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# The well-posedness of an anisotropic parabolic equation based on the partial boundary value condition

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## Abstract

Consider the anisotropic parabolic equation with the variable exponent

$$u_t = \sum_{i=1}^N (a_i(x) |u_{x_i}|^{p_i(x)-2} u_{x_i})_{x_i},$$

with  $a_i(x), p_i(x) \in C^1(\overline{\Omega})$ ,  $p_i(x) > 1$ ,  $a_i(x) \geq 0$ . If some of  $\{a_i(x)\}$  are degenerate on the boundary, a partial boundary value condition is imposed, the stability of weak solutions can be proved based on the partial boundary value condition.

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**Keywords:** anisotropic parabolic equation; partial boundary value condition; the well-posedness

## 1 Introduction

The equation

$$u_t = \operatorname{div}(a(x) |\nabla u|^{p(x)-2} \nabla u), \quad (x, t) \in Q_T, \quad (1.1)$$

comes from the so-called electrorheological fluids theory (see [1, 2]), where  $\Omega \subset \mathbb{R}^N$  is a bounded domain with smooth boundary  $\partial\Omega$ ,  $p(x) > 1$  is a measurable function. If  $a(x) \equiv 1$ , there are many related papers to study equation (1.1) with the usual initial-boundary value conditions

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (1.2)$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T), \quad (1.3)$$

one can see [3–5] and the references therein.

If  $a(x) > 0$  when  $x \in \Omega$  but  $a(x)|_{x \in \partial\Omega} = 0$ , then the stability of weak solutions can be proved without the boundary value condition (1.3), provided that the diffusion coefficient  $a(x)$  satisfies some other restrictions. One can see our previous works [6–8].

In this paper, we will consider an anisotropic parabolic equation of the type

$$u_t = \sum_{i=1}^N (a_i(x) |u_{x_i}|^{p_i(x)-2} u_{x_i})_{x_i}, \quad (x, t) \in Q_T. \tag{1.4}$$

We denote that

$$p_i^+ = \max_{x \in \bar{\Omega}} p_i(x), \quad p_i^- = \min_{x \in \bar{\Omega}} p_i(x)$$

for any  $i \in \{1, 2, \dots, N\}$  and denote that

$$p_0 = \min\{p_1^-, p_2^-, \dots, p_{N-1}^-, p_N^-\}, \quad p_0 > 1,$$

$$p^0 = \max\{p_1^+, p_2^+, \dots, p_{N-1}^+, p_N^+\}.$$

If  $a_i(x) \equiv 1$ , the existence of a weak solution was proved in [9]. Also, one can refer to the excellent papers [10–12].

Let  $I = \{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, N\}$ ,  $J = \{j_1, j_2, \dots, j_l\} \subset \{1, 2, \dots, N\}$ ,  $k + l = N$ ,  $I \cap J = \emptyset$ . Not only we assume that  $a_i(x) \in C(\bar{\Omega})$ , and when  $x \in \Omega$ ,  $a_i(x) > 0$ , but we also assume that

$$a_{i_1}(x) \geq c_1 > 0, \quad a_{i_2}(x) \geq c_2 > 0, \quad \dots, \quad a_{i_k}(x) \geq c_k > 0, \quad x \in \bar{\Omega}, \tag{1.5}$$

$$a_{j_1}(x) = 0, \quad a_{j_2}(x) = 0, \quad \dots, \quad a_{j_l}(x) = 0, \quad x \in \partial\Omega. \tag{1.6}$$

Besides the initial value condition (1.2), instead of the usual boundary value condition (1.3), by assumptions (1.5)-(1.6), only a partial boundary value condition

$$u(x, t) = 0, \quad (x, t) \in \Sigma_1 \times (0, T) \tag{1.7}$$

should be imposed. To see that, let us give a simple example to show what  $\Sigma_1$  is. Let  $N = 2$ ,  $p_1(x) = p_2(x) \equiv p(x)$ , the domain  $\Omega$  be a square,

$$\Omega = \{(x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < 1\}.$$

Consider the equation

$$u_t = \frac{\partial}{\partial x_1} (a_1(x) |u_{x_1}|^{p(x)-2} u_{x_1}) + \frac{\partial}{\partial x_2} (|u_{x_1}|^{p(x)-2} u_{x_2}). \tag{1.8}$$

Then we conjecture that

$$\Sigma_1 = \{(x_1, x_2) : 0 < x_1 < 1, x_2 = 0\} \cup \{(x_1, x_2) : 0 < x_1 < 1, x_2 = 1\}. \tag{1.9}$$

This conjecture was proved in [13] recently.

However, in general, it is difficult to depict out the geometric character of  $\Sigma_1$ . We have tried to depict out  $\Sigma_1$  by the Fichera function in [14], but it seems not so successful. In this short paper,

$$\Sigma_1 = \left\{ x \in \partial\Omega : \frac{|\prod_{j=1}^l a_j(x)_{x_{i_r}}|^{p_{i_r}(x)}}{[\prod_{j=1}^l a_j(x)]^{p_{i_r}(x)-1}} \neq 0 \right\},$$

we will study the well-posedness of the equation basing on the partial boundary value condition (1.7). Also, we denote that

$$\Sigma_2 = \partial\Omega \setminus \Sigma_1.$$

**Definition 1.1** A function  $u(x, t)$  is said to be a weak solution of equation (1.4) with the initial value condition (1.2) if

$$u \in L^\infty(Q_T), \quad \frac{\partial u}{\partial t} \in L^2(Q_T), \quad a_i(x)|u_{x_i}|^{p_i(x)} \in L^2(0, T; L^1(\Omega)), \quad (1.10)$$

and for any function  $\varphi_1 \in C_0^1(Q_T)$ ,  $\varphi_2 \in L^\infty(Q_T)$  and  $\varphi_{2x_i} \in L^2(0, T; W_{loc}^{1,p_i(x)}(\Omega))$  such that

$$\iint_{Q_T} \left[ \frac{\partial u}{\partial t}(\varphi_1\varphi_2) + \sum_{i=1}^N a_i(x)|u_{x_i}|^{p_i(x)-2}u_{x_i}(\varphi_1\varphi_2)_{x_i} \right] dx dt = 0. \quad (1.11)$$

The initial value condition (1.2) is satisfied in the sense of

$$\lim_{t \rightarrow 0} \int_{\Omega} |u(x, t) - u_0(x)| dx = 0. \quad (1.12)$$

Besides, if the partial boundary value condition (1.7) is satisfied in the sense of the trace, then we say that there is a weak solution of the initial-boundary value problem (1.4)-(1.2)-(1.7).

In this paper, we first study the existence of the weak solution.

**Theorem 1.2** *If  $p_0 > 1$ ,  $a_i(x)$  satisfies conditions (1.5), (1.6),*

$$u_0 \in L^\infty(\Omega), \quad u_{0x_i} \in L^{p_i(x)}(\Omega), \quad (1.13)$$

*then there is a solution of equation (1.4) with the initial value (1.2). Moreover, if for every  $1 \leq r \leq l$ ,  $\int_{\Omega} a_{j_r}^{-1/(p_{j_r}(x)-1)}(x) dx < \infty$ , then the initial-boundary value problem (1.4)-(1.2)-(1.7) has a solution.*

Here,  $L^{p_i(x)}(\Omega)$  is the variable exponent space, its definition is given in Section 2.

Secondly, we will study the stability of weak solutions to the initial-boundary value problem (1.4)-(1.2)-(1.7).

**Theorem 1.3** *If  $u$  and  $v$  are two solutions of equation (1.4) with the same partial boundary value condition (1.7) and with the initial values  $u_0(x)$ ,  $v_0(x)$ , respectively, if  $l > 1$  and for every  $1 \leq r \leq l$ ,*

$$n \left( \int_{\Omega \setminus \Omega_n} a_{j_r}(x) \left| \left( \prod_{s=1}^l a_{j_s}(x) \right)_{x_{j_r}} \right|^{p_{j_r}(x)} dx \right)^{\frac{1}{p_{j_r}^+}} \leq c, \quad (1.14)$$

*then*

$$\int_{\Omega} |u(x, t) - v(x, t)| dx \leq \int_{\Omega} |u_0(x) - v_0(x)| dx. \quad (1.15)$$

*Here,  $\Omega_n = \{x \in \Omega : \prod_{s=1}^l a_{j_s}(x) > \frac{1}{n}\}$ .*

If  $l = 1$  in Theorem 1.3, without loss of generality, we may assume that

$$a_1(x) = 0, \quad x \in \partial\Omega, \tag{1.16}$$

while  $a_i(x) > 0$  for  $i > 1$ . Then we have the following.

**Theorem 1.4** *If (1.16) is true and  $a_i(x) > 0$  for  $i > 1$ ,  $u$  and  $v$  are two solutions of equation (1.4) with the same partial boundary value condition (1.7) and with the initial values  $u_0(x)$ ,  $v_0(x)$ , respectively, if for every  $i \geq 2$*

$$n \left( \int_{\Omega_n} |a_{1x_i}(x)|^{p_i(x)} dx \right)^{\frac{1}{p_i}} \leq c \tag{1.17}$$

and

$$n \left( \int_{\Omega \setminus \Omega_n} a_1(x) |a_{1x_1}(x)|^{p_1(x)} dx \right)^{\frac{1}{p_1}} \leq c, \tag{1.18}$$

then the stability of weak solutions (1.15) is true. Here,  $a_{1x_1}(x) = \frac{\partial a_1(x)}{\partial x_1}$ ,  $a_{1x_i}(x) = \frac{\partial a_1(x)}{\partial x_i}$  as usual,  $\Omega_n = \{x \in \Omega : a_1(x) > \frac{1}{n}\}$ .

At the end of the introduction, we would like to suggest that there are many papers devoted to the anisotropic elliptic equations, for examples, one can see [15–25] and the references therein. For example, Fu and Shan studied the problem of removable isolated singularities for elliptic equations with variable exponents in [23]. They gave a sufficient condition for removability of the isolated singular point for the equations in  $W^{1,p(x)}(\Omega)$ . Cencelj and Repovš studied the perturbation by a critical term and a superlinear subcritical nonlinearity of a quasilinear elliptic equation containing a singular potential in [24]. By means of variational arguments and a version of the concentration-compactness principle in the singular case, they proved the existence of solutions for positive values of the parameter under the principal eigenvalue of the associated singular eigenvalue problem. Konaté and Ouaro studied nonlinear anisotropic problems with bounded Radon diffuse measure and variable exponent in [25]. They proved the existence and uniqueness of an entropy solution. By the way, the definition of weak solutions and the method used in [13] are different from the ones in this paper. Moreover, only the case when the domain is the  $n$ -dimensional unit cube is considered in [13], and the diffusion coefficient  $a_i(x) = a_i(x_i)$  is restricted only dependent on the single variable  $x_i$ .

## 2 The existence

We firstly give some basic concepts about the exponent variable spaces.

1.  $L^{p(x)}(\Omega)$  space.

$$L^{p(x)}(\Omega) = \left\{ u : u \text{ is a measurable real-valued function, } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}.$$

The space  $L^{p(x)}(\Omega)$  is equipped with the following Luxemburg norm:

$$|u|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

The space  $(L^{p(x)}(\Omega), |\cdot|_{L^{p(x)}(\Omega)})$  is a separable, uniformly convex Banach space.

2.  $W^{1,p(x)}(\Omega)$  space.

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

endowed with the following norm:

$$|u|_{W^{1,p(x)}} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

We use  $W_0^{1,p(x)}(\Omega)$  to denote the closure of  $C_0^\infty(\Omega)$  in  $W^{1,p(x)}$ .

**Lemma 2.1** ([26–28]) *The spaces  $(L^{p(x)}(\Omega), |\cdot|_{L^{p(x)}(\Omega)})$ ,  $(W^{1,p(x)}(\Omega), |\cdot|_{W^{1,p(x)}(\Omega)})$  and  $W_0^{1,p(x)}(\Omega)$  are reflexive Banach spaces.*

**Lemma 2.2** ([13]) *If, for any given  $i \in \{1, 2, \dots, N\}$ ,  $\int_{\Omega} a_i^{-\frac{1}{p_i(x)-1}}(x) dx < \infty$ , then*

$$\int_{\Omega} |u_{x_i}| dx \leq c. \tag{2.1}$$

By this lemma, one can see that if  $a_i(x)$  satisfies (1.5), (1.6) and if for every  $1 \leq r \leq l$ ,  $\int_{\Omega} a_{j_r}^{-1/(p_{j_r(x)}-1)}(x) dx < \infty$ , then (2.1) is satisfied. Thus, we can define the trace of  $u$  on the boundary  $\partial\Omega$ .

*Proof of Theorem 1.2* Consider the partially regularized equation

$$u_t = \sum_{i=1}^N (a_i(x) |u_{x_i}|^{p_i(x)-2} u_{x_i})_{x_i} + \varepsilon \Delta u, \quad (x, t) \in Q_T \tag{2.2}$$

with the initial boundary conditions

$$u(x, 0) = u_{0\varepsilon}(x), \quad x \in \Omega, \tag{2.3}$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T). \tag{2.4}$$

Here, we let  $u_{0\varepsilon}(x) \in C_0^\infty(\Omega)$  and strongly convergent to  $u_0(x)$  in  $W_0^{1,p_0}(\Omega)$ .

Since  $a_i(x)$  satisfies (1.5) and (1.6), similar to the proof of the usual evolutionary  $p$ -Laplacian equation, we can prove that there is a solution  $u_\varepsilon \in L^2(0, T; W_0^{1,p_0}(\Omega))$  of the initial-boundary value problem (2.2)-(2.4), which satisfies

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leq c, \quad \|u_{\varepsilon t}\|_{L^2(Q_T)} \leq c. \tag{2.5}$$

Multiplying (2.1) by  $u_\varepsilon$  and integrating it over  $Q_T$ , we have

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} u_\varepsilon^2 dx + \sum_{i=1}^n \iint_{Q_T} a_i(x) |u_{\varepsilon x_i}|^{p_i(x)} dx dt + \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^2 dx dt \\ &= \frac{1}{2} \int_{\Omega} u_0^2(x) dx, \end{aligned} \tag{2.6}$$

then

$$\varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^2 dx \leq c \tag{2.7}$$

and

$$\sum_{i=1}^N \iint_{Q_T} a_i(x) |u_{\varepsilon x_i}|^{p_i(x)} dx dt \leq c. \tag{2.8}$$

Hence, by (2.5), (2.7), (2.8), there exists a function  $u$  and an  $n$ -dimensional vector  $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$  satisfying that  $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$

$$u \in L^\infty(Q_T), \quad \frac{\partial u}{\partial t} \in L^2(Q_T), \quad \zeta_i \in L^1(0, T; L^{\frac{p_i(x)}{p_i(x)-1}}(\Omega)),$$

and  $u_\varepsilon \rightarrow u$  a.e.  $\in Q_T$ ,

$$\begin{aligned} & u_\varepsilon \rightharpoonup u, \quad \text{weakly star in } L^\infty(Q_T), \\ & u_\varepsilon \rightarrow u, \quad \text{in } L^2(0, T; L^r_{\text{loc}}(\Omega)), \\ & \frac{\partial u_\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^2(Q_T), \\ & \varepsilon \nabla u_\varepsilon \rightarrow 0, \quad \text{in } L^2(Q_T), \\ & a_i(x) |u_{\varepsilon x_i}|^{p_i(x)-2} u_{\varepsilon x_i} \rightharpoonup \zeta_i \quad \text{in } L^1(0, T; L^{\frac{p_i(x)}{p_i(x)-1}}(\Omega)). \end{aligned}$$

Here,  $r < \frac{Np_0}{N-p_0}$ .

Now, similar to the general evolutionary  $p$ -Laplacian equation, we are able to prove that (the details are omitted here)

$$\lim_{t \rightarrow 0} \int_{\Omega} |u(x, t) - u_0(x)| dx = 0$$

and

$$\sum_{i=1}^N \iint_{Q_T} a_i(x) |u_{x_i}|^{p_i(x)-2} u_{x_i} \varphi_{x_i} dx dt = \sum_{i=1}^N \iint_{Q_T} \zeta_i(x) \varphi_{x_i} dx dt \tag{2.9}$$

for any function  $\varphi \in C^1_0(Q_T)$ . By a process of the limit [8], we can show that (2.9) is also true for any  $\varphi = \varphi_1 \varphi_2$ , where  $\varphi_1 \in C^1_0(Q_T)$ ,  $\varphi_2 \in L^\infty$  and  $\varphi_{2x_i} \in L^2(0, T; W^{1,p_i(x)}_{\text{loc}}(\Omega))$ . Then

$u$  satisfies equation (1.4) with the initial value (1.2) in the sense of Definition 1.1. At last, according to Lemma 2.2, the partial boundary value condition (1.7) is satisfied in the sense of trace. Theorem 1.2 is proved.  $\square$

### 3 The stability

**Lemma 3.1** ([26–28])

- (i)  *$p(x)$ -Hölder's inequality.* Let  $q_1(x)$  and  $q_2(x)$  be real functions with  $\frac{1}{q_1(x)} + \frac{1}{q_2(x)} = 1$  and  $q_1(x) > 1$ . Then the conjugate space of  $L^{q_1(x)}(\Omega)$  is  $L^{q_2(x)}(\Omega)$ . And, for any  $u \in L^{q_1(x)}(\Omega)$  and  $v \in L^{q_2(x)}(\Omega)$ , we have

$$\left| \int_{\Omega} uv \, dx \right| \leq 2 \|u\|_{L^{q_1(x)}(\Omega)} \|v\|_{L^{q_2(x)}(\Omega)}.$$

- (ii)
- $$|u|_{L^{p(x)}(\Omega)} = 1, \quad \text{then } \int_{\Omega} |u|^{p(x)} \, dx = 1,$$
- $$|u|_{L^{p(x)}(\Omega)} > 1, \quad \text{then } |u|_{L^{p(x)}(\Omega)}^{p^-} \leq \int_{\Omega} |u|^{p(x)} \, dx \leq |u|_{L^{p(x)}(\Omega)}^{p^+},$$
- $$|u|_{L^{p(x)}(\Omega)} < 1, \quad \text{then } |u|_{L^{p(x)}(\Omega)}^{p^+} \leq \int_{\Omega} |u|^{p(x)} \, dx \leq |u|_{L^{p(x)}(\Omega)}^{p^-}.$$

Now, we will prove Theorem 1.3. For any given positive integer  $n$ , let  $g_n(s)$  be an odd function, and

$$g_n(s) = \begin{cases} 1, & s > \frac{1}{n}, \\ n^2 s^2 e^{1-n^2 s^2}, & 0 \leq s \leq \frac{1}{n}. \end{cases}$$

Clearly,

$$\lim_{n \rightarrow 0} g_n(s) = \text{sign}(s), \quad s \in (-\infty, +\infty). \tag{3.1}$$

*Proof of Theorem 1.3* Let  $u$  and  $v$  be two weak solutions of equation (1.4) with the initial values  $u(x, 0)$ ,  $v(x, 0)$ , respectively.

Let  $\Omega_n = \{x \in \Omega : \prod_{r=1}^l a_{j_r}(x) > \frac{1}{n}\}$ , and

$$\phi_n(x) = \begin{cases} 1, & \text{if } x \in \Omega_n, \\ n \prod_{r=1}^l a_{j_r}(x), & \text{if } x \in \Omega \setminus \Omega_n. \end{cases} \tag{3.2}$$

Obviously,  $\phi_{nx_i} = n(\prod_{r=1}^l a_{j_r}(x))_{x_i}$  when  $x \in \Omega \setminus \Omega_n$ , in other places, it is identical to zero. We can choose  $\varphi_1 = \chi_{[\tau, s]} \phi_n$ ,  $\varphi_2 = g_n(u - v)$ ,  $\varphi = \chi_{[\tau, s]} \phi_n g_n(u - v)$  as the test function, then

$$\begin{aligned} & \int_{\tau}^s \int_{\Omega} \phi_n g_n(u - v) \frac{\partial(u - v)}{\partial t} \, dx \, dt \\ & + \sum_{i=1}^N \int_{\tau}^s \int_{\Omega} a_i(x) (|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i}) (u_{x_i} - v_{x_i}) g'_n(u - v) \phi_n(x) \, dx \, dt \\ & + \sum_{r=1}^k \int_{\tau}^s \int_{\Omega} a_{i_r}(x) (|u_{x_{i_r}}|^{p_{i_r}(x)-2} u_{x_{i_r}} - |v_{x_{i_r}}|^{p_{i_r}(x)-2} v_{x_{i_r}}) g_n(u - v) \phi_{nx_{i_r}} \, dx \, dt \end{aligned}$$

$$\begin{aligned}
 & + \sum_{r=l}^l \int_{\tau}^s \int_{\Omega} a_{j_r}(x) (|u_{x_{j_r}}|^{p_{j_r}(x)-2} u_{x_{j_r}} - |v_{x_{j_r}}|^{p_{j_r}(x)-2} v_{x_{j_r}}) g_n(u-v) \phi_{nx_{j_r}} dx dt \\
 & = 0.
 \end{aligned} \tag{3.3}$$

In the first place, as usual, we have

$$\int_{\Omega} a_i(x) (|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i}) (u_{x_i} - v_{x_i}) g'_n(u-v) \phi_n(x) dx \geq 0, \tag{3.4}$$

and since  $u_t \in L^2(Q_T)$ , using the Lebesgue dominated theorem, we have

$$\begin{aligned}
 & \lim_{\eta \rightarrow 0} \int_{\tau}^s \int_{\Omega} \phi_n(x) g_n(u-v) \frac{\partial(u-v)}{\partial t} dx dt \\
 & = \int_{\Omega} |u-v|(x, s) dx - \int_{\Omega} |u-v|(x, \tau) dx.
 \end{aligned} \tag{3.5}$$

In the second place, we deal with the third term on the left-hand side of (3.3). For simplicity, in what follows, we denote  $\prod_{r=1}^l a_{j_r}(x)$  as  $\prod_{j=1}^l a_j(x)$ ,

$$\begin{aligned}
 & \left| \int_{\Omega} a_{i_r}(x) (|u_{x_{i_r}}|^{p_{i_r}(x)-2} u_{x_{i_r}} - |v_{x_{i_r}}|^{p_{i_r}(x)-2} v_{x_{i_r}}) g_n(u-v) \phi_{nx_{i_r}} dx \right| \\
 & = \left| \int_{\Omega \setminus \Omega_n} a_{i_r}(x) (|u_{x_{i_r}}|^{p_{i_r}(x)-2} u_{x_{i_r}} - |v_{x_{i_r}}|^{p_{i_r}(x)-2} v_{x_{i_r}}) g_n(u-v) \phi_{nx_{i_r}} dx \right| \\
 & \leq \int_{\Omega \setminus \Omega_n} a_{i_r}(x) (|u_{x_{i_r}}|^{p_{i_r}(x)-1} + |v_{x_{i_r}}|^{p_{i_r}(x)-1}) |g_n(u-v) \phi_{nx_{i_r}}| dx \\
 & \leq \int_{\Omega \setminus \Omega_n} \left[ n a_{i_r}(x) \prod_{j=1}^l a_j(x) \right]^{\frac{p_{i_r}(x)-1}{p_{i_r}(x)}} (|u_{x_{i_r}}|^{p_{i_r}(x)-1} + |v_{x_{i_r}}|^{p_{i_r}(x)-1}) \\
 & \quad \cdot (a_{i_r}(x))^{\frac{1}{p_{i_r}(x)}} \left| g_n(u-v) \frac{n^{\frac{1}{p_{i_r}(x)}} [\prod_{j=1}^l a_j(x)]_{x_{i_r}}}{[\prod_{j=1}^l a_j(x)]^{\frac{p_{i_r}(x)-1}{p_{i_r}(x)}}} \right| dx \\
 & \leq \left( \int_{\Omega \setminus \Omega_n} n \prod_{j=1}^l a_j(x) a_{i_r}(x) (|u_{x_{i_r}}|^{p_{i_r}(x)} + |v_{x_{i_r}}|^{p_{i_r}(x)}) dx \right)^{\frac{1}{q_{i_r}^+}} \\
 & \quad \cdot \left( \int_{\Omega \setminus \Omega_n} n a_{i_r}(x) |g_n(u-v)|^{p_{i_r}(x)} \frac{([\prod_{j=1}^l a_j(x)]_{x_{i_r}})^{p_{i_r}(x)}}{([\prod_{j=1}^l a_j(x)]^{p_{i_r}(x)-1}} dx \right)^{\frac{1}{p_{i_r}^+}} \\
 & \leq c \left( \int_{\Omega \setminus \Omega_n} a_{i_r}(x) (|u_{x_{i_r}}|^{p_{i_r}(x)} + |v_{x_{i_r}}|^{p_{i_r}(x)}) dx \right)^{\frac{1}{q_{i_r}^+}} \\
 & \quad \cdot \left( n \int_{\Omega \setminus \Omega_n} |g_n(u-v)|^{p_{i_r}(x)} \frac{([\prod_{j=1}^l a_j(x)]_{x_{i_r}})^{p_{i_r}(x)}}{([\prod_{j=1}^l a_j(x)]^{p_{i_r}(x)-1}} dx \right)^{\frac{1}{p_{i_r}^+}}.
 \end{aligned} \tag{3.6}$$

Here,  $p_{i_r}^{\pm} = p_{i_r}^+ \text{ or } p_{i_r}^-$  according to (ii) of Lemma 3.1.  $q_{i_r}(x) = \frac{p_{i_r}(x)}{p_{i_r}(x)-1}$ ,  $q_{i_r}^{\pm}$  has a similar sense. If we denote that

$$\Omega_{n1} = \{x \in \Omega \setminus \Omega_n : \text{dist}(x, \Sigma_2) > \text{dist}(x, \Sigma_1)\},$$



$$\Omega_{n2} = \{x \in \Omega \setminus \Omega_n : \text{dist}(x, \Sigma_2) \leq \text{dist}(x, \Sigma_1)\},$$

then

$$\begin{aligned} & n \int_{\Omega \setminus \Omega_n} |g_n(u - v)|^{p_{ir}(x)} \frac{|\prod_{j=1}^l a_j(x)_{x_{ir}}|^{p_{ir}(x)}}{[\prod_{j=1}^l a_j(x)]^{p_{ir}(x)-1}} dx \\ & \leq n \int_{\Omega_{n1}} |g_n(u - v)|^{p_{ir}(x)} \frac{|\prod_{j=1}^l a_j(x)_{x_{ir}}|^{p_{ir}(x)}}{[\prod_{j=1}^l a_j(x)]^{p_{ir}(x)-1}} dx \\ & \quad + n \int_{\Omega_{n2}} |g_n(u - v)|^{p_{ir}(x)} \frac{|\prod_{j=1}^l a_j(x)_{x_{ir}}|^{p_{ir}(x)}}{[\prod_{j=1}^l a_j(x)]^{p_{ir}(x)-1}} dx. \end{aligned} \tag{3.7}$$

Since

$$u = v = 0, \quad x \in \Sigma_1,$$

by the definition of the trace, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} n \int_{\Omega_{n1}} |g_n(u - v)|^{p_{ir}(x)} \frac{|\prod_{j=1}^l a_j(x)_{x_{ir}}|^{p_{ir}(x)}}{[\prod_{j=1}^l a_j(x)]^{p_{ir}(x)-1}} dx \\ & = \int_{\Sigma_1} \text{sign}(u - v) \frac{|\prod_{j=1}^l a_j(x)_{x_{ir}}|^{p_{ir}(x)}}{[\prod_{j=1}^l a_j(x)]^{p_{ir}(x)-1}} d\Sigma = 0. \end{aligned} \tag{3.8}$$

Moreover, since

$$\begin{aligned} & \frac{|\prod_{j=1}^l a_j(x)_{x_{ir}}|^{p_{ir}(x)}}{[\prod_{j=1}^l a_j(x)]^{p_{ir}(x)-1}} = 0, \quad x \in \Sigma_2, \\ & \lim_{n \rightarrow \infty} n \int_{\Omega_{n2}} |g_n(u - v)|^{p_{ir}(x)} \frac{|\prod_{j=1}^l a_j(x)_{x_{ir}}|^{p_{ir}(x)}}{[\prod_{j=1}^l a_j(x)]^{p_{ir}(x)-1}} dx \\ & \leq \lim_{n \rightarrow \infty} n \int_{\Omega_{n2}} \frac{|\prod_{j=1}^l a_j(x)_{x_{ir}}|^{p_{ir}(x)}}{[\prod_{j=1}^l a_j(x)]^{p_{ir}(x)-1}} dx \\ & = \int_{\Sigma_2} \frac{|\prod_{j=1}^l a_j(x)_{x_{ir}}|^{p_{ir}(x)}}{[\prod_{j=1}^l a_j(x)]^{p_{ir}(x)-1}} d\Sigma = 0. \end{aligned} \tag{3.9}$$

By (3.6)-(3.9), we conclude that

$$\lim_{n \rightarrow 0} \left| \int_{\Omega} a_{ir}(x) (|u_{x_{ir}}|^{p_{ir}(x)-2} u_{x_{ir}} - |v_{x_{ir}}|^{p_{ir}(x)-2} v_{x_{ir}}) g_n(u - v) \phi_{nx_{ir}} dx \right| = 0. \tag{3.10}$$

In the third place, we deal with the last term on the left-hand side of (3.3)

$$\begin{aligned} & \left| \int_{\Omega} a_{jr}(x) (|u_{x_{jr}}|^{p_{jr}(x)-2} u_{x_{jr}} - |v_{x_{jr}}|^{p_{jr}(x)-2} v_{x_{jr}}) g_n(u - v) \phi_{nx_{jr}} dx \right| \\ & = \left| \int_{\Omega \setminus \Omega_n} a_{jr}(x) (|u_{x_{jr}}|^{p_{jr}(x)-2} u_{x_{jr}} - |v_{x_{jr}}|^{p_{jr}(x)-2} v_{x_{jr}}) g_n(u - v) \phi_{nx_{jr}} dx \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \int_{\Omega \setminus \Omega_n} a_{j_r}(x) (|u_{x_{j_r}}|^{p_{j_r}(x)-1} + |v_{x_{j_r}}|^{p_{j_r}(x)-1}) \left( \prod_{j=1}^l a_j(x) \right)_{x_{j_r}} g_n(u-v) dx \\
 &\leq c \left( \int_{\Omega \setminus \Omega_n} a_{j_r}(x) |u_{x_{j_r}}|^{p_{j_r}(x)} + |v_{x_{j_r}}|^{p_{j_r}(x)} dx \right)^{\frac{1}{q_r^+}} \\
 &\quad \times n \left( \int_{\Omega \setminus \Omega_n} a_{j_r}(x) \left| \left( \prod_{j=1}^l a_j(x) \right)_{x_i} \right|^{p_{j_r}(x)} dx \right)^{\frac{1}{p_{j_r}^+}}. \tag{3.11}
 \end{aligned}$$

Here,  $q_r(x) = \frac{p_{j_r}(x)}{p_{j_r}(x)-1}$ ,  $q_r^+ = \max_{x \in \bar{\Omega}} q_r(x)$ .

By assumption (1.14)

$$n \left( \int_{\Omega \setminus \Omega_n} a_{j_r}(x) \left| \left( \prod_{j=1}^l a_j(x) \right)_{x_{j_r}} \right|^{p_{j_r}(x)} dx \right)^{\frac{1}{p_{j_r}^+}} \leq c.$$

Then

$$\lim_{n \rightarrow \infty} \left| \int_{\Omega} a_{j_r}(x) (|u_{x_{j_r}}|^{p_{j_r}(x)-2} u_{x_{j_r}} - |v_{x_{j_r}}|^{p_{j_r}(x)-2} v_{x_{j_r}}) g_n(u-v) \phi_{nx_{j_r}} dx \right| = 0. \tag{3.12}$$

Now, let  $n \rightarrow \infty$  in (3.3). Then

$$\int_{\Omega} |u(x, s) - v(x, s)| dx \leq \int_{\Omega} |u(x, \tau) - v(x, \tau)| dx, \tag{3.13}$$

by the arbitrariness of  $\tau$ , we have

$$\int_{\Omega} |u(x, s) - v(x, s)| dx \leq \int_{\Omega} |u_0(x) - v_0(x)| dx. \quad \square$$

#### 4 The case of $l = 1$

*Proof of Theorem 1.3* Let  $u$  and  $v$  be two weak solutions of equation (1.4) with the initial values  $u(x, 0)$ ,  $v(x, 0)$ , respectively. Since  $a_1(x)$  satisfies (1.16) and for  $i \geq 2$ ,  $a_i(x) > 0$ ,  $x \in \bar{\Omega}$ , we can let  $\Omega_n = \{x \in \Omega : a_1(x) > \frac{1}{n}\}$  and

$$\phi_n(x) = \begin{cases} 1, & \text{if } x \in \Omega_n, \\ na_1(x), & \text{if } x \in \Omega \setminus \Omega_n. \end{cases} \tag{4.1}$$

Obviously,  $\phi_{nx_i} = na_{1x_i}$  when  $x \in \Omega \setminus \Omega_n$ , in other places, it is identical to zero.

We can choose  $\chi_{[\tau, s]} \phi_n g_n(u-v)$  as the test function, then

$$\begin{aligned}
 &\int_{\tau}^s \int_{\Omega} \phi_n g_n(u-v) \frac{\partial(u-v)}{\partial t} dx dt \\
 &\quad + \sum_{i=1}^N \int_{\tau}^s \int_{\Omega} a_i(x) (|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i}) (u_{x_i} - v_{x_i}) g_n'(u-v) \phi_n(x) dx dt \\
 &\quad + \int_{\tau}^s \int_{\Omega} a_1(x) (|u_{x_1}|^{p_1(x)-2} u_{x_1} - |v_{x_1}|^{p_1(x)-2} v_{x_1}) g_n(u-v) \phi_{nx_1} dx dt
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=2}^N \int_{\tau}^s \int_{\Omega} a_i(x) (|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i}) g_n(u-v) \phi_{nx_i} dx dt \\
 & = 0.
 \end{aligned} \tag{4.2}$$

Certainly, we have

$$\int_{\Omega} a_i(x) (|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i}) (u_{x_i} - v_{x_i}) g'_n(u-v) \phi_n(x) dx \geq 0 \tag{4.3}$$

and

$$\begin{aligned}
 & \lim_{n \rightarrow 0} \int_{\tau}^s \int_{\Omega} \phi_n(x) \phi_n g_n(u-v) \frac{\partial(u-v)}{\partial t} dx dt \\
 & = \int_{\Omega} |u-v|(x, s) dx - \int_{\Omega} |u-v|(x, \tau) dx.
 \end{aligned} \tag{4.4}$$

Now, we deal with the third term on the left-hand side of (4.2).

$$\begin{aligned}
 & \left| \int_{\Omega} a_1(x) (|u_{x_1}|^{p_1(x)-2} u_{x_1} - |v_{x_1}|^{p_1(x)-2} v_{x_1}) g_n(u-v) \phi_{nx_1} dx \right| \\
 & = \left| \int_{\Omega \setminus \Omega_n} a_1(x) (|u_{x_1}|^{p_1(x)-2} u_{x_1} - |v_{x_1}|^{p_1(x)-2} v_{x_1}) g_n(u-v) \phi_{nx_1} dx \right| \\
 & \leq \int_{\Omega \setminus \Omega_n} a_1(x) (|u_{x_1}|^{p_1(x)-1} + |v_{x_1}|^{p_1(x)-1}) |g_n(u-v) \phi_{nx_1}| dx \\
 & \leq n \int_{\Omega \setminus \Omega_n} a_1(x) (|u_{x_1}|^{p_1(x)-1} + |v_{x_1}|^{p_1(x)-1}) |g_n(u-v) a_{x_1}(x)| dx \\
 & \leq cn \left( \int_{\Omega \setminus \Omega_n} a_1(x) (|u_{x_1}|^{p_1(x)} + |v_{x_1}|^{p_1(x)}) dx \right)^{\frac{1}{q_1}} \\
 & \quad \times \left( \int_{\Omega \setminus \Omega_n} a_1(x) |g_n(u-v)|^{p_1(x)} |a_{x_1}(x)|^{p_1(x)} dx \right)^{\frac{1}{p_1}} \\
 & \leq cn^{1-\frac{1}{p_1}-\frac{1}{q_1}} \left( n \int_{\Omega \setminus \Omega_n} a_1(x) (|u_{x_1}|^{p_1(x)} + |v_{x_1}|^{p_1(x)}) dx \right)^{\frac{1}{q_1}} \\
 & \quad \times \left( \int_{\Omega \setminus \Omega_n} |g_n(u-v)|^{p_1(x)} |a_{x_1}(x)|^{p_1(x)} dx \right)^{\frac{1}{p_1}}.
 \end{aligned} \tag{4.5}$$

If we denote that

$$d_n = \sup_{x \in \Omega \setminus \Omega_n} \text{dist}(x, \partial\Omega) \tag{4.6}$$

and

$$\Omega_{n1} = \{x \in \Omega \setminus \Omega_n : \text{dist}(x, \Sigma_2) > d_n\}, \quad \Omega_{n2} = (\Omega \setminus \Omega_n) \setminus \overline{\Omega_{n1}},$$

then

$$\begin{aligned}
 & n \left( \int_{\Omega \setminus \Omega_n} a_{i_r}(x) |g_n(u - v)|^{p_{i_r}(x)} \left| \left( \prod_{j=1}^l a_j(x) \right) \Big|_{x_{i_r}} \right|^{p_{i_r}(x)} dx \right)^{\frac{1}{p_{i_r}^*}} \\
 & \leq n \left( \int_{\Omega_{n1}} a_{i_r}(x) |g_n(u - v)|^{p_{i_r}(x)} \left| \left( \prod_{j=1}^l a_j(x) \right) \Big|_{x_{i_r}} \right|^{p_{i_r}(x)} dx \right)^{\frac{1}{p_{i_r}^*}} \\
 & \quad + n \left( \int_{\Omega_{n2}} a_{i_r}(x) |g_n(u - v)|^{p_{i_r}(x)} \left| \left( \prod_{j=1}^l a_j(x) \right) \Big|_{x_{i_r}} \right|^{p_{i_r}(x)} dx \right)^{\frac{1}{p_{i_r}^*}} \\
 & \leq cn \left( \int_{\Omega_{n1}} |g_n(u - v)|^{p_{i_r}(x)} dx \right)^{\frac{1}{p_{i_r}^*}} \\
 & \quad + n \left( \int_{\Omega_{n2}} a_{i_r}(x) \left| \left( \prod_{j=1}^l a_j(x) \right) \Big|_{x_{i_r}} \right|^{p_{i_r}(x)} dx \right)^{\frac{1}{p_{i_r}^*}}. \tag{4.7}
 \end{aligned}$$

By the definition of the trace, we have

$$\lim_{n \rightarrow \infty} n \int_{\Omega_{n1}} |g_n(u - v)|^{p_{i_r}} dx = 0. \tag{4.8}$$

Moreover, since  $a_i(x) \in C^1(\overline{\Omega})$  and  $l > 1$ , by (1.6), we always have

$$\left( \prod_{j=1}^l a_j(x) \right) \Big|_{x_{i_r} |_{\partial\Omega}} = 0. \tag{4.9}$$

Thus, by the fact that  $a_i(x) \in C^1(\overline{\Omega})$ , we get

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} n \left( \int_{\Omega_{n2}} a_{i_r}(x) \left| \left( \prod_{j=1}^l a_j(x) \right) \Big|_{x_{i_r}} \right|^{p_{i_r}(x)} dx \right)^{\frac{1}{p_{i_r}^*}} \\
 & \leq c \max_{x \in \Sigma_2} \left| \left( \prod_{j=1}^l a_j(x) \right) \Big|_{x_{i_r}} \right| = 0. \tag{4.10}
 \end{aligned}$$

By (4.6)-(4.10), we conclude that

$$\lim_{n \rightarrow 0} \left| \int_{\Omega} a_{i_r}(x) (|u_{x_{i_r}}|^{p_{i_r}(x)-2} u_{x_{i_r}} - |v_{x_{i_r}}|^{p_{i_r}(x)-2} v_{x_{i_r}}) g_n(u - v) \phi_{nx_{i_r}} dx \right| = 0. \tag{4.11}$$

In the third place, we deal with the last term on the left-hand side of (4.2)

$$\begin{aligned}
 & \left| \int_{\Omega} a_{j_r}(x) (|u_{x_{j_r}}|^{p_{j_r}(x)-2} u_{x_{j_r}} - |v_{x_{j_r}}|^{p_{j_r}(x)-2} v_{x_{j_r}}) g_n(u - v) \phi_{nx_{j_r}} dx \right| \\
 & = \left| \int_{\Omega \setminus \Omega_n} a_{j_r}(x) (|u_{x_{j_r}}|^{p_{j_r}(x)-2} u_{x_{j_r}} - |v_{x_{j_r}}|^{p_{j_r}(x)-2} v_{x_{j_r}}) g_n(u - v) \phi_{nx_{j_r}} dx \right|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \int_{\Omega \setminus \Omega_n} a_{j_r}(x) (|u_{x_{j_r}}|^{p_{j_r}(x)-1} + |v_{x_{j_r}}|^{p_{j_r}(x)-1}) \left( \prod_{j=1}^l a_j(x) \right)_{x_{j_r}} g_n(u-v) dx \\
 &\leq c \left( \int_{\Omega \setminus \Omega_n} a_{j_r}(x) (|u_{x_{j_r}}|^{p_{j_r}(x)} + |v_{x_{j_r}}|^{p_{j_r}(x)}) dx \right)^{\frac{1}{q_{j_r}^+}} \\
 &\quad \times n \left( \int_{\Omega \setminus \Omega_n} a_{j_r}(x) \left| \left( \prod_{j=1}^l a_j(x) \right)_{x_i} \right|^{p_{j_r}(x)} dx \right)^{\frac{1}{p_{j_r}^+}}.
 \end{aligned} \tag{4.12}$$

Here,  $q_{j_r}(x) = \frac{p_{j_r}(x)}{p_{j_r}(x)-1}$ ,  $q_{j_r}^+ = \max_{x \in \bar{\Omega}} q_{j_r}(x)$ .

By assumption (1.14),

$$n \left( \int_{\Omega \setminus \Omega_n} a_{j_r}(x) \left| \left( \prod_{j=1}^l a_j(x) \right)_{x_{j_r}} \right|^{p_{j_r}(x)} dx \right)^{\frac{1}{p_{j_r}^+}} \leq c.$$

Then

$$\lim_{n \rightarrow \infty} \left| \int_{\Omega} a_{j_r}(x) (|u_{x_{j_r}}|^{p_{j_r}(x)-2} u_{x_{j_r}} - |v_{x_{j_r}}|^{p_{j_r}(x)-2} v_{x_{j_r}}) g_n(u-v) \phi_{nx_{j_r}} dx \right| = 0. \tag{4.13}$$

Now, let  $n \rightarrow \infty$  in (4.2). Then

$$\int_{\Omega} |u(x, s) - v(x, s)| dx \leq \int_{\Omega} |u(x, \tau) - v(x, \tau)| dx, \tag{4.14}$$

by the arbitrariness of  $\tau$ , we have

$$\int_{\Omega} |u(x, s) - v(x, s)| dx \leq \int_{\Omega} |u_0(x) - v_0(x)| dx. \quad \square$$

### 5 Conclusion

The anisotropic parabolic equation is considered in this paper. If in some directions the diffusion coefficients are degenerate on the boundary, while in other directions they are not degenerate, how to give a suitable partial boundary value condition to match the equation was studied by the author in [13]. If a partial boundary value condition is imposed, only when the domain is an  $N$ -dimensional cube, the stability of weak solutions is proved [13]. This short paper solves the problem when the domain is a usual bounded domain, gives a complete supplement of the paper [13].

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#### Authors' contributions

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