabla\big(\varpi(t_k) - \varpi^k\big)\|_0 = \|\nabla e^k\|_0 \leq C\tau, \quad k = 1, 2, \ldots, N, \tag{17}$$

where $C^2 = 2\tau\varepsilon^{-1}\sum_{i=1}^{k}(4\|(I - \varepsilon\Delta)\varpi_{tt}(\xi_2^i)\|_0^2 + \|(I - \varepsilon\Delta)\varpi_{tt}(\xi_1^i)\|_0^2) \cdot \exp(T)$. This finishes the proof of Theorem 1. □

3 Natural boundary reduction on the outside circle area and error estimate for the fully discretized NBE solutions

We define $\mu := (\sqrt{(\varepsilon + \tau\beta\gamma)})^{-1}$, $\widetilde{\varpi}^k := (I - \varepsilon\Delta)\varpi^{k-1} + \tau(I - \varepsilon\Delta)(1-\beta)\dot{\varpi}^{k-1}$, $\widetilde{f} := -\widetilde{\varpi}^k - \tau\beta f^k$.

The procedure for solving the above semi-discretized Problem III is given as follows.
(1) Prediction

$$\widetilde{\varpi}^k := (I - \varepsilon\Delta)\varpi^{k-1} + \tau(I - \varepsilon\Delta)(1 - \beta)\dot{\varpi}^{k-1}. \tag{18}$$

(2) Solve the problem

$$\Delta\varpi^k - \mu^2\varpi^k = \mu^2\widetilde{f}^k, \quad \boldsymbol{x} \in \Theta^c, \tag{19}$$

$$\frac{\partial\varpi^k}{\partial n} = g^k, \quad \boldsymbol{x} \in \Gamma, \tag{20}$$

$$\left|\varpi^k\right| < +\infty, \quad |\boldsymbol{x}| \to +\infty. \tag{21}$$

(3) Update value

$$\dot{\varpi}^k = \frac{1}{\tau\beta}\left(\varpi^k - \widetilde{\varpi}^k\right). \tag{22}$$

From the above procedure, it is not difficult to find that our main task is to solve elliptic boundary value problems at each time level.

Let $I_n(x)$ and $K_n(x)$ ($n = 0, 1, 2, \ldots$) be, severally, the first and second type of modified Bessel functions (see [17]), and $\Re_\mu$ and $\Im_\mu$ be, respectively, the natural and Poisson integral operators (see [1, 10]). From the NBE method (see [1, 18, 19]), the Dirichlet boundary $\hat{\varpi}_0^k$ and the Neumann boundary $\frac{\partial\varpi^k}{\partial\boldsymbol{n}}$ satisfy the following relationship:

$$\frac{\partial\varpi^k}{\partial\boldsymbol{n}} + \mathcal{N}\left(\mu, r; \widetilde{f}^k, \theta\right) = \Re_\mu\hat{\varpi}_0^k, \tag{23}$$

and the relationship between the solution ϖ^k of Problem III with its Dirichlet boundary value $\hat{\varpi}_0^k$ is as follows:

$$\varpi^k = \Im_\mu\hat{\varpi}_0^k + \mathcal{F}\left(\mu, r; \widetilde{f}^k, R, \theta\right), \tag{24}$$

where

$$\mathcal{N}\left(\mu, r; \tilde{f}^k, \theta\right) = \frac{\mu^2}{2}\sum_{n=0}^{+\infty}\xi_n\int_r^{+\infty}\bar{G}_n(\mu, r; \sigma)\left[\tilde{f}_n^{k,c}(\sigma)\cos n\theta + \tilde{f}_n^{k,s}(\sigma)\sin n\theta\right]\mathrm{d}\sigma,$$

$$\xi_0 = 1; \xi_n = 2, n = 1, 2, \ldots,$$

$$\bar{G}_n(\mu, r; \sigma) = -\frac{K_n(\mu\sigma)}{K_n(\mu r)}\cdot\frac{\sigma}{r}, \quad n = 0, 1, 2, \ldots,$$

$$\mathcal{F}\left(\mu, r; \tilde{f}^k, R, \theta\right) = \frac{\mu^2}{2}\sum_{n=0}^{+\infty}\xi_n\int_r^{+\infty}\sigma^2 G_n(R, \sigma)\left[\tilde{f}_n^{k,c}(\sigma)\cos n\theta + \tilde{f}_n^{k,s}(\sigma)\sin n\theta\right]\mathrm{d}\sigma,$$

$$\sigma^2 G_n(R, \sigma) = \begin{cases} \frac{\phi_n(\sigma)\psi_n(R)}{E_n(\sigma)}, & R \le \sigma, \\ \frac{\psi_n(\sigma)\phi_n(R)}{E_n(\sigma)}, & R \ge \sigma, \end{cases} \quad n = 0, 1, 2, \ldots,$$

$$\phi_n(\sigma) = K_n(\mu\sigma), \qquad \psi_n(\sigma) = I_n(\mu\sigma)K_n(\mu r) - K_n(\mu\sigma)I_n(\mu r), \quad n = 0, 1, 2, \ldots,$$

$$E_n(\sigma) = \psi_n(\sigma)\phi_n'(\sigma) - \phi_n(\sigma)\psi_n'(\sigma), \quad n = 0,1,2,\ldots,$$

$$\tilde{f}_n^{k,c}(\sigma) = \frac{1}{\pi}\int_0^{2\pi} \tilde{f}^k(\sigma,\theta)\cos n\theta \,\mathrm{d}\theta, \quad n = 0,1,2,\ldots,$$

$$\tilde{f}_n^{k,s}(\sigma) = \frac{1}{\pi}\int_0^{2\pi} \tilde{f}^k(\sigma,\theta)\sin n\theta \,\mathrm{d}\theta, \quad n = 1,2,\ldots.$$

The above (23) and (24) are named the natural and the Poisson integral equations, respectively. Thus, (23) is equivalent to the following variation form:

Find $\varpi_0^k \in H^{\frac{1}{2}}(\Gamma)$ $(1 \le k \le N)$ that satisfy

$$\hat{B}(\varpi_0^k, \nu^k) = \langle g^k(r,\theta) + \mathcal{N}(\mu, r; \widetilde{f^k}, \theta), \nu^k\rangle, \quad \forall \nu^k \in H^{\frac{1}{2}}(\Gamma), \tag{25}$$

where $\hat{B}(\varpi_0^k, \nu^k) = \langle \Re_\mu \hat{\varpi}_0^k, \nu^k\rangle =: \int_\Gamma (\Re_\mu \hat{\varpi}_0^k)\nu^k \,\mathrm{d}s$ and $\langle \omega, \ell\rangle =: \int_\Gamma \omega\ell \,\mathrm{d}s$.

3.1 Natural boundary reduction on the external circle area

Now, let the domain Θ be a circle with radius r and center at origin. For convenience, we also suppose that the solutions ϖ^k of Problem III are appropriately smooth. Under the polar coordinates, $\Gamma = \{(R,\theta) : R = r, \theta \in [0,2\pi]\}$, $\Theta^c = \{(R,\theta) : R = |\boldsymbol{x}| > r, \theta \in [0,2\pi]\}$, and the external normal derivative operator onto Γ satisfies $\frac{\partial}{\partial \boldsymbol{n}} = -\frac{\partial}{\partial \boldsymbol{R}}$. The solution of equations (19)-(21) can be expressed with the following form in the polar coordinates:

$$\varpi^k(R,\theta) = \frac{1}{2}a_0(R) + \sum_{n=1}^{+\infty}\bigl[a_n(R)\cos n\theta + b_n(R)\sin n\theta\bigr], \tag{26}$$

where

$$a_n(R) = \frac{1}{\pi}\int_0^{2\pi} \varpi^k(R,\theta)\cos n\theta \mathrm{d}\theta, \quad n = 0,1,2,\ldots;$$

$$b_n(R) = \frac{1}{\pi}\int_0^{2\pi} \varpi^k(R,\theta)\sin n\theta \mathrm{d}\theta, \quad n = 1,2,\ldots.$$

By calculating, we get the solution $\varpi^k(R,\theta)$ of equations (19)-(21) as follows.

$$\begin{aligned}\varpi^k(R,\theta) = {}& \frac{1}{2\pi}\sum_{n=0}^{+\infty}\xi_n \int_0^{2\pi} \frac{K_n(\mu R)}{K_n(\mu r)}\cos n(\theta - \theta')\varpi^k(r,\theta')\,\mathrm{d}\theta' \\ & + \mathcal{F}(\mu, r; \tilde{f}^k, R, \theta), \quad R > r,\end{aligned} \tag{27}$$

$$\frac{\partial \varpi^k(r,\theta)}{\partial \boldsymbol{n}} + \mathcal{N}(\mu, r; \tilde{f}^k, \theta) = \frac{\mu}{2\pi}\int_0^{2\pi} \bar{K}_n(\mu, r; \theta - \theta')\varpi^k(r,\theta')\,\mathrm{d}\theta', \tag{28}$$

where $\bar{K}_n(\mu, r; \theta - \theta') = -\sum_{n=0}^{+\infty}\xi_n \cos n(\theta - \theta') \cdot K_n'(\mu r)/K_n(\mu r)$.

Remark 1 The formats (27) and (28) are, respectively, the Poisson and the natural integral equations. We can attain the solution $\varpi^k(r,\theta)$ from the natural integral equation (28) and then obtain the solution of the original boundary value, i.e., Problem I, by the Poisson integral formula (27). But the solution of Problem I can be acquired directly by the Poisson integral formula (27) for the Cauchy-Dirichlet initial boundary value problem, because the function $\varpi^k(r,\theta)$ is known.

Remark 2 In numerical computations, we can limit $r \leq R < +\infty$ on the $r \leq R \leq R_{\max}$ and use the corresponding numerical integral to calculate the integral calculation of the $\mathcal{N}(\mu, r; \tilde{f}^k, \theta)$ and $\mathcal{F}(\mu, r; \tilde{f}^k, R, \theta)$. Meanwhile, the expressions of an infinite series summation can be substituted with a finite series summation in the practical applications.

3.2 Error analysis of NBE solutions

We divide the circumference Γ into some finite elements, which satisfy regular conditions. For the sake of simplicity, we take uniform subdivision. Let $\mathcal{S}_h(\Gamma) \subset H^{1/2}(\Gamma)$ be a finite element subspace spanned by appropriate basis functions. Thus, the NBE approximation to Problem II is as follows.

Problem IV Find $\varpi_{0h}^k \in \mathcal{S}_h(\Gamma)$ $(1 \leq k \leq N)$ that satisfy

$$\hat{B}\big(\varpi_{0h}^k, \nu^k\big) = \big\langle g^k(r,\theta) + \mathcal{N}\big(\mu, r; \widetilde{f^k}, \theta\big), \nu^k\big\rangle, \quad \forall \nu^k \in \mathcal{S}_h(\Gamma), \tag{29}$$

and

$$\begin{aligned} \varpi_h^k(R,\theta) = {} & \frac{1}{2\pi} \sum_{n=0}^{+\infty} \xi_n \int_0^{2\pi} \frac{K_n(\mu R)}{K_n(\mu r)} \cos n\big(\theta - \theta'\big) \varpi_{0h}^k\big(r,\theta'\big) \, \mathrm{d}\theta' \\ & + \mathcal{F}\big(\mu, r; \tilde{f}^k, R, \theta\big), \quad R > r. \end{aligned} \tag{30}$$

The above formula (30) is the approximate expression of Poisson integral formula (27). In order to analyze the errors of NBE solutions of Problem IV, it is necessary to introduce the following L^2-projection and its property (see [16]).

Definition 1 An operator $P_h : L_2(\Gamma) \to \mathcal{S}_h(\Gamma)$ (where $\mathcal{S}_h(\Gamma) \subset L^2(\Gamma)$ is a finite element space) is known as an L^2-projection if, for any $\upsilon \in L^2(\Gamma)$, there exists unique $P_h\upsilon \in \mathcal{S}_h(\Gamma)$ satisfying

$$\langle \upsilon - P_h\upsilon, \upsilon_h \rangle = 0, \quad \forall \upsilon_h \in \mathcal{S}_h(\Gamma).$$

Lemma 1 *If $\mathcal{S}_h(\Gamma)$ is a subspace spanned by piecewise linear polynomials and $\upsilon \in H^2(\Gamma)$, then the L^2-projection P_h satisfies*

$$\|\upsilon - P_h\upsilon\|_s \leq Ch^{2-s} \|\upsilon\|_{2,\Gamma}, \quad s = -1, 0, 1,$$

where C used next represents a generic positive real independent of τ and h.

Problem IV possesses the following conclusion.

Theorem 2 *Let $\varpi_0^k \in H^{\frac{1}{2}}(\Gamma)$ and ϖ_{0h}^k be, respectively, solutions to* (25) *together with* (29), *$\tau = O(h^2)$, and $\mathcal{S}_h(\Gamma)$ be the piecewise linear polynomial subspace. Then we have the following error estimates*:

$$\big\|\varpi_0^k - \varpi_{0h}^k\big\|_{0,\Gamma} \leq Ch^2, \quad k = 1, 2, \ldots, N. \tag{31}$$

Proof By subtracting (29) from (25) taking $v^k = v_h^k$, we obtain

$$\begin{aligned}
&\hat{B}\big(\varpi_0^k - \varpi_{0h}^k, v_h^k\big) \\
&\quad = \left\langle \frac{\mu^2}{2\pi} \sum_{n=0}^{+\infty} \xi_n \int_r^{+\infty} \bar{G}_n(\mu, r; \sigma) \left[\int_0^{2\pi} \big[-\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big) \right.\right. \\
&\qquad + \varepsilon \Delta\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big) - \tau(1-\beta)\gamma \Delta\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big)\big] \cos n\hat{\theta}\, \mathrm{d}\hat{\theta} \cos n\theta \\
&\qquad + \int_0^{2\pi} \big[-(I - \varepsilon\Delta)\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big) - \tau\gamma\Delta\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big) \\
&\qquad \left.\left. + \tau\beta\gamma\Delta\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big)\big] \sin n\hat{\theta}\, \mathrm{d}\hat{\theta} \sin n\theta \right] \mathrm{d}\sigma, v_h^k \right\rangle, \quad \forall v^k \in \mathcal{S}_h(\Gamma).
\end{aligned} \tag{32}$$

Due to $\hat{B}(\cdot,\cdot)$ being positive definite on $H^{\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ (see [1]), by using the Hölder inequality, we have

$$\begin{aligned}
&M\big\|\varpi_0^k - \varpi_{0h}^k\big\|_{0,\Gamma}^2 \\
&\quad \le \big|\hat{B}\big(\varpi_0^k - \varpi_{0h}^k, \varpi_0^k - \varpi_{0h}^k\big)\big| \\
&\quad \le \big|\hat{B}\big(\varpi_0^k - \varpi_{0h}^k, \varpi_0^k - P_h\varpi_0^k\big)\big| + \big|\hat{B}\big(\varpi_0^k - \varpi_{0h}^k, P_h\varpi_0^k - \varpi_{0h}^k\big)\big| \\
&\quad \le \big\|\varpi_0^k - \varpi_{0h}^k\big\|_{0,\Gamma} \big\|\varpi_0^k - P_h\varpi_0^k\big\|_{0,\Gamma} + \left|\left\langle \frac{\mu^2}{2\pi} \sum_{n=0}^{+\infty} \xi_n \int_r^{+\infty} \bar{G}_n(\mu, r; \sigma) \right.\right. \\
&\qquad \times \left[\int_0^{2\pi} \big[-\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big) + \varepsilon\Delta\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big) \right. \\
&\qquad - \tau(1-\beta)\gamma\Delta\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big)\big] \cos n\hat{\theta}\, \mathrm{d}\hat{\theta} \cos n\theta \\
&\qquad + \int_0^{2\pi} \big[-(I - \varepsilon\Delta)\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big) \\
&\qquad \left.\left.\left. - \tau(1-\beta)\gamma\Delta\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big)\big] \sin n\hat{\theta}\, \mathrm{d}\hat{\theta} \sin n\theta \right] \mathrm{d}\sigma, P_h\varpi_0^k - \varpi_{0h}^k \right\rangle\right| \\
&\quad \le \big\|\varpi_0^k - \varpi_{0h}^k\big\|_{0,\Gamma} \big\|\varpi_0^k - P_h\varpi_0^k\big\|_{0,\Gamma} + C\big[\big\|\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big\|_{0,\Gamma} \big\|P_h\varpi_0^k - \varpi_0^k\big\|_{0,\Gamma} \\
&\qquad + \big(\varepsilon\big\|\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big\|_{0,\Gamma} + \tau\big\|\nabla\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big)\big\|_{0,\Gamma}\big) \big\|P_h\varpi_0^k - \varpi_0^k\big\|_{0,\Gamma} \\
&\qquad - \tau\beta\big\|\nabla\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big)\big\|_{0,\Gamma} \big\|P_h\varpi_0^k - \varpi_0^k\big\|_{0,\Gamma} \\
&\qquad + (1+\varepsilon)\big\|\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big\|_{0,\Gamma} \big\|\varpi_0^k - \varpi_{0h}^k\big\|_{0,\Gamma} \\
&\qquad + \tau\gamma(1-\beta)\big\|\nabla\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big)\big\|_{0,\Gamma} \big\|\varpi_0^k - \varpi_{0h}^k\big\|_{0,\Gamma}\big].
\end{aligned}$$

Further, by the Cauchy-Schwarz inequality and Lemma 1 as well as the inverse estimating theorem (see [16]), we acquire

$$\big\|\varpi_0^k - \varpi_{0h}^k\big\|_{0,\Gamma}^2 \le C\big(\big\|\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big\|_{0,\Gamma}^2 + h^4\big).$$

Further, we acquire

$$\|\varpi_0^k - \varpi_{0h}^k\|_{0,\Gamma} \le Ch^2.$$

This finishes the proof of Theorem 2. □

The solutions for approximate expression (30) have the following error estimates.

Theorem 3 *Let ϖ^k and ϖ_h^k be, respectively, the solutions of* (27) *and* (30) *and* $\tau = O(h^2)$. *Then we have the error estimates*:

$$\|\varpi^k - \varpi_h^k\|_{0,\infty,\Theta^c} \le Ch^2. \tag{33}$$

Proof By the literature [17], we can immediately derive

$$K_n(x) = \sqrt{\frac{\pi}{2n}}\left(\frac{2n}{ex}\right)^n\left[1 + o\left(n^{-1}\right)\right], \quad n \to +\infty.$$

Therefore, we have $K_n(\mu R)/K_n(\mu r) \le 1$ $(r < R)$. Thus, there is real $\tilde{M} \ge 0$ that satisfies

$$\left|\sum_{n=0}^{+\infty}\xi_n\frac{K_n(\mu R)}{K_n(\mu r)}\right| \le \tilde{M}.$$

By using (18), we have $\tilde{f}^k = [-(I - \varepsilon\Delta) - \tau(1-\beta)\gamma\Delta]\varpi^{k-1} - \tau\beta f^k - \tau(1-\beta)f^{k-1}$. Thus, from (27) and (30), we have

$$\begin{aligned}
&\left|\varpi^k - \varpi_h^k\right| \\
&\quad\le \frac{1}{2\pi}\left|\sum_{n=0}^{+\infty}\xi_n\int_0^{2\pi}\sum_{n=0}^{+\infty}\frac{K_n(\mu R)}{K_n(\mu r)}\cos n\left(\theta-\theta'\right)\cdot\left(\varpi_0^k - \varpi_{0h}^k\right)\mathrm{d}\theta'\right| \\
&\qquad + \frac{\mu^2}{2\pi}\sum_{n=0}^{+\infty}\xi_n\int_r^{+\infty}\sigma^2 G_n(R,\sigma)\left\{\left[\int_0^{2\pi}\left|-(I-\varepsilon\Delta)\left(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\right)\cos n\hat{\theta}\right|\mathrm{d}\hat{\theta}\right.\right. \\
&\qquad \left. + \int_0^{2\pi}\left|-\tau(1-\beta)\gamma\Delta\left(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\right)\cos n\hat{\theta}\right|\mathrm{d}\hat{\theta}\right]\cos n\theta \\
&\qquad + \left[\int_0^{2\pi}\left|-(I-\varepsilon\Delta)\left(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\right)\sin n\hat{\theta}\right|\mathrm{d}\hat{\theta}\right. \\
&\qquad \left.\left. + \int_0^{2\pi}\left|-\tau(1-\beta)\gamma\Delta\left(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\right)\sin n\hat{\theta}\right|\mathrm{d}\hat{\theta}\right]\sin n\theta\right\}\mathrm{d}\sigma \\
&\quad\le \left|\frac{1}{2\pi}\sum_{n=0}^{+\infty}\xi_n\frac{K_n(\mu R)}{K_n(\mu r)}\right|\left(\int_0^{2\pi}\cos^2 n\left(\theta-\theta'\right)\mathrm{d}\theta'\right)^{\frac{1}{2}}\cdot\left\|\varpi_0^k - \varpi_{0h}^k\right\|_{0,\Gamma} \\
&\qquad + \frac{\mu^2}{2\pi}\sum_{n=0}^{+\infty}\xi_n\int_r^{+\infty}\sigma^2 G_n(R,\sigma)\cdot\left\{\left[\left\|\varpi_0^{k-1} - \varpi_{0h}^{k-1}\right\|_{0,\Gamma}\left(\int_0^{2\pi}\cos^2 n\hat{\theta}\,\mathrm{d}\hat{\theta}\right)^{\frac{1}{2}}\right.\right. \\
&\qquad + \varepsilon\left\|\left(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\right)\right\|_{0,\Gamma}\|\Delta\cos n\hat{\theta}\|_0
\end{aligned}$$

$$
\begin{aligned}
&+ \tau(1-\beta)\gamma \big\| \nabla\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big)\big\|_{0,\Gamma} \|\nabla \cos n\hat{\theta}\|_0 \bigg] \cos n\theta \\
&+ \bigg[\big\|\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big\|_{0,\Gamma} \bigg(\int_0^{2\pi} \sin^2 n\hat{\theta}\, \mathrm{d}\hat{\theta} \bigg)^{\frac{1}{2}} \\
&+ \varepsilon \big\| \big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big)\big\|_{0,\Gamma} \|\Delta \sin n\hat{\theta}\|_0 \\
&+ \tau(1-\beta)\gamma \big\| \nabla\big(\varpi_0^{k-1} - \varpi_{0h}^{k-1}\big)\big\|_{0,\Gamma} \|\nabla \sin n\hat{\theta}\|_0 \bigg] \sin n\theta \bigg\} \,\mathrm{d}\sigma \\
\le{}& Ch^2 + Ch^2 \frac{\mu^2}{2\pi} \sum_{n=0}^{+\infty} \varepsilon_n \int_r^{+\infty} \sigma^2 G_n(R,\sigma)(\sin n\theta + \cos n\theta)\, \mathrm{d}\sigma \\
\le{}& Ch^2.
\end{aligned}
\tag{34}
$$

From (34), we immediately gain (33). This finishes the proof of Theorem 3. □

For the fully discretized NBE format, Problem IV, we have the following conclusion.

Theorem 4 *If* $f^k \in L^2(\Theta^c)$, $g \in L^0(\Gamma)$, *and* $\vartheta_0 \in H^1(\Theta^c)$, *Problem* IV *has a unique solution* $\varpi_{0h}^k \in \mathcal{S}_h(\Gamma)$ *satisfying*

$$
\big\|\varpi_{0h}^k\big\|_0 \le C\Bigg[\big\|u_h^0\big\|_{0,\Gamma} + \sum_{i=1}^{k} \big(\big\|g^i\big\|_{0,\Gamma} + \tau\beta \big\|f^i\big\|_{0,\Theta^c}\big) \Bigg] \cdot \exp\big((1+\varepsilon+\beta\gamma)T\big),
\tag{35}
$$

where $u_h^0 = P_h u_0(\boldsymbol{x})$. *This signifies that the solutions of Problem* IV *are steady and consecutively rely on the source function* f, *boundary value function* g, *and initial value function* ϑ_0. *Furthermore, when* $\tau = O(h^2)$, *we have the error estimates.*

$$
\big\|\varpi(t_k) - \varpi_h^k\big\|_{0,\Theta^c} \le C\big(\tau + h^2\big), \quad k = 1,2,\ldots,N.
\tag{36}
$$

Proof Due to $\hat{B}(\cdot,\cdot)$ being symmetrical, continuous, and positive definitive on $H^{\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ (see [1, 10]), by the Lax-Milgram theorem (see [1, 10, 16]), we know that Problem IV has a unique set of solutions.

Then, by taking $v_h^k = \varpi_{0h}^k$ in (29) and using the Hölder inequality, we have

$$
\begin{aligned}
M\big\|\varpi_{0h}^k\big\|_{0,\Gamma}^2 \le{}& \big|\hat{B}\big(\varpi_{0h}^k, \varpi_{0h}^k\big)\big| = \big|\big\langle g^k(r,\theta) + \mathcal{N}\big(\mu, r; \tilde{f}^k, \theta\big), \varpi_{0h}^k\big\rangle\big| \\
\le{}& \Bigg[\big\|g^k\big\|_{0,\Gamma} + \Bigg\| \frac{\mu^2}{2\pi} \sum_{n=0}^{+\infty} \xi_n \int_r^{+\infty} \bar{G}_n(\mu, r;\sigma) \bigg\{ \int_0^{2\pi} \big[\big(-(I-\varepsilon\Delta) \\
&- \tau(1-\beta)\gamma\Delta\big)\varpi_{0h}^{k-1} - \tau(1-\beta) f^{k-1} - \tau\beta f^k\big] \cdot \cos n\hat{\theta}\, \mathrm{d}\hat{\theta} \cos n\theta \\
&+ \int_0^{2\pi} \big[\big(-(I-\varepsilon\Delta) - \tau(1-\beta)\gamma\Delta\big)\varpi_{0h}^{k-1} \\
&- \tau(1-\beta) f^{k-1} - \tau\beta f^k\big] \sin n\hat{\theta}\, \mathrm{d}\hat{\theta} \sin n\theta \bigg\} \,\mathrm{d}\sigma \Bigg\|_{0,\Gamma} \Bigg] \big\|\varpi_{0h}^k\big\|_{0,\Gamma} \\
\le{}& C\big[(1+\varepsilon+\beta\tau)\big\|\varpi_{0h}^{k-1}\big\|_{0,\Gamma} + \tau\beta\big\|f^k\big\|_{0,\Theta^c} + \big\|g^k\big\|_{0,\Gamma}\big]\big\|\varpi_{0h}^k\big\|_{0,\Gamma}.
\end{aligned}
\tag{37}
$$

Furthermore, we have

$$\|\varpi_{0h}^{k}\|_{0} \le C\big((1+\varepsilon+\beta\tau)\|\varpi_{0h}^{k-1}\|_{0} + \|g^{k}\|_{0,\Gamma} + \tau\beta\|f^{k}\|_{0,\Theta^{c}}\big). \tag{38}$$

By summing (38) from 1 to k and using Gronwall's lemma (see [15, 16]), we get

$$\|\varpi_{0h}^{k}\|_{0} \le C\left[\|u_{0}\|_{0,\Gamma} + \sum_{i=1}^{k}\big(\|g^{i}\|_{0,\Gamma} + \tau\beta\|f^{i}\|_{0,\Theta^{c}}\big)\right]\cdot\exp\big((1+\varepsilon+\beta\gamma)T\big). \tag{39}$$

By using the triangle inequality,

$$\|\varpi(t_{k}) - \varpi_{h}^{k}\|_{0,\Theta^{c}} \le \|\varpi(t_{k}) - \varpi^{k}\|_{0,\Theta^{c}} + C\|\varpi^{k} - \varpi_{h}^{k}\|_{0,\infty,\Theta^{c}}, \tag{40}$$

and combining (7) and (33) with (40), we can acquire (36). This finishes the proof of Theorem 4. □

4 Some numerical experiments

At the moment, we utilize some numerical experimentations to validate that the numerical computational conclusions coincide with the theoretical ones and that the NBE format is effective and feasible for solving the Sobolev equation in the 2D unbounded domain.

Let Θ^{c} be the external region outside the unit circle. The source term is chosen as

$$\begin{aligned} f(\mathbf{x},t) = {}& 3\big[\big(R^{-1} - 2.25\pi^{2}R^{-1} - R^{-3}\big)\sin(1.5\pi R) + 1.5\pi^{2}R^{-2}\cos(1.5\pi R)\big]\cos(3t) \\ & + \big[R^{-3} - 2.25\pi^{2}R^{-1}\sin(1.5\pi R) - 1.5\pi R^{-2}\cos(1.5\pi R)\big]\sin(3t), \end{aligned}$$

where $R = |\mathbf{x}| = \sqrt{x^{2}+y^{2}} \ge 1$ and take $\varepsilon = \gamma = 1$. The boundary and initial functions are, respectively, chosen as $g(\mathbf{x},t) = -\sin(3t)$ and $u_{0}(\mathbf{x}) = 0$ as in [1]. We approximately replace $\sum_{n=1}^{+\infty}$ with $\sum_{n=1}^{M}$ and adopt the numerical integral to calculate $\mathcal{N}(\mu, r; \tilde{f}^{k}, \theta)$ and $\mathcal{F}(\mu, r; \tilde{f}^{k}, R, \theta)$ in our numerical experimentations.

We divide the circumference Γ into 64 arc paragraphs with side length $\triangle\theta = \pi/32$, which satisfies usual regular conditions, and take $M = 120$ and the time step size $\tau = 0.0125$. We find exact solution ϖ_{0}^{k} and numerical solutions ϖ_{0h}^{k} at time $t = 1, 10, 30, 60, 90, 120, 180$, which are shown in Photo (a)'s and (b)'s of Figs. 1-7, severally. Whereas the errors between the exact solution and numerical solutions at $t = 1, 10, 30, 60, 90, 120, 180$ are exhibited graphically in Photo (c)'s of Figs. 1-7, severally. From each group of photos in

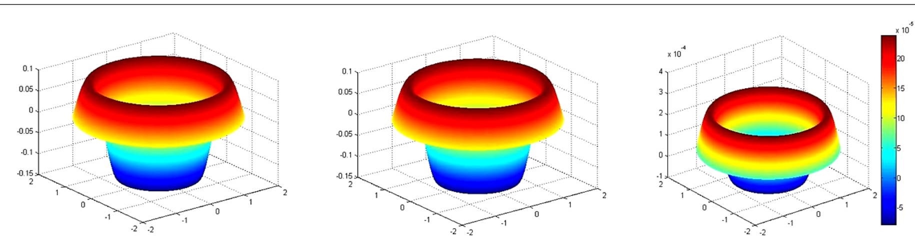

Figure 1 The exact and NBE solutions and the error between them at $t = 1$. Photos **(a)** and **(b)** are severally the exact and NBE solutions and Photo **(c)** is the error between them.

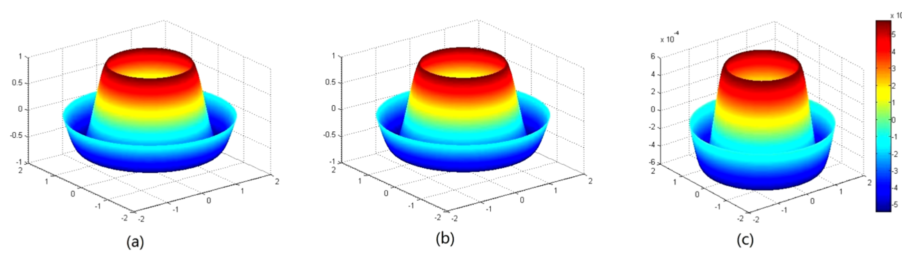

Figure 2 The exact and NBE solutions and the error between them at $t = 10$. Photos **(a)** and **(b)** are severally the exact and NBE solutions and Photo **(c)** is the error between them.

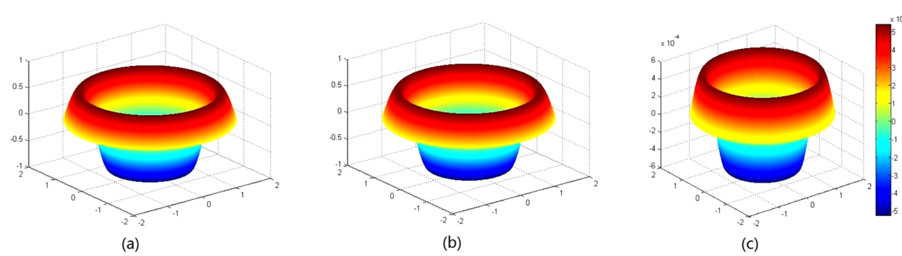

Figure 3 The exact and NBE solutions and the error between them at $t = 30$. Photos **(a)** and **(b)** are severally the exact and NBE solutions and Photo **(c)** is the error between them.

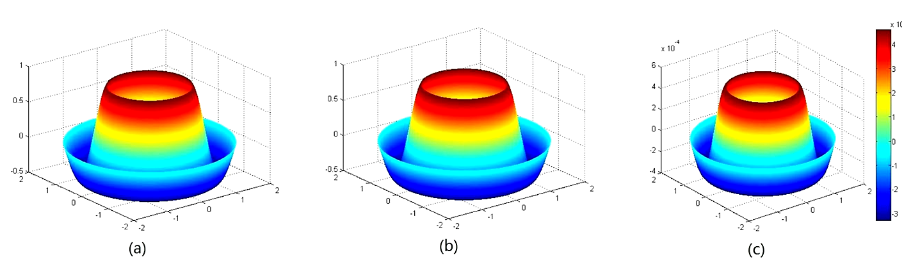

Figure 4 The exact and NBE solutions and the error between them at $t = 60$. Photos **(a)** and **(b)** are severally the exact and NBE solutions and Photo **(c)** is the error between them.

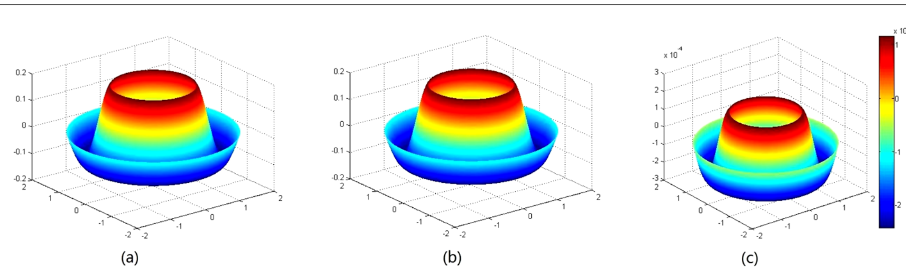

Figure 5 The exact and NBE solutions and the error between them at $t = 90$. Photos **(a)** and **(b)** are severally the exact and NBE solutions and Photo **(c)** is the error between them.

Figs. 1-6, we can clearly see that the exact solutions are basically the same as the numerical solutions. In particular, the error photos indicated that the numerical computing consequences are consistent with the theoretical ones since both theoretical and numerical

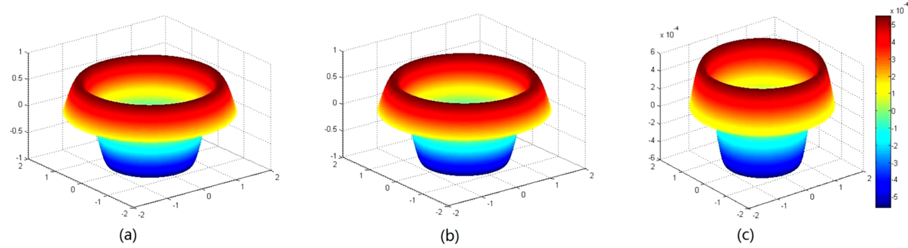

Figure 6 The exact and NBE solutions and the error between them at t = 120. Photos **(a)** and **(b)** are severally the exact and NBE solutions and Photo **(c)** is the error between them.

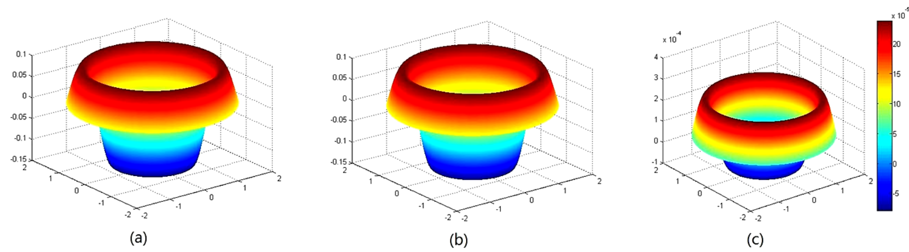

Figure 7 The exact and NBE solutions and the error between them at t = 180. Photos **(a)** and **(b)** are severally the exact and NBE solutions and Photo **(c)** is the error between them.

errors are $O(10^{-4})$. Especially, it is super-convergent about time accuracy. Even if $t = 300$, the numerical solution still converges and maintains accuracy $O(10^{-4})$. These sufficiently signify that the NBE method is very effective and feasible for solving the Sobolev equation in the 2D unbounded domain.

5 Conclusions

In this article, we have established the semi-discretized format about time for the Sobolev equation in the 2D unbounded domain by the Newmark method and gained the error estimates of super-convergence of the semi-discretized solutions about time. Especially, we have built the fully discretized NBE format and analyzed the errors between the analytical solution and the fully discretized NBE solutions. We have also provided some numerical experiments to validate that our method is effective and feasible. The most important thing is that the NBE method applied to solve the Sobolev equation in the 2D unbounded domain is first presented, it is new and original. Moreover, the method can also solve many practical problems.

Acknowledgements
This research was supported by the Fundamental Research Funds for the Central Universities grant 2017XS067 and National Science Foundation of China grant 11671106.

Availability of data and materials
The authors declare that all the data and material in the article are available and veritable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors wrote, read, and approved the final manuscript.

Author details
[1]School of Control and Computer Engineering, North China Electric Power University, No. 2, Bei Nong Road, Beijing, Changping District 102206, China. [2]School of Mathematics and Physics, North China Electric Power University, No. 2, Bei Nong Road, Beijing, Changping District 102206, China.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 5 October 2017 Accepted: 21 November 2017 Published online: 28 November 2017

References

1. Du, QK, Chen, JR: Development Equation Boundary Element Method and Its Application. Science Press, Beijing (2013) (in Chinese)
2. Zhu, CJ, Du, QK: Natural boundary element method for parabolic equations in an unbounded domain. J. Math. Res. Exposition **22**(2), 177-188 (2002)
3. Abdelwahed, M, Chorfi, N, Radulescu, V: Numerical solutions to heat equations via the spectral method. Electron. J. Differ. Equ. **2016**, 68 (2016)
4. Zhan, H: Reaction diffusion equations with boundary degeneracy. Electron. J. Differ. Equ. **2016**, 81 (2016)
5. Shi, DM: On the initial boundary value problem of nonlinear the equation of the moisture in soil. Acta Math. Appl. Sin. **13**(1), 33-40 (1990)
6. Ting, TW: A cooling process according to two-temperature theory of heat conduction. J. Math. Anal. Appl. **45**(1), 23-31 (1974)
7. Feng, K: Finite element method and natural boundary reduction. In: Proceedings of the International Congress of Mathematicians. Polish Academy Press, Warszawa (1983)
8. Feng, K, Yu, DH: Canonical integral equations of elliptic boundary value problem and their numerical solutions. In: Proceeding of China-France Symposium on the Finite Element Method, pp. 211-252. Science Press, Beijing (1983)
9. Yu, DH: Mathematical Theory of Natural Boundary Element Method. Science Press, Beijing (1993)
10. Yu, DH: Numerical solutions of harmonic and biharmonic canonical integral equations in interior or exterior circular domains. J. Comput. Math. **1**(1), 52-62 (1983)
11. Feng, K: Canonical Boundary Reduction and Finite Element Method. Science Press, Beijing (1982)
12. Barenblett, GI, Zheltov, IP, Kochina, IN: Basic concepts in the theory of homogeneous liquids in fissured rocks. J. Appl. Math. Mech. **24**(5), 1286-1303 (1990)
13. Wood, WL: Practical Time-Stepping Schemes. Clarendon, London (1990)
14. Adams, RA: Sobolev Spaces. Academic Press, New York (1975)
15. Girault, V, Raviant, PA: Finite Element Approximations of the Navier-Stokes Equations, Theorem and Algorithms. Springer, New York (1986)
16. Luo, ZD: Mixed Finite Element Methods and Applications. Science Press, Beijing (2006) (in Chinese)
17. Wang, ZX, Guo, DR: Special Functions. World Scientific, Singapore (1989)
18. Courant, R, Hilbert, D: Methods of Mathematical Physics (II). Wiley, New York (1962)
19. Francis, B, Hilde, B: Advance Calculus for Applications, 2nd edn. Prentice Hall, New York (1976)