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Existence of nonconstant periodic solutions for a class of second-order systems with $p(t)$-Laplacian

Yukun An[1*], Yuanfang Ru[2,3] and Fanglei Wang[3]

*Correspondence:
anykna@nuaa.edu.cn
[1]College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing, 210016, P.R. China
Full list of author information is available at the end of the article

Abstract
In this paper, we investigate a class of second-order $p(t)$-Laplacian systems with local 'superquadratic' potential. By using the generalized mountain pass theorem, we obtain an existence result for nonconstant periodic solutions.

Keywords: $p(t)$-Laplacian; generalized Sobolev space; generalized mountain pass theorem; periodic solution

1 Introduction

This paper is concerned with the existence of periodic solutions for the following $p(t)$-Laplacian system:

$$\begin{cases} (|u'(t)|^{p(t)-2}u'(t))' + \nabla F(t,u(t)) = 0, & t \in [0,T], \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{1}$$

where $T > 0$, $u \in R^N$. $F(t,u)$ and $p(t)$ satisfy the following conditions:

(F_0) $F : [0,T] \times R^N \to R$ is measurable and T-periodic in t for each $u \in R^N$ and continuously differentiable in u for a.e. $t \in [0,T]$, and there exist $a \in C(R^+,R^+)$ and $b \in L^1([0,T],R^+)$ such that

$$\left|F(t,u)\right| \le a\left(|u|\right)b(t), \qquad \nabla\left|F(t,u)\right| \le a\left(|u|\right)b(t)$$

for all $u \in R^N$ and a.e. $t \in [0,T]$.

(P) $p(t) \in C([0,T],R^+)$, $p(t) = p(t+T)$ and

$$1 < p^- := \min p(t) \le p^+ := \max p(t) < +\infty.$$

The $p(t)$-Laplacian system can be applied to describe the physical phenomena with 'pointwise different properties' which first arose from the nonlinear elasticity theory (see [1]).

If $p(t) = p$ is a constant, system (1) reduces to the p-Laplacian system

$$\begin{cases} (|u'(t)|^{p-2}u'(t))' + \nabla F(t,u(t)) = 0, & t \in [0,T], \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{2}$$

Especially, when $p = 2$, system (1) or (2) becomes the well-known second-order Hamiltonian system

$$\begin{cases} u''(t) + \nabla F(t,u(t)) = 0, & t \in [0,T], \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{3}$$

In 1978, Rabinowitz [2] published his pioneer paper on the existence of periodic solutions for problem (3) under the following Ambrosetti-Rabinowitz superquadratic condition:

(AR) There exist $\mu > 2$ and $L^* > 0$ such that

$$0 < \mu F(t,u) \leq \big(\nabla F(t,u), u\big)$$

for all $|u| \geq L^*$ and a.e. $t \in [0,T]$.

From then on, many researchers have tried to replace the Ambrosetti-Rabinowitz (shortened AR) condition by other superquadratic conditions. Some new superquadratic conditions under which there exist periodic solutions for problem (3) have been discovered in literature, see, for example, the references [3–5]. In [5], the authors obtained the following existence theorem for (3) under the 'local superquadratic conditions'.

Theorem A ([5], Theorem 1.1) *Suppose that $F(t,u)$ satisfies (F_0) and the following conditions*:

(V_1) $F(t,u) \geq 0$ *for all* $t \in [0,T]$ *and* $u \in R^N$;

(V_2) *There are constants* $m > 0$ *and* $\alpha \leq \frac{6m^2}{T^2}$ *such that* $F(t,u) \leq \alpha$ *for all* $u \in R^N$, $|u| < m$ *and a.e.* $t \in [0,T]$.

(V_3) *There are constants* $\mu > 2$, $1 \leq \gamma < 2$, $G > 0$ *and the function* $d(t) \in L^1([0,T],R^+)$ *such that*

$$\mu F(t,u) \leq \big(\nabla F(t,u), u\big) + d(t)|u|^\gamma$$

for all $u \in R^N$, $|u| \geq G$ *and a.e.* $t \in [0,T]$.

(V_4) *There exist a constant* $M > 0$ *and a subset* E *of* $[0,T]$ *with* meas$(E) > 0$ *such that*
   (a) $\liminf_{|u|\to\infty} \frac{F(t,u)}{|u|^2} > 0$ *uniformly for a.e.* $t \in E$;
   (b) $d(t) \leq M$ *for a.e.* $t \in E$.

Then problem (3) *has at least one nonconstant* T*-periodic solution*.

Recently, in [6], the authors extended the above result of [5] to system (2) and got the following theorem for (2).

Theorem B ([6], Theorem 1.4) *Suppose that* $F(t,u)$ *satisfies* (F_0), (V_1) *and the following conditions*:

(H_1) $\liminf_{|u|\to 0} \frac{F(t,u)}{|u|^p} = 0$ *uniformly for a.e.* $t \in [0,T]$;

(H_2) *There are constants* $\mu > p$, $G > 0$ *and the function* $d(t) \in L^1([0,T],R)$ *such that*

$$\mu F(t,u) - \big(\nabla F(t,u), u\big) \le d(t)|u|^p$$

for all $u \in R^N$, $|u| \ge G$ *and a.e.* $t \in [0,T]$, *and*

$$\limsup_{|u|\to\infty} \frac{\mu F(t,u) - (\nabla F(t,u), u)}{|u|^p} \le 0$$

uniformly for a.e. $t \in [0,T]$;

(H_3) *There exists a subset* E *of* $[0,T]$ *with* $\operatorname{meas}(E) > 0$ *such that*

$$\liminf_{|u|\to\infty} \frac{F(t,u)}{|u|^p} > 0$$

uniformly for a.e. $t \in E$.

Then problem (2) *has at least one nonconstant* T*-periodic solution.*

On the other hand, the $p(t)$-Laplacian system has been studied by many authors in the last two decades, see, for example, [7–11] and the references cited therein. In [8], by using linking methods, the authors obtained an existence result under the AR condition as follows.

Theorem C ([8], Theorem 4.1) *Suppose that conditions* (P) *and* (F_0) *hold and* $F(t,u)$ *satisfies the following conditions*:

(A_1) $F(0,0) = 0$ *and* $F(t,u) \ge 0$ *for all* $t \in [0,T]$ *and* $u \in R^N$;

(A_2) *There are constants* $\mu > P^+$ *and* $G > 0$ *such that*

$$\mu F(t,u) \le \big(\nabla F(t,u), u\big)$$

for all $u \in R^N$, $|u| \ge G$ *and a.e.* $t \in [0,T]$;

(A_3) *There exist* $\nu > P^+$ *and* $g \in C([0,T],R)$ *such that*

$$\limsup_{|u|\to\infty} \frac{F(t,u)}{|u|^\nu} \le \big|g(t)\big|.$$

Then problem (1) *has at least one periodic solution.*

Moreover, in [12–14], the authors studied a superlinear elliptic equation with $p(x)$-Laplacian without the AR condition and obtained some existence results.

Motivated by the papers [3, 5, 6, 8, 12], we aim in this paper to study the existence of nonconstant periodic solutions of system (1) with local 'superquadratic' potential and without the AR condition. We get an existence result which generalizes the above Theorem A and Theorem B and extends Theorem C. That is the following theorem.

Theorem 1 *Suppose that conditions* (P) *and* (F_0) *hold and, in addition,* $F(t,u)$ *satisfies the following conditions*:

(F_1) $F(t,u) \geq 0$ *for all* $t \in [0,T]$ *and* $u \in R^N$;

(F_2) $\liminf_{|u|\to 0} \frac{F(t,u)}{|u|^{p^+}} = 0$ *uniformly for a.e.* $t \in [0,T]$;

(F_3) *There are constants* $\mu > p^+$, $G > 0$ *and the function* $d(t) \in L^1([0,T],R)$ *such that*

$$\mu F(t,u) - \big(\nabla F(t,u),u\big) \leq d(t)|u|^{p^-}$$

for all $u \in R^N$, $|u| \geq G$ *and a.e.* $t \in [0,T]$, *and*

$$\limsup_{|u|\to\infty} \frac{\mu F(t,u) - (\nabla F(t,u),u)}{|u|^{p^-}} \leq 0$$

uniformly for a.e. $t \in [0,T]$;

(F_4) *There exists a subset* Ω *of* $[0,T]$ *with* $\operatorname{meas}(\Omega) > 0$ *such that*

$$\liminf_{|u|\to\infty} \frac{F(t,u)}{|u|^{p^+}} > 0$$

uniformly for a.e. $t \in \Omega$.

Then problem (1) has at least one nonconstant T-periodic solution.

Example Set $p(t) = \frac{5}{2} + \sin(\frac{2\pi}{T}t - \frac{\pi}{2})$, then $p(t)$ satisfies condition (P) and $p^- = \frac{3}{2}$, $p^+ = \frac{7}{2}$. Let

$$\psi(t) = \begin{cases} \sin(\frac{2\pi}{T}t), & t \in [0,\frac{T}{2}], \\ 0, & t \in [\frac{T}{2},T], \end{cases}$$

$$\phi(u) = \begin{cases} |u|^4, & |u| \leq 1, u \in R^N, \\ \frac{16}{5}|u|^{\frac{5}{4}}, & |u| > 1, u \in R^N, \end{cases}$$

$$F(t,u) = \psi(t)|u|^4 + \phi(u), \quad t \in [0,T], u \in R^N.$$

It is clear that (F_1) and (F_2) hold. Let $\mu = 4$, $G = 1$, then (F_3) holds. And take $\Omega = [\frac{T}{8}, \frac{3T}{8}]$, then (F_4) holds for $t \in \Omega$. Therefore, F satisfies all the conditions of our Theorem 1. Moreover, it is easy to verify that the function $F(t,u)$ does not satisfy the AR condition A_2 in Theorem C for $t \in [\frac{T}{2}, T]$.

2 Preliminaries

For the reader's convenience, we first give some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces. We can refer the reader to [8, 15–19]. In the following, we use $|\cdot|$ to denote the Euclidean norm in R^N.

Let $p(t)$ satisfy condition (P) and define

$$L^{p(t)}\big([0,T];R^N\big) = \left\{u \in L^1\big([0,T];R^N\big) : \int_0^T |u|^{p(t)}\,dt < \infty\right\},$$

with the norm

$$|u|_{L^{p(t)}} = |u|_{p(t)} = \inf\left\{\lambda > 0 : \int_0^T \left|\frac{u}{\lambda}\right|^{p(t)} dt \le 1\right\}.$$

Define

$$C_T^\infty = C_T^\infty(R;R^N) = \{u \in C^\infty(R;R^N) : u \text{ is } T\text{-periodic}\}.$$

For $u \in L^1([0,T];R^N)$, if there exists $v \in L^1([0,T];R^N)$ satisfying

$$\int_0^T v\varphi\,dt = -\int_0^T u\varphi'\,dt, \quad \forall \varphi \in C_T^\infty,$$

then v is called the T-weak derivative of u and is denoted by u'. Define

$$W_T^{1,p(t)}([0,T];R^N) = \{u \in L^{p(t)}([0,T];R^N) : u' \in L^{p(t)}([0,T];R^N)\}$$

with the norm

$$\|u\|_{W_T^{1,p(t)}} = \|u\| = |u|_{p(t)} + |u'|_{p(t)}.$$

For $u \in W_T^{1,p(t)}([0,T];R^N)$, let

$$\bar{u} = \frac{1}{T}\int_0^T x(s)\,ds, \qquad \tilde{u}(t) = u(t) - \bar{u}$$

and

$$\widetilde{W}_T^{1,p(t)}([0,T];R^N) = \left\{x \in W_T^{1,p(t)}([0,T];R^N) : \int_0^T x(s)\,ds = 0\right\},$$

then

$$W_T^{1,p(t)}([0,T];R^N) = \widetilde{W}_T^{1,p(t)}([0,T];R^N) \oplus R^N.$$

In the following we use $L^{p(t)}$, $W_T^{1,p(t)}$, $\widetilde{W}_T^{1,p(t)}$ to denote $L^{p(t)}([0,T];R^N)$, $W_T^{1,p(t)}([0,T];R^N)$, $\widetilde{W}_T^{1,p(t)}([0,T];R^N)$, respectively.

Proposition 1 ([7]) *For $u \in L^{p(t)}$, one has*

$$(1) \quad |u|_{p(t)} < 1 (= 1; > 1) \quad \Leftrightarrow \quad \int_0^T |u(t)|^{p(t)} dt < 1 (= 1; > 1);$$

$$(2) \quad |u|_{p(t)} > 1 \quad \Rightarrow \quad |u|_{p(t)}^{p^-} \le \int_0^T |u(t)|^{p(t)} dt \le |u|_{p(t)}^{p^+};$$

$$|u|_{p(t)} < 1 \quad \Rightarrow \quad |u|_{p(t)}^{p^+} \le \int_0^T |u(t)|^{p(t)} dt \le |u|_{p(t)}^{p^-};$$

$$(3)\quad |u|_{p(t)} \to 0 \quad \Leftrightarrow \quad \int_0^T \left|u(t)\right|^{p(t)}\,dt \to 0;$$
$$|u|_{p(t)} \to \infty \quad \Leftrightarrow \quad \int_0^T \left|u(t)\right|^{p(t)}\,dt \to \infty.$$

Proposition 2 ([7]) *The spaces $L^{p(t)}$ and $W_T^{1,p(t)}$ are separable and reflexive Banach spaces when $p^- > 1$.*

Proposition 3 ([7]) *There is a continuous embedding $W_T^{1,p(t)} \hookrightarrow C([0,T];R^N)$; when $p^- > 1$, it is a compact embedding.*

Proposition 4 ([7]) *For every $u \in \widetilde{W}_T^{1,p(t)}$, there is a constant C independent of u such that*

$$\|u\|_\infty \le C\left|u'\right|_{p(t)}.$$

Proposition 5 ([7]) *Let $u = \bar{u} + \tilde{u} \in W_T^{1,p(t)}$, then the norm $|\tilde{u}'|_{p(t)}$ is an equivalent norm on $\widetilde{W}_T^{1,p(t)}$ and $|\bar{u}| + |u'|_{p(t)}$ is an equivalent norm on $W_T^{1,p(t)}$.*

To prove the main theorem of the paper, we need the following generalized mountain pass theorem.

Lemma 1 ([20]) *Let E be a real Banach space with $E = V \oplus X$, where $V \neq 0$ is finite dimensional. Suppose $\varphi \in C^1(E,R)$ satisfies the (PS) condition, and*

(a) *There exist $\rho, \alpha > 0$ such that $\varphi|_{\partial B_\rho \cap X} \ge \alpha$, where $B_\rho = \{u \in E | \|u\|_E \le \rho\}$, ∂B_ρ denotes the boundary of B_ρ;*
(b) *There exist $e \in \partial B_1 \cap X$ and $r > \rho$ such that if $Q \equiv (\bar{B}_r \cap V) \oplus \{se | 0 \le s \le r\}$, then $\varphi|_{\partial Q} \le \frac{\alpha}{2}$.*

Then φ possesses a critical value $c \ge \alpha$ which can be characterized as

$$c \equiv \inf_{h\in\Gamma} \max_{u\in Q} \varphi\bigl(h(u)\bigr),$$

where $\Gamma = \{h \in C(\bar{Q}, E) | h = id \text{ on } \partial Q\}$, and id denotes the identity operator.

3 Proof of Theorem 1

Define a functional φ on $W_T^{1,p(t)}$ by

$$\varphi(u) = \int_0^T \frac{1}{p(t)}\left|u'(t)\right|^{p(t)}\,dt - \int_0^T F\bigl(t,u(t)\bigr)\,dt$$

for each $u \in W_T^{1,p(t)}$. It follows from assumption (F_0) that the functional φ is continuously differentiable on $W_T^{1,p(t)}$. Moreover, we have

$$\bigl\langle \varphi'(u), v\bigr\rangle = \int_0^T \bigl(\left|u'(t)\right|^{p(t)-2}u'(t), v'(t)\bigr)\,dt - \int_0^T \bigl(\nabla F\bigl(t,u(t)\bigr), v(t)\bigr)\,dt$$

for all $u, v \in W_T^{1,p(t)}$. And it is well known (see [7]) that the problem of finding a T-periodic solution of system (1) is equal to that of finding the critical of functional φ.

We shall apply Lemma 1 to φ to prove Theorem 1.

For the convenience to verify the (PS) condition, we need the following lemma. The proof can be found in [7] or [8].

Lemma 2 *Let* $J(u) = \int_0^T \frac{1}{p(t)}|u'(t)|^{p(t)}\,dt$ *for* $u \in W_T^{1,p(t)}$. *Then* $\langle J'(u), v\rangle = \int_0^T |(|u'(t)|^{p(t)-2} \times u'(t), v'(t))\,dt$ *for all* $u, v \in W_T^{1,p(t)}$. *And* J' *is a mapping of type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *and* $\limsup_{n\to\infty}\langle J'(u_n) - J'(u), u_n - u\rangle \le 0$, *then* $\{u_n\}$ *has a convergent subsequence in* $W_T^{1,p(t)}$.

In the following lemma we will show that φ satisfies the (PS) condition.

Lemma 3 *The functional* φ *satisfies the* (PS) *condition, i.e., for every sequence* $\{u_n\} \in W_T^{1,p(t)}$, $\{u_n\}$ *has a convergent subsequence if*

$$\big\{\varphi(u_n)\big\} \quad \text{is bounded and} \quad \varphi'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{4}$$

Proof First we prove that $\{u_n\}$ is a bounded sequence in $W_T^{1,p(t)}$. Otherwise, $\{u_n\}$ would be unbounded. Passing to a subsequence, we may assume that $\|u_n\| \to \infty$. Let $w_n = \frac{u_n}{\|u_n\|}$, so that $\|w_n\| = 1$. By Proposition 3, also passing to a subsequence, we can suppose that

$$\begin{aligned} &w_n \rightharpoonup w \quad \text{weakly in } W_T^{1,p(t)}, \\ &w_n \to w \quad \text{strongly in } C\big([0,T]; R^N\big) \end{aligned}$$

as $n \to \infty$. Moreover, we have

$$\bar{w}_n = \frac{1}{T}\int_0^T w_n(t)\,dt \to \frac{1}{T}\int_0^T w(t)\,dt = \bar{w} \tag{5}$$

as $n \to \infty$. By (4) there exists a constant $C_1 > 0$ such that

$$\begin{aligned} &\int_0^T \left(\frac{\mu}{p(t)} - 1\right)\big|u_n'(t)\big|^{p(t)}\,dt \\ &\quad = \mu\varphi(u_n) - \big\langle\varphi'(u_n), u_n\big\rangle + \int_0^T \Big(\mu F\big(t, u_n(t)\big) - \big(\nabla F\big(t, u_n(t)\big), u_n(t)\big)\Big)\,dt \\ &\quad \le C_1\big(1 + \|u_n\|\big) + \int_0^T \Big(\mu F\big(t, u_n(t)\big) - \big(\nabla F\big(t, u_n(t)\big), u_n(t)\big)\Big)\,dt. \end{aligned}$$

Notice that $\|u_n\| \to \infty$, we have

$$\begin{aligned} \left(\frac{\mu}{p^+} - 1\right)\int_0^T \big|w_n'(t)\big|^{p(t)}\,dt &= \left(\frac{\mu}{p^+} - 1\right)\int_0^T \frac{|u_n'(t)|^{p(t)}}{\|u_n\|^{p(t)}}\,dt \\ &\le \int_0^T \left(\frac{\mu}{p(t)} - 1\right)\frac{|u_n'(t)|^{p(t)}}{\|u_n\|^{p^-}}\,dt. \end{aligned}$$

So, we obtain

$$\begin{aligned} &\left(\frac{\mu}{p^+} - 1\right)\int_0^T \big|w_n'(t)\big|^{p(t)}\,dt \\ &\quad \le \frac{C_1(1 + \|u_n\|)}{\|u_n\|^{p^-}} + \int_0^T \frac{(\mu F(t, u_n(t)) - (\nabla F(t, u_n(t)), u_n(t)))}{\|u_n\|^{p^-}}\,dt. \end{aligned} \tag{6}$$

In view of (F_0) and (F_3), there exists $\Omega_0 \subset [0,T]$ with $\text{meas}(\Omega_0) = 0$ such that

$$\bigl|F(t,u)\bigr| \le a\bigl(|u|\bigr)b(t), \qquad \nabla\bigl|F(t,u)\bigr| \le a\bigl(|u|\bigr)b(t) \tag{7}$$

for all $u \in R^N$ and $t \in [0,T] \setminus \Omega_0$ and

$$\limsup_{|u|\to\infty} \frac{\mu F(t,u) - (\nabla F(t,u),u)}{|u|^{p^-}} \le 0$$

uniformly for $t \in [0,T] \setminus \Omega_0$. This yields

$$\limsup_{n\to\infty} \frac{\mu F(t,u_n(t)) - (\nabla F(t,u_n(t)),u_n(t))}{\|u_n\|^{p^-}} \le 0 \tag{8}$$

for $t \in [0,T] \setminus \Omega_0$. Otherwise, there exist $t_0 \in [0,T] \setminus \Omega_0$ and a subsequence of u_n, still denoted by u_n, such that

$$\limsup_{n\to\infty} \frac{\mu F(t_0,u_n(t_0)) - (\nabla F(t_0,u_n(t_0)),u_n(t_0))}{\|u_n\|^{p^-}} > 0. \tag{9}$$

If $\{u_n(t_0)\}$ is bounded, then there exists a positive constant C_2 such that $|u_n(t_0)| \le C_2$ for all $n \in \mathbf{N}$. By (7) we find

$$\begin{aligned}
&\frac{\mu F(t_0,u_n(t_0)) - (\nabla F(t_0,u_n(t_0)),u_n(t_0))}{\|u_n\|^{p^-}} \\
&\quad \le \frac{(\mu + C_2)\max_{0\le s\le C_2} a(s)b(t_0)}{\|u_n\|^{p^-}} \to 0
\end{aligned}$$

as $n \to \infty$, which contradicts (9). So, there is a subsequence of $u_n(t_0)$, still denoted by $u_n(t_0)$, such that $|u_n(t_0)| \to \infty$ as $n \to \infty$.

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\mu F(t_0,u_n(t_0)) - (\nabla F(t_0,u_n(t_0)),u_n(t_0))}{\|u_n\|^{p^-}} \\
&\quad = \limsup_{n\to\infty} \frac{\mu F(t_0,u_n(t_0)) - (\nabla F(t_0,u_n(t_0)),u_n(t_0))}{|u_n(t_0)|^{p^-}} \bigl|w_n(t_0)\bigr|^{p^-} \\
&\quad = \limsup_{n\to\infty} \frac{\mu F(t_0,u_n(t_0)) - (\nabla F(t_0,u_n(t_0)),u_n(t_0))}{|u_n(t_0)|^{p^-}} \lim_{n\to\infty} \bigl|w_n(t_0)\bigr|^{p^-} \\
&\quad \le 0.
\end{aligned}$$

This contradicts (9). Thus, (8) holds. From (6) and (8) we obtain

$$\limsup_{n\to\infty} \left(\frac{\mu}{p^+} - 1\right) \int_0^T \bigl|w_n'(t)\bigr|^{p(t)}\,dt \le 0.$$

Since $\mu > p^+$, we get

$$\lim_{n\to\infty} \left(\frac{\mu}{p^+} - 1\right) \int_0^T \bigl|w_n'(t)\bigr|^{p(t)}\,dt = 0. \tag{10}$$

Combining with (5), this yields

$$w_n \to \bar{w} \quad \text{as } n \to \infty,$$

which means that

$$w = \bar{w} \quad \text{and} \quad T|\bar{w}| = \|w\| = 1.$$

Then we have

$$u_n(t) \to \infty \quad \text{as } n \to \infty$$

uniformly for a.e. $t \in [0, T]$. And we get from (F_1), (F_4) and Fatou's lemma that

$$\begin{aligned}
&\liminf_{n\to\infty} \int_0^T \frac{F(t,u_n(t))}{\|u_n\|^{p^+}}\,dt \\
&\quad \geq \int_0^T \liminf_{n\to\infty} \frac{F(t,u_n(t))}{\|u_n\|^{p^+}}\,dt \\
&\quad = \int_0^T \liminf_{n\to\infty} \frac{F(t,u_n(t))}{|u_n(t)|^{p^+}} \left|w_n(t)\right|^{p^+}\,dt \\
&\quad \geq \int_\Omega \liminf_{n\to\infty} \frac{F(t,u_n(t))}{|u_n(t)|^{p^+}} |w_0|^{p^+}\,dt > 0.
\end{aligned} \tag{11}$$

On the other hand, we have

$$\begin{aligned}
\int_0^T \frac{F(t,u_n(t))}{\|u_n\|^{p^+}}\,dt &= \int_0^T \frac{1}{p(t)} \frac{|u_n'(t)|^{p(t)}}{\|u_n\|^{p^+}}\,dt - \frac{\varphi(u_n)}{\|u_n\|^{p^+}} \\
&\leq \frac{1}{p^-} \int_0^T \left|\frac{u_n'(t)}{\|u_n\|}\right|^{p(t)}\,dt - \frac{\varphi(u_n)}{\|u_n\|^{p^+}} \\
&= \frac{1}{p^-} \int_0^T \left|w_n'(t)\right|^{p(t)}\,dt - \frac{\varphi(u_n)}{\|u_n\|^{p^+}}.
\end{aligned}$$

Therefore, combining (4) and (10), we obtain that

$$\liminf_{n\to\infty} \int_0^T \frac{F(t,u_n(t))}{\|u_n\|^{p^+}}\,dt \leq 0,$$

which contradicts (11). Hence, $\{u_n\}$ is a bounded sequence in $W_T^{1,p(t)}$.

By Proposition 2 and Proposition 3, $\{u_n\}$ has a subsequence, again denoted by $\{u_n\}$, such that

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{weakly in } W_T^{1,p(t)}, \\
&u_n \to u \quad \text{strongly in } C\big([0,T];R^N\big).
\end{aligned} \tag{12}$$

Now, we will show that $\{u_n\}$ has a subsequence convergent strongly to u in $W_T^{1,p(t)}$. From Lemma 2 it suffices to prove that $\limsup_{n\to\infty}\langle J'(u_n) - J'(u), u_n - u\rangle \leq 0$.

It follows from Proposition 3 that $\max_{0\le t\le T}|u_n(t)|\le C_3\|u_n\|$, which implies

$$\left|u_n(t)\right|\le C_4 \quad \text{for all } t\in[0,T]. \tag{13}$$

From (12), (13) and (F_0), we get

$$\begin{aligned}
&\left|\int_0^T \left(\nabla F\left(t,u_n(t)\right),u_n(t)-u(t)\right)dt\right| \\
&\quad\le \int_0^T \left|\nabla F\left(t,u_n(t)\right)\right|\left|u_n(t)-u(t)\right|dt \\
&\quad\le \|u_n-u\|_\infty \int_0^T a\left(\left|u_n(t)\right|\right)b(t)\,dt \\
&\quad\le C_5\|u_n-u\|_\infty \int_0^T b(t)\,dt.
\end{aligned}$$

Thus, from (12), we obtain

$$\left|\int_0^T \left(\nabla F\left(t,u_n(t)\right),u_n(t)-u(t)\right)dt\right|\to 0. \tag{14}$$

By (4) and (13), we have

$$\left\langle\varphi'(u_n),u_n-u\right\rangle\to 0. \tag{15}$$

Then it follows from (14) and (15) that

$$\begin{aligned}
\left\langle J'(u_n),u_n-u\right\rangle &= \int_0^T \left(\left|u_n'(t)\right|^{p(t)-2}u_n'(t),u_n'(t)-u'(t)\right)dt \\
&= \left\langle\varphi'(u_n),u_n-u\right\rangle + \int_0^T \left(\nabla F\left(t,u_n(t)\right),u_n(t)-u(t)\right)dt \\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned} \tag{16}$$

Moreover, since $J'(u)\in(W_T^{1,p(t)})^*$, we get $\langle\varphi'(u),u_n-u\rangle\to 0$, which combined with (16) implies that

$$\lim_{n\to\infty}\left\langle J'(u_n)-J'(u),u_n-u\right\rangle\le 0.$$

Hence, from Lemma 2, $\{u_n\}$ has a subsequence convergent strongly to u in $W_T^{1,p(t)}$. The proof of the lemma is completed. □

The following result establishes the generalized mountain pass geometry for the functional $\varphi(u)$.

Lemma 4 *Let* $W_T^{1,p(t)}=R^N\oplus\widetilde{W}_T^{1,p(t)}$, $\mathbf{B}_r=\{u\in W_T^{1,p(t)}\,|\,\|u\|\le r\}$, $\mathbf{S}_r=\widetilde{W}_T^{1,p(t)}\cap\partial\mathbf{B}_r$. *Then there exist* $\rho>0$ *and* $\alpha>0$ *such that*

$$\inf_{u\in\mathbf{S}_\rho}\varphi(u)>\alpha.$$

And there exist $r_1 > 0, r_2 > \rho$ *and* $e \in \widetilde{W}_T^{1,p(t)}$ *such that*

$$\sup_{u \in \partial \mathbf{Q}} \varphi(u) \leq 0,$$

where $\mathbf{Q} = \{u + se | u \in R^N \cap \mathbf{B}_{r_1}, s \in [0, r_2]\}$.

Proof Firstly, we show that there exists $\rho > 0$ such that $\inf_{u \in \mathbf{S}_\rho} \varphi(u) > 0$. Let C be the constant in Proposition 4. By condition (F_2), we know that for any positive constant $\epsilon < \min\{C, \frac{1}{p^+ TC^{p^+}}\}$, there exists $\delta \in (0, \epsilon)$ such that

$$\left|F(t,u)\right| \leq \epsilon |u|^{p^+} \tag{17}$$

for all $|u| \leq \delta$ and a.e. $t \in [0, T]$. Let $0 < \rho \leq \frac{\delta}{C}$ and by Proposition 5 set $\mathbf{S}_\rho = \{u \in \widetilde{W}_T^{1,p(t)} | |u'|_{p(t)} = \rho\}$. By Proposition 4, we get $|u(t)| \leq C|u'|_{p(t)} = C\rho = \delta$. Since $\rho < 1$, then it follows from Proposition 1 and (17) that

$$\begin{aligned}
\varphi(u) &= \int_0^T \frac{1}{p(t)} \left|u'(t)\right|^{p(t)} dt - \int_0^T F\bigl(t, u(t)\bigr)\, dt \\
&\geq \frac{1}{p^+} \int_0^T \left|u'(t)\right|^{p(t)} dt - \epsilon \int_0^T \left|u(t)\right|^{p^+} dt \\
&\geq \frac{1}{p^+} \left|u'\right|_{p(t)}^{p^+} - \epsilon TC^{p^+} \left|u'\right|_{p(t)}^{p^+} \\
&= \left(\frac{1}{p^+} - \epsilon TC^{p^+}\right) \rho^{p^+} = \alpha > 0.
\end{aligned}$$

Secondly, we prove that there exist $r_1 > 0$, $r_2 > \rho$ and $e \in \widetilde{W}_T^{1,p(t)}$ such that $\sup_{u \in \partial \mathbf{Q}} \varphi(u) \leq 0$. By (F_3) and (F_4) there exist constants $C_6 > \max\{1, G\}$, $\eta > 0$ and a subset of Ω, still denoted by Ω, with $|\Omega| = \text{meas}(\Omega) > 0$ such that

$$\mu F(t,u) - \bigl(\nabla F(t,u), u\bigr) \leq \eta |u|^{p^-} \tag{18}$$

and

$$F(t,u) > \frac{2\eta}{\mu - p^-} |u|^{p^+} \tag{19}$$

for all $|u| \geq C_6$ and $t \in \Omega$. For $u \in R^N \setminus \{0\}$ and $t \in \Omega$, let $f(s) = F(t, su)$ for all $s \geq \frac{C_6}{|u|}$. We deduce from (18) that

$$\begin{aligned}
f'(s) &= \frac{1}{s} \bigl(\nabla F(t, su), su\bigr) \\
&\geq \frac{\mu}{s} F(t, su) - \eta s^{p^- - 1} |u|^{p^-} \\
&= \frac{\mu}{s} f(s) - \eta s^{p^- - 1} |u|^{p^-},
\end{aligned}$$

which yields

$$g(s) = f'(s) - \frac{\mu}{s} f(s) + \eta s^{p^- - 1} |u|^{p^-} \geq 0 \tag{20}$$

for all $s \geq \frac{C_6}{|u|}$. From (20) we have

$$f(s) = \left(\int_{\frac{C_6}{|u|}}^{s} \frac{g(r) - \eta r^{p^- -1}|u|^{p^-}}{r^{\mu}}\,dr + \left(\frac{|u|}{C_6}\right)^{\mu} f\left(\frac{C_6}{|u|}\right)\right)s^{\mu} \tag{21}$$

for all $s \geq \frac{C_6}{|u|}$. It follows from (21) and (20) that

$$\begin{aligned} f(s) &\geq \left(\left(\frac{|u|}{C_6}\right)^{\mu} f\left(\frac{C_6}{|u|}\right) + \frac{\eta|u|^{p^-}}{(\mu - p^-)s^{\mu-p^-}} - \frac{\eta|u|^{\mu}}{(\mu - p^-)C_6^{\mu-p^-}}\right)s^{\mu} \\ &\geq \left(F\left(t, \frac{C_6}{|u|}u\right) - \frac{\eta C_6^{p^-}}{\mu - p^-}\right)\left(\frac{|u|}{C_6}\right)^{\mu} s^{\mu}. \end{aligned}$$

Combining this with (19) yields

$$\begin{aligned} F(t,u) &= f(1) \geq \left(F\left(t, \frac{C_6}{|u|}u\right) - \frac{\eta C_6^{p^-}}{\mu - p^-}\right)\left(\frac{|u|}{C_6}\right)^{\mu} \\ &\geq \left(\frac{2\eta C_6^{p^+-\mu}}{\mu - p^-} - \frac{\eta C_6^{p^--\mu}}{\mu - p^-}\right)|u|^{\mu} \\ &\geq C_7|u|^{\mu} \end{aligned}$$

for all $|u| \geq C_6$ and $t \in \Omega$, where $C_7 = \frac{\eta}{\mu-p^-}(\frac{2}{C_6^{\mu-p^+}} - \frac{1}{C_6^{\mu-p^-}}) > 0$. So, notice that $F(t,u) \geq 0$, we get

$$F(t,u) \geq C_7\left(|u|^{\mu} - C_6^{\mu}\right) = C_7|u|^{\mu} - C_8 \tag{22}$$

for all $u \in R^N$ and $t \in \Omega$.

Choose $e(t) \in \widetilde{W}_T^{1,p(t)}$ with $\|e(t)\| = 1$ such that $e(t) = 0$ for all $t \in [0,t] \setminus \Omega$. Therefore, one has

$$\int_{\Omega} e(t)\,dt = \int_0^T e(t)\,dt - \int_{[0,t]\setminus\Omega} e(t)\,dt = 0,$$

which implies that

$$\int_{\Omega} \big(u, e(t)\big)\,dt = \int_0^T \big(u, e(t)\big)\,dt - \int_{[0,t]\setminus\Omega} \big(u, e(t)\big)\,dt = 0 \tag{23}$$

for all $u \in R^N$. Let $\overline{W}_T^{1,p(t)} = R^N \oplus \operatorname{span}\{e(t)\}$. Since $\dim(\overline{W}_T^{1,p(t)}) < \infty$, all the norms are equivalent. For any $v = u + se(t) \in \overline{W}_T^{1,p(t)}$, there exists a positive constant K such that

$$\|v\|_{L^{\mu}(\Omega)} \geq K\|v\|_{L^2(\Omega)}. \tag{24}$$

Denoting $E_1 = \int_0^T |e'(t)|^{p(t)}\,dt$, $E_2 = \int_\Omega |e(t)|^2\,dt$, by (22), (23), (24) and (F_1), we get

$$\begin{aligned}
\varphi(u+se) &= \int_0^T \frac{1}{p(t)}\left|se'(t)\right|^{p(t)}\,dt - \int_0^T F\bigl(t,u+se(t)\bigr)\,dt \\
&\le \frac{1}{p^-}\int_0^T \left|se'(t)\right|^{p(t)}\,dt - \int_\Omega F\bigl(t,u+se(t)\bigr)\,dt \\
&\le \frac{1}{p^-}\int_0^T \left|se'(t)\right|^{p(t)}\,dt - C_7\int_\Omega \left|u+se(t)\right|^{\mu}\,dt + C_8|\Omega| \\
&\le \frac{1}{p^-}\int_0^T \left|se'(t)\right|^{p(t)}\,dt - C_7K^{\mu}\left(\int_\Omega \left|u+se(t)\right|^2\,dt\right)^{\frac{\mu}{2}} + C_8|\Omega| \\
&= \frac{1}{p^-}\int_0^T \left|se'(t)\right|^{p(t)}\,dt - C_7K^{\mu}\left(\int_\Omega \left(|u|^2 + s^2\left|e(t)\right|^2\right)\,dt\right)^{\frac{\mu}{2}} + C_8|\Omega| \\
&\le \frac{1}{p^-}\int_0^T \left|se'(t)\right|^{p(t)}\,dt - C_7K^{\mu}|u|^{\mu}|\Omega|^{\frac{\mu}{2}} - C_7K^{\mu}s^{\mu}|E_2|^{\frac{\mu}{2}} + C_8|\Omega|.
\end{aligned}$$

Therefore, when $s > 1$, we have

$$\varphi(u+se) \le \frac{E_1}{p^-}s^{p^+} - C_7K^{\mu}|E_2|^{\frac{\mu}{2}}s^{\mu} + C_8|\Omega|.$$

Since $\mu > p^+$, there exists $r_2 > \max\{1,\rho\}$ such that

$$\varphi(u+se) \le 0 \quad \text{for all } u \in R^N \text{ and } s = r_2. \tag{25}$$

Moreover, for all $u \in R^N$ and $0 \le s \le r_2$, we have

$$\varphi(u+se) \le \frac{E_1 r_2^{p^+}}{p^-} - C_7K^{\mu}|u|^{\mu}|\Omega|^{\frac{\mu}{2}} + C_8|\Omega|.$$

This deduces that

$$\varphi(u+se) \le 0 \quad \text{when } |u|^{\mu} \ge \frac{E_1 r_2^{p^+} + C_8|\Omega|p^-}{C_7K^{\mu}|\Omega|^{\frac{\mu}{2}}p^-}.$$

Let $u \in R^N$, $|u| \ge 1$, from Proposition 4, we know that

$$|u|^{\mu}T = \int_0^T |u|^{\mu}\,dt \ge \int_0^T |u|^{p(t)}\,dt \ge |u|_{p(t)}^{p^-}.$$

So, let r_1 satisfy

$$r_1^{p^-} \ge \max\left\{1, \frac{E_1 r_2^{p^+} + C_8|\Omega|p^-}{C_7K^{\mu}|\Omega|^{\frac{\mu}{2}}p^-}T\right\},$$

then, when $u \in R^N$, $\|u\| = |u|_{p(t)} = r_1$, we obtain

$$\varphi(u+se) \le 0 \quad \text{for all } s \in [0, r_2]. \tag{26}$$

On the other hand, if $s = 0$, by (F_1), we get

$$\varphi(u+se) = -\int_0^T F(t,u)\,dt \leq 0 \quad \text{for all } u \in R^N. \tag{27}$$

Setting $\mathbf{Q} = \{u + se | u \in R^N \cap \mathbf{B}_{r_1}, s \in [0, r_2]\}$, by (25),(26) and (27), we have

$$\sup_{u\in\partial\mathbf{Q}} \varphi(u) \leq 0. \tag{28}$$

The proof of Lemma 4 is completed. □

Proof of Theorem 1 By Lemma 3 and Lemma 4, applying Lemma 1, then φ possesses a critical point $u(t)$ whose critical value c satisfies $c \geq \alpha > 0$. By F_1, we can see that $u(t)$ is nonconstant. Hence, problem (1) has at least one nonconstant T-periodic solution in $\mathbf{W}_T^{1,p(t)}$. □

4 Conclusions

In this work, we have established an existence result for nonconstant periodic solutions of a class of second-order systems with $p(t)$-Laplacian. For $p(x)$ is a constant p, it is easy to see that the conditions and conclusion in Theorem 1 are the same as those in Theorem 1.4 in [6]. Thus Theorem 1 generalizes Theorem 1.4 in [6] and Theorem 1.1 in [5]. Furthermore, obviously, conditions (F_2) and (F_3) of Theorem 1 are weaker than (A_2) and (A_3) of Theorem 4.1 in [8]. Therefore, Theorem 1 extends Theorem 4.1 in [8].

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Author details
[1]College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing, 210016, P.R. China. [2]College of Science, China Pharmaceutical University, Nanjing, 211198, P.R. China. [3]College of Science, Hohai University, Nanjing, 210098, P.R. China.

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References

1. Zhikov, VV: Averaging of functionals of the calculus of variations and elasticity theory. Math. USSR, Izv. **9**, 33-66 (1987)
2. Rabinowitz, PH: Periodic solutions of Hamiltonian systems. Commun. Pure Appl. Math. **31**, 157-184 (1978)
3. Schechter, M: Periodic non-autonomous second-order dynamical systems. J. Differ. Equ. **223**, 290-302 (2006)
4. Schechter, M: Periodic solution of second order non-autonomous dynamical systems. Bound. Value Probl. **2006**, 25104 (2006)
5. Wang, ZY, Zhang, JH, Zhang, ZT: Periodic solutions of second order non-autonomous Hamiltonian systems with local superquadratic potential. Nonlinear Anal. **70**, 3672-3681 (2009)
6. Li, C, Agarwal, RP, Pu, Y, Tang, CL: Nonconstant periodic solutions for a class of ordinary p-Laplacian systems. Bound. Value Probl. **2016**, 213 (2016)
7. Fan, XL, Fan, X: A Knobloch-type result for $p(t)$-Laplacian systems. J. Math. Anal. Appl. **282**, 453-464 (2003)
8. Wang, XJ, Yuan, R: Existence of periodic solutions for $p(t)$-Laplacian systems. Nonlinear Anal. **70**, 866-880 (2009)
9. Ayazoglu, R, Avci, M: Positive periodic solutions of nonlinear differential equations system with nonstandard growth. Appl. Math. Lett. **43**, 5-9 (2015)
10. Zhang, SG: Periodic solutions for a class of second order Hamiltonian systems with $p(t)$-Laplacian. Bound. Value Probl. **2016**, 211 (2016)
11. Zhang, L, Tang, XH, Chen, J: Infinitely many periodic solutions for some differential systems with $p(t)$-Laplacian. Bound. Value Probl. **2011**, 33 (2011)
12. Tan, Z, Fang, F: On superlinear $p(x)$-Laplacian problems without Ambrosetti and Rabinowitz condition. Nonlinear Anal. **75**, 3902-3915 (2012)
13. Avci, CM: Existence and multiplicity of solutions for Dirichlet problems involving the p(x)-Laplace operator. Electron. J. Differ. Equ. **14**, 197 (2013)
14. Yucedag, Z: Existence of solutions for p(x)-Laplacian equations without Ambrosetti-Rabinowitz type condition. Bull. Malays. Math. Sci. Soc. **38**(3), 1023-1033 (2015)
15. Acerbi, E, Mingione, G: Regularity results for a class of functionals with nonstandard growth. Arch. Ration. Mech. Anal. **156**, 121-140 (2001)
16. Fan, XL, Zhao, D: On the spaces $L^{p(x)}$ and $W^{m,p(x)}$. J. Math. Anal. Appl. **263**, 424-446 (2001)
17. Fan, XL, Shen, JS, Zhao, D: Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$. J. Math. Anal. Appl. **262**, 749-760 (2001)
18. Fan, XL, Zhao, YZ, Zhao, D: Compact imbedding theorems with symmetry of Strauss-Lions type for the space $W^{1,p(x)}(\Omega)$. J. Math. Anal. Appl. **225**, 333-348 (2001)
19. Edmunds, DE, Rakosnik, J: Sobolev embeddings with variable exponent. Math. Nachr. **246-247**, 53-67 (2002)
20. Rabinowitz, PH: Minimax methods in critical point theory with applications to differential equations. In: CBMS Reg. Conf. Ser. in Math., vol. 65. Am. Math. Soc., Providence (1986)