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Positive periodic solutions for p-Laplacian neutral differential equations with a singularity

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Abstract

In this paper, we study the positive periodic solutions of a kind of p-Laplacian neutral differential equation with a singularity. By applying the continuation theorem and some analytic techniques, we shall establish several new criteria for the existence of positive periodic solutions for the considered problem. Some recent results in the literature are generalized and improved. Three examples are given to illustrate the effectiveness of our results.

Keywords: neutral differential equations; positive periodic solutions; continuation theorem; p-Laplacian; singularity

1 Introduction

In the past years, the study of periodic solutions for some types of p-Laplacian neutral differential equations has attracted much attention from researchers; see [1–12] and the references cited therein. For example, in [11] and [12], Zhu and Lu studied the periodic solutions for p-Laplacian neutral functional differential equations as follows:

$$\left(\varphi_p\left(x(t)-cx(t-\tau)\right)'\right)' = g\left(t,x\left(t-\tau(t)\right)\right) + e(t) \tag{1.1}$$

and

$$\left(\varphi_p\left(x(t)-cx(t-\delta)\right)'\right)' = f\left(t,x(t)\right)x'(t) + \sum_{j=1}^{n}\beta_j g\left(t,x\left(t-\tau_j(t)\right)\right) + p(t), \tag{1.2}$$

respectively.

On the basis of work of [11] and [12], Wang and Zhu [7] further discussed existence of periodic solutions for a fourth-order p-Laplacian neutral functional differential equation of the form:

$$\left(\varphi_p\left(x(t)-cx(t-\delta)\right)''\right)'' = f\left(x(t)\right)x'(t) + g\left(t,x\left(t-\tau\left(t,|x|_\infty\right)\right)\right) + e(t), \tag{1.3}$$

where $p>1$, $\varphi_p:\mathbb{R}\to\mathbb{R}$, $\varphi_p(u)=|u|^{p-2}u$ for $u\neq 0$ and $\varphi_p(0)=0$; $f\in C(\mathbb{R},\mathbb{R})$; $g,\tau\in C(\mathbb{R}^2,\mathbb{R})$ with $g(t+T,\cdot)=g(t,\cdot)$; $\tau(t+T)\equiv\tau(t)$; $e\in C(\mathbb{R},\mathbb{R})$ with $e(t+T)\equiv e(t)$; $T>0$, $|c|\neq 1$ and δ are given constants.

In recent years, by applying the method of coincidence degree, many good results of the existence of positive periodic solutions for some types of differential equations with a singularity have been obtained; see [13–22] and the references cited therein. For example, Wang in [21] studied the periodic solutions for the Liénard equation with a singularity and a deviating argument, which extends the results of Zhang in [22]

$$x''(t) + f\big(x(t)\big)x'(t) + g\big(t, x(t-\sigma)\big) = 0, \tag{1.4}$$

where $g : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is an L^2-Carathéodory funtion, $g(t, x)$ is a T-periodic function in the first argument and can be singular at $x = 0$.

As usual, we say that g has a singularity of repulsive type, if

$$\lim_{x\to 0^+} g(t,x) = -\infty, \quad \text{uniformly for a.e. } t \in (-\infty, +\infty),$$

and g has a singularity of attractive type, if

$$\lim_{x\to 0^+} g(t,x) = +\infty, \quad \text{uniformly for a.e. } t \in (-\infty, +\infty).$$

Compared with the classical p-Laplacian neutral differential problems or the singular problems, p-Laplacian neutral differential problems with singular effects have been scarcely studied, not to mention the high-order p-Laplacian neutral differential equations with a singularity. This motivated us to carry out a study. In this paper, we consider the following high-order p-Laplacian neutral differential equation with a singularity and a deviating argument:

$$\big[\varphi_p\big(\big(x(t) - cx(t-\gamma)\big)^{(m)}\big)\big]^{(m)} + f\big(x(t)\big)x'(t) + g\big(t, x(t-\tau)\big) = e(t), \tag{1.5}$$

where $\varphi_p : \mathbb{R} \to \mathbb{R}$, $\varphi_p(u) = |u|^{p-2}u$, $p \geq 2$; m is a positive integer; c is a constant with $|c| < 1$; $0 \leq \gamma, \tau < T$; $f : \mathbb{R} \to \mathbb{R}$ is continuous, $g : [0, T] \times (0, +\infty) \to \mathbb{R}$ is a continuous function and can be singular at $u = 0$, $e(t)$ is T-periodic with $\int_0^T e(t)\,dt = 0$.

Remark 1.1 Obviously, if there is no singularity, (1.1) is the specially case of (1.5). The equations in [1, 3–6, 10] are also the specially cases of (1.5) if $m \equiv 1$ and there are no singularities. So we extend the result in [1, 3–7, 9–11]. Therefore, we extend the corresponding results in literature to the high-order case and the singular case.

The paper is organized as follows. Section 2 is devoted to introducing some definitions and recalling some preliminary results that will be extensively used. The existence results will be obtained in Section 3. Finally, two examples are given to illustrate the effectiveness of our result in Section 4.

2 Preliminaries

Let

$$C_T = \big\{x \mid x \in C(\mathbb{R}, \mathbb{R}), x(t+T) \equiv x(t)\big\}$$

with the norm $|x|_0 = \max_{t\in[0,T]} |x(t)|$, and

$$C_T^1 = \{x \mid x \in C^1(\mathbb{R},\mathbb{R}), x(t+T) \equiv x(t)\}$$

with the norm $\|x\| = \max\{|x|_0, |x'|_0\}$.

Define a linear operator

$$A: \quad C_T \to C_T, \qquad (Ax)(t) = x(t) - cx(t-\gamma).$$

Lemma 2.1 ([23]) *If $|c| < 1$, then A has continuous bounded inverse on X, and*

(1) $\|A^{-1}x\| \le \frac{|x|_0}{|1-|c||}$, $\forall x \in X$;

(2) $\int_0^T |(A^{-1}f)(t)|\,dt \le \frac{1}{1-|c|}\int_0^T |f(t)|\,dt$, $\forall f \in X$.

Lemma 2.2 ([24]) *Let $T > 0$ be a constant, $x \in C^m(\mathbb{R},\mathbb{R})$, $m \ge 2$, and $x(t+T) \equiv x(t)$, $|x^{(i)}|_0 = \max_{t\in[0,T]} |x^{(i)}(t)|$, then there are $M_i(m) > 0$ independent of x such that*

$$\left|x^{(i)}\right|_0 \le M_i(m)\int_0^T \left|x^{(m)}(t)\right|dt, \quad i = 1,2,\dots,m-1,$$

where if m is an even integer,

$$M_i(m) = \begin{cases} M_{2s-1}(m) = T^{m-2s}\sqrt{-B_{2m-4s}/12(2m-4s)!}, & s = 1,2,\dots,\frac{m}{2}-1, \\ M_{2s}(m) = \frac{(-1)^{\frac{m-2s}{2}+1}T^{m-2s-1}B_{m-2s}}{(m-2s)!}, & s = 1,\dots,\frac{m}{2}-1, \\ M_{m-1}(m) = \frac{1}{2}, & \end{cases}$$

if m is an odd integer,

$$M_i(m) = \begin{cases} M_{2s+1}(m) = \frac{(-1)^{\frac{m-2s-1}{2}+1}T^{m-2s-2}B_{m-2s-1}}{(m-2s-1)!}, & s = 0,1,\dots,\frac{m+1}{2}-2, \\ M_{2s}(m) = T^{m-2s-1}\sqrt{-B_{2m-4s-2}/12(2m-4s-2)!}, & s = 1,2,\dots,\frac{m+1}{2}-2, \\ M_{m-1}(m) = \frac{1}{2}, & \end{cases}$$

and B_{m-2s}, B_{2m-4s}, B_{m-2s-1}, $B_{2m-4s-2}$ are Bernoulli numbers, which can be calculated by the following recursion formula:

$$B_0 = 1, \qquad B_p = \frac{-\sum_{i=0}^{p-1} C_{p+1}^i B_i}{p+1},$$

where C_{p+1}^i is the number of combinations.

Lemma 2.3 ([25]) *Let X and Y be two real Banach spaces, $L: D(L) \subset X \to Y$ be a Fredholm operator with index zero, $\Omega \subset X$ be an open bounded set, and $N: \bar{\Omega} \subset X \to Y$ be L-compact on $\bar{\Omega}$. Suppose that all of the following conditions hold:*

(1) $Lx \ne \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\forall \lambda \in (0,1)$;

(2) $QNx \ne 0$, $\forall x \in \partial\Omega \cap \ker L$;

(3) $\deg\{JQN, \Omega \cap \ker L, 0\} \ne 0$, *where $J: \operatorname{Im} Q \to \ker L$ is an homeomorphism map.*

Then the equation $Lx = Nx$ has at least one solution on $D(L) \cap \bar{\Omega}$.

In order to use the continuation theorem to study the positive T-periodic solutions for equation (1.5), we consider the following system:

$$\begin{cases} u^{(m)}(t) = [A^{-1}\varphi_q(v)](t), \\ v^{(m)}(t) = -f(u(t))u'(t) - g(t, u(t-\tau)) + e(t), \end{cases} \tag{2.1}$$

where $q > 1$ is a constant with $1/p + 1/q = 1$. Clearly, if $x(t) = (u(t), v(t))^\top$ is a T-periodic solution of system (2.1), then $u(t)$ must be a T-periodic solution of equation (1.5). Thus, the problem of finding a positive T-periodic solution for (1.5) reduces to finding one for (2.1).

Define

$$X = \{x = (u,v)^\top \in C^1(\mathbb{R}, \mathbb{R}^2) : x(t+T) \equiv x(t)\}$$

with the norm $\|x\|_X = \max\{\|u\|; \|v\|\}$,

$$Y = \{x = (u,v)^\top \in C(\mathbb{R}, \mathbb{R}^2) : x(t+T) \equiv x(t)\}$$

with the norm $\|x\|_Y = \max\{|u|_0; |v|_0\}$. Clearly, X and Y are two Banach spaces.

Define the linear operator

$$L: \quad D(L) \subset X \to Y, \qquad Lx = \left(u^{(m)}, v^{(m)}\right)^\top, \tag{2.2}$$

and

$$\begin{aligned} &N: \quad D(N) \subset X \to Y, \\ &(Nx)(t) = \begin{pmatrix} [A^{-1}\varphi_q(v)](t) \\ -f(u(t))u'(t) - g(t, u(t-\tau)) + e(t) \end{pmatrix}, \quad \forall t \in \mathbb{R}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} &D(L) = \{x \in C^m(\mathbb{R}, \mathbb{R}^2), x(t+T) \equiv x(t)\}, \\ &D(N) = \{x = (u,v)^\top \in X : u(t) > 0, t \in [0,T]\}. \end{aligned}$$

Then we can see that equation (2.1) can be converted to the abstract equation $Lx = Nx$. Moreover, from the definition of L, we can have

$$\ker L = \mathbb{R}^2, \quad \text{and} \quad \operatorname{Im} L = \left\{y \in Y, \int_0^T y(s)\,ds = 0\right\}.$$

Clearly, L is a Fredholm operator with index zero.

Let projectors $P : X \to \ker L$ and $Q : Y \to Y/\operatorname{Im} L$ be defined by

$$Px = x(0); \qquad Qy = \frac{1}{T}\int_0^T y(s)\,ds,$$

and let K represent the inverse of $L_{\ker P\cap D(L)}$. Obviously, $\ker Q = \operatorname{Im} L = \mathbb{R}^2$ and

$$(Ky)(t) = \sum_{i=1}^{m-1} \frac{1}{i!} x^{(i)}(0)t^i + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} y(s)\,ds, \tag{2.4}$$

where $x^{(i)}(0)$ $(i = 1, 2, \dots, m-1)$ are defined by the equation

$$\mathbf{AX} = \mathbf{D},$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ c_1 & 1 & 0 & \dots & 0 & 0 \\ c_2 & c_1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{m-3} & c_{m-4} & c_{m-5} & \dots & 1 & 0 \\ c_{m-2} & c_{m-3} & c_{m-4} & \dots & c_1 & 1 \end{pmatrix},$$

$\mathbf{X} = (x^{(m-1)}(0), x^{(m-2)}(0), \dots, x''(0), x'(0))^\top$, $\mathbf{D} = (d_1, d_2, \dots, d_{m-2}, d_{m-1})^\top$, $d_i = -\frac{1}{i!T}\int_0^T (T - s)^i y(s)\,ds$, $i = 1, 2, \dots, m-1$, and $c_j = \frac{T^j}{(j+1)!}$, $j = 1, 2, \dots, m-2$.

For the sake of convenience, we list the following assumptions:

[H_1] There exist positive constants D_1 and D_2 with $D_1 < D_2$ such that

(1) for each positive continuous T-periodic function $x(t)$ satisfying $\int_0^T g(t, x(t))\,dt = 0$, then there exists a positive point $\sigma \in [0, T]$ such that

$$D_1 \le x(\sigma) \le D_2;$$

(2) $\bar{g}(x) < 0$ for all $x \in (0, D_1)$ and $\bar{g}(x) > 0$ for all $x > D_2$, where $\bar{g}(x) = \frac{1}{T}\int_0^T g(t, x)\,dt$, $x > 0$.

[H_2] $g(t, x) = g_1(t, x) + g_0(x)$ for all $u \in (0, +\infty)$, where $g_0 : (0, +\infty) \to \mathbb{R}$ is a continuous function, $g_1 : [0, T] \times [0, +\infty) \to \mathbb{R}$ is a continuous function and

(1) there exist positive constants m_0 and m_1 such that

$$g(t, x) \le m_0 x^{p-1} + m_1, \quad \text{for all } (t, x) \in [0, T] \times (0, +\infty);$$

(2) $\int_0^1 g_0(x)\,dx = -\infty$.

[H_3] There exist positive constants α, β such that

$$f(x) \le \alpha x^{p-2} + \beta, \quad \text{for all } x \in (0, +\infty).$$

3 Main results

Theorem 3.1 *Suppose that conditions* [H_1]-[H_3] *hold and*

$$\frac{M_1^p(m)T^{2p-1}(1 + |c|)(2m_0T + \alpha)}{(1 - |c|)^p} < 1,$$

then there exist positive constants A_1, A_2, A_3 and ρ, which are independent of λ such that

$$A_1 \le u(t) \le A_2, \qquad |v|_0 \le A_3,$$

where $u(t)$ is any solution to the equation $Lx = \lambda Nx$, $\lambda \in (0,1)$.

Proof Consider the following operator equation:

$$Lx = \lambda Nx, \quad \lambda \in (0,1),$$

where L and N is defined by (2.2) and (2.3), respectively.

Define

$$\Omega_1 = \Big\{(u,v)^\top \in X : \min_{t\in[0,T]} u(t) > 0, Lx = \lambda Nx, \lambda \in (0,1)\Big\}.$$

If $x = (u,v)^\top \in \Omega_1$, then (u,v) satisfies

$$\begin{cases} u^{(m)}(t) = \lambda[A^{-1}\varphi_q(v)](t), \\ v^{(m)}(t) = -\lambda f(u(t))u'(t) - \lambda g(t,u(t-\tau)) + \lambda e(t). \end{cases} \tag{3.1}$$

From the first equation of (3.1), we get $v(t) = \varphi_p(\lambda^{-1}(Au)^{(m)}(t))$, and combining with the second equation of (3.1) yields

$$\big(\varphi_p\big((Au)^{(m)}(t)\big)\big)^{(m)} + \lambda^p f\big(u(t)\big)u'(t) + \lambda^p g\big(t,u(t-\tau)\big) = \lambda^p e(t). \tag{3.2}$$

Integrating equation (3.2) on the interval $[0,T]$, we have

$$\int_0^T g\big(t,u(t-\tau)\big)\,dt = 0. \tag{3.3}$$

It follows from condition (1) in assumption $[H_1]$ that there exist positive constants D_1, D_2 and $\sigma \in [0,T]$ such that

$$D_1 \le u(\sigma) \le D_2, \tag{3.4}$$

then we get

$$\begin{aligned} |u|_0 = \max_{t\in[0,T]} \big|u(t)\big| &\le \max_{t\in[0,T]} \left|u(\sigma) + \int_\sigma^t u'(s)\,ds\right| \\ &\le D_2 + \int_0^T \big|u'(s)\big|\,ds. \end{aligned} \tag{3.5}$$

Multiplying both sides of (3.2) by $(Au)(t)$ and integrating on the interval $[0,T]$, we get

$$\begin{aligned}\int_0^T \left|(Au)^{(m)}(t)\right|^p dt &\le \left(1+|c|\right)|u|_0 \int_0^T \left|f\left(u(t)\right)\right|\left|u'\right| dt \\ &\quad + \left(1+|c|\right)|u|_0 \int_0^T \left|g\left(t,u(t-\tau)\right)\right| dt \\ &\quad + \left(1+|c|\right)|u|_0 \int_0^T \left|e(t)\right| dt,\end{aligned}$$

which combining with $[H_3]$ yields

$$\begin{aligned}\int_0^T \left|(Au)^{(m)}(t)\right|^p dt &\le \left(1+|c|\right)|u|_0 \int_0^T \left(\alpha\left|u(t)\right|^{p-2}+\beta\right)\left|u'(t)\right| dt \\ &\quad + \left(1+|c|\right)|u|_0 \int_0^T \left|g\left(t,u(t-\tau)\right)\right| dt + \left(1+|c|\right)|u|_0|e|_0 T \\ &\le \left(1+|c|\right)\alpha|u|_0^{p-1} \int_0^T \left|u'(t)\right| dt + \left(1+|c|\right)\beta|u|_0 \int_0^T \left|u'(t)\right| dt \\ &\quad + \left(1+|c|\right)|u|_0 \int_0^T \left|g\left(t,u(t-\tau)\right)\right| dt + \left(1+|c|\right)|u|_0|e|_0 T. \qquad (3.6)\end{aligned}$$

Write

$$\begin{aligned}I_+ &= \left\{t\in[0,T] : g\left(t,u(t-\tau)\right)\ge 0\right\}; \\ I_- &= \left\{t\in[0,T] : g\left(t,u(t-\tau)\right)\le 0\right\}.\end{aligned}$$

Then it follows from (3.3) and $[H_2](1)$ that

$$\begin{aligned}\int_0^T \left|g\left(t,u(t-\tau)\right)\right| dt &= \int_{I_+} g\left(t,u(t-\tau)\right) dt - \int_{I_-} g\left(t,u(t-\tau)\right) dt \\ &= 2\int_{I_+} g\left(t,u(t-\tau)\right) dt \\ &\le 2m_0 \int_0^T u^{p-1}(t-\tau)\, dt + 2\int_0^T m_1\, dt \\ &\le 2m_0 T|u|_0^{p-1} + 2Tm_1. \qquad (3.7)\end{aligned}$$

Substituting (3.7) into (3.6), combining with (3.5) and Lemma 2.2, we can have

$$\begin{aligned}&\int_0^T \left|(Au)^{(m)}(t)\right|^p dt \\ &\quad\le 2\left(1+|c|\right)m_0 T|u|_0^p + \left(1+|c|\right)\alpha|u|_0^{p-1} \int_0^T \left|u'(t)\right| dt \\ &\qquad + \left(1+|c|\right)\beta|u|_0 \int_0^T \left|u'(t)\right| dt + 2\left(1+|c|\right)m_1 T|u|_0 + \left(1+|c|\right)|e|_0 T|u|_0 \\ &\quad\le 2\left(1+|c|\right)m_0 T^{p+1} M_1^p(m)\left(\frac{D_2}{TM_1(m)} + \int_0^T \left|u^{(m)}(t)\right| dt\right)^p\end{aligned}$$

$$\begin{aligned}
&+ (1+|c|)\alpha T^p M_1^p(m)\left(\frac{D_2}{TM_1(m)} + \int_0^T \left|u^{(m)}(t)\right| dt\right)^{p-1} \cdot \int_0^T \left|u^{(m)}(t)\right| dt \\
&+ (1+|c|)\beta T^2 M_1^2(m)\left(\frac{D_2}{TM_1(m)} + \int_0^T \left|u^{(m)}(t)\right| dt\right) \cdot \int_0^T \left|u^{(m)}(t)\right| dt \\
&+ 2(1+|c|)m_1 T^2 M_1(m)\left(\frac{D_2}{TM_1(m)} + \int_0^T \left|u^{(m)}(t)\right| dt\right) \\
&+ (1+|c|)|e|_0 T^2 M_1(m)\left(\frac{D_2}{TM_1(m)} + \int_0^T \left|u^{(m)}(t)\right| dt\right).
\end{aligned} \tag{3.8}$$

It follows from conclusion (2) of Lemma 2.1 and by applying the Hölder inequality that

$$\begin{aligned}
\int_0^T \left|u^{(m)}(t)\right| dt &= \int_0^T \left|\left(A^{-1}(Au)^{(m)}\right)(t)\right| dt \\
&\le \frac{\int_0^T |(Au)^{(m)}(t)|\, dt}{1-|c|}.
\end{aligned} \tag{3.9}$$

From the above inequality, we consider the following two cases:

Case 1 If $\int_0^T |(Au)^{(m)}(t)|\, dt = 0$, then $\int_0^T |u^{(m)}(t)|\, dt = 0$, it follows from (3.5) and Lemma 2.2 that

$$\begin{aligned}
|u|_0 &\le D_2 + \int_0^T \left|u'(s)\right| ds \le D_2 + T\left|u'\right|_0 \\
&\le D_2 + TM_1(m)\int_0^T \left|u^{(m)}(t)\right| dt \\
&= D_2.
\end{aligned} \tag{3.10}$$

Case 2 If $\int_0^T |(Au)^{(m)}(t)|\, dt > 0$, then substituting (3.9) into (3.8), we can have

$$\begin{aligned}
&\int_0^T \left|(Au)^{(m)}(t)\right|^p dt \\
&\le 2(1+|c|)m_0 T^{p+1} M_1^p(m)\left(\frac{D_2}{TM_1(m)} + \frac{\int_0^T |(Au)^{(m)}(t)|\, dt}{1-|c|}\right)^p \\
&\quad + (1+|c|)\alpha T^p M_1^p(m)\left(\frac{D_2}{TM_1(m)} + \frac{\int_0^T |(Au)^{(m)}(t)|\, dt}{1-|c|}\right)^{p-1} \cdot \frac{\int_0^T |(Au)^{(m)}(t)|\, dt}{1-|c|} \\
&\quad + (1+|c|)\beta T^2 M_1^2(m)\left(\frac{D_2}{TM_1(m)} + \frac{\int_0^T |(Au)^{(m)}(t)|\, dt}{1-|c|}\right) \cdot \frac{\int_0^T |(Au)^{(m)}(t)|\, dt}{1-|c|} \\
&\quad + 2(1+|c|)m_1 T^2 M_1(m)\left(\frac{D_2}{TM_1(m)} + \frac{\int_0^T |(Au)^{(m)}(t)|\, dt}{1-|c|}\right) \\
&\quad + (1+|c|)|e|_0 T^2 M_1(m)\left(\frac{D_2}{TM_1(m)} + \frac{\int_0^T |(Au)^{(m)}(t)|\, dt}{1-|c|}\right).
\end{aligned} \tag{3.11}$$

Furthermore,

$$\begin{aligned}
&\left(\frac{D_2}{TM_1(m)}+\frac{\int_0^T|(Au)^{(m)}(t)|\,dt}{1-|c|}\right)^p \\
&\quad=\frac{1}{(1-|c|)^p}\left(\int_0^T\left|(Au)^{(m)}(t)\right|dt\right)^p\left(1+\frac{\frac{D_2}{TM_1(m)}(1-|c|)}{\int_0^T|(Au)^{(m)}(t)|\,dt}\right)^p. \qquad (3.12)
\end{aligned}$$

By classical elementary inequalities, we see that there exists a $l(p)>0$ which is dependent on p only, such that

$$(1+x)^p<1+(1+p)x,\quad x\in\bigl(0,l(p)\bigr]. \qquad (3.13)$$

If $\frac{\frac{D_2}{TM_1(m)}(1-|c|)}{\int_0^T|(Au)^{(m)}(t)|\,dt}>l(p)$, then $\int_0^T|(Au)^{(m)}(t)|\,dt<\frac{\frac{D_2}{TM_1(m)}(1-|c|)}{l(p)}$. It follows from (3.5), (3.9) and Lemma 2.2 that

$$|u|_0\le D_2+TM_1(m)\int_0^T\left|u^{(m)}(t)\right|\le D_2+\frac{D_2}{l(p)}:=M_1. \qquad (3.14)$$

If $\frac{\frac{D_2}{TM_1(m)}(1-|c|)}{\int_0^T|(Au)^{(m)}(t)|\,dt}\le l(p)$, then it follows from (3.12) and (3.13) that

$$\begin{aligned}
&\left(\frac{D_2}{TM_1(m)}+\frac{\int_0^T|(Au)^{(m)}(t)|\,dt}{1-|c|}\right)^p \\
&\quad\le\frac{1}{(1-|c|)^p}\left(\int_0^T\left|(Au)^{(m)}(t)\right|dt\right)^p\left(1+\frac{(p+1)\frac{D_2}{TM_1(m)}(1-|c|)}{\int_0^T|(Au)^{(m)}(t)|\,dt}\right) \\
&\quad=\frac{(\int_0^T|(Au)^{(m)}(t)|\,dt)^p}{(1-|c|)^p}+\frac{(p+1)\frac{D_2}{TM_1(m)}}{(1-|c|)^{p-1}}\cdot\left(\int_0^T\left|(Au)^{(m)}(t)\right|dt\right)^{p-1}. \qquad (3.15)
\end{aligned}$$

Substituting (3.15) into (3.11) and by applying the Hölder inequality, we can see that

$$\begin{aligned}
&\int_0^T\left|(Au)^{(m)}(t)\right|^p dt \\
&\quad\le\frac{M_1^p(m)T^{2p-1}(1+|c|)(2m_0T+\alpha)}{(1-|c|)^p}\int_0^T\left|(Au)^{(m)}(t)\right|^p dt \\
&\qquad+\frac{T^{\frac{2p^2-3p+1}{p}}M_1^{p-1}(m)D_2(1+|c|)[2m_0T(p+1)+\alpha p]}{(1-|c|)^{p-1}}\cdot\left(\int_0^T\left|(Au)^{(m)}(t)\right|^p dt\right)^{\frac{p-1}{p}} \\
&\qquad+\frac{(1+|c|)\beta T^{4p-2}M_1^2(m)}{(1-|c|)^2}\left(\int_0^T\left|(Au)^{(m)}(t)\right|^p dt\right)^{\frac{2}{p}} \\
&\qquad+\frac{(1+|c|)T^{\frac{2p-1}{p}}M_1(m)[\beta D_2+2m_1T+|e|_0T]}{1-|c|}\cdot\left(\int_0^T\left|(Au)^{(m)}(t)\right|^p dt\right)^{\frac{1}{p}} \\
&\qquad+\bigl(1+|c|\bigr)TD_2\bigl(2m_1+|e|_0\bigr). \qquad (3.16)
\end{aligned}$$

It follows from $\frac{M_1^p(m)T^{2p-1}(1+|c|)(2m_0T+\alpha)}{(1-|c|)^p} < 1$ and $p \geq 2$ that there exists a positive constant $M_2 > 0$ such that

$$\int_0^T \left|(Au)^{(m)}(t)\right|^p dt \leq M_2.$$

Then by (3.9), we can see that

$$\begin{aligned}\int_0^T \left|u^{(m)}(t)\right| dt &\leq \frac{\int_0^T |(Au)^{(m)}(t)|\,dt}{1-|c|} \\ &\leq \frac{T^{\frac{p-1}{p}}(\int_0^T |(Au)^{(m)}(t)|^p\,dt)^{\frac{1}{p}}}{1-|c|} \\ &\leq \frac{T^{\frac{p-1}{p}}M_2^{\frac{1}{p}}}{1-|c|} := M_3, \end{aligned} \tag{3.17}$$

which together with (3.5) and Lemma 2.2 yields

$$\begin{aligned}|u|_0 &\leq D_2 + \int_0^T \left|u'(s)\right| ds \leq D_2 + T\left|u'\right|_0 \\ &\leq D_2 + TM_1(m)\int_0^T \left|u^{(m)}(t)\right| dt \\ &\leq D_2 + TM_1(m)M_3 := M_4.\end{aligned}$$

Therefore, in both Case 1 and Case 2, we obtain

$$|u|_0 \leq M_4. \tag{3.18}$$

From the second equation of (3.1), we can get

$$\begin{aligned}\int_0^T \left|v^{(m)}(t)\right| dt \leq \lambda\biggl[&\int_0^T \left|f\bigl(u(t)\bigr)\right|\left|u'(t)\right| dt + \int_0^T \left|g\bigl(t,u(t-\tau)\bigr)\right| dt \\ &+ \int_0^T \left|e(t)\right| dt\biggr],\end{aligned}$$

from which by applying $[H_3]$ and (3.7), we have

$$\begin{aligned}\int_0^T \left|v^{(m)}(t)\right| dt \leq \lambda\bigl[&\alpha T|u|_0^{p-2}\left|u'\right|_0 + \beta T\left|u'\right|_0 + 2m_0T|u|_0^{p-1} \\ &+ 2Tm_1 + |e|_0T\bigr].\end{aligned} \tag{3.19}$$

By Lemma 2.2, (3.17) and (3.18), we obtain

$$\begin{aligned}\int_0^T \left|v^{(m)}(t)\right| dt \leq \lambda\bigl[&\alpha T|u|_0^{p-2}\left|u'\right|_0 + \beta T\left|u'\right|_0 + 2m_0T|u|_0^{p-1} \\ &+ 2Tm_1 + |e|_0T\bigr]\end{aligned}$$

$$\begin{aligned}&\le\lambda\big[\alpha TM_4^{p-2}M_1(m)M_3+\beta TM_1(m)M_3+2m_0TM_4^{p-1}\\&\quad+2Tm_1+|e|_0T\big].\end{aligned}\tag{3.20}$$

Moreover, integrating the first equation of (3.1) on the interval $[0,T]$, we have

$$\int_0^T\big|v(t)\big|^{q-2}v(t)\,dt=0,$$

which implies that there exists $\eta\in[0,T]$ such that $v(\eta)=0$. Thus,

$$\big|v(t)\big|=\left|\int_\eta^t v'(s)\,ds+v(\eta)\right|\le\int_0^T\big|v'(s)\big|\,ds\le T\big|v'\big|_0,$$

by Lemma 2.2 and (3.20), we can obtain

$$\begin{aligned}|v|_0&=\max_{t\in[0,T]}\big|v(t)\big|\\&\le T\big|v'\big|_0\le TM_1(m)\int_0^T\big|v^{(m)}(t)\big|\,dt\\&\le\lambda TM_1(m)\big[\alpha TM_4^{p-2}M_1(m)M_3+\beta TM_1(m)M_3+2m_0TM_4^{p-1}\\&\quad+2Tm_1+|e|_0T\big]\\&\le TM_1(m)\big[\alpha TM_4^{p-2}M_1(m)M_3+\beta TM_1(m)M_3+2m_0TM_4^{p-1}\\&\quad+2Tm_1+|e|_0T\big]\\&:=A_3.\end{aligned}\tag{3.21}$$

On the other hand, from the second equation of (3.1) and $[H_2]$, we can see that

$$\begin{aligned}v^{(m)}(t+\tau)=-\lambda f\big(u(t+\tau)\big)u'(t+\tau)\\-\lambda\big[g_1\big(t+\tau,u(t)\big)+g_0\big(u(t)\big)\big]\\+\lambda e(t+\tau).\end{aligned}\tag{3.22}$$

Furthermore, multiplying both sides of equation (3.22) by $u'(t)$, we have

$$\begin{aligned}v^{(m)}(t+\tau)u'(t)=-\lambda f\big(u(t+\tau)\big)u'(t+\tau)u'(t)\\-\lambda\big[g_1\big(t+\tau,u(t)\big)+g_0\big(u(t)\big)\big]u'(t)\\+\lambda e(t+\tau)u'(t).\end{aligned}\tag{3.23}$$

Let $\sigma\in[0,T]$ be as in (3.4). For any $t\in[\sigma,T]$, integrating equation (3.23) on the interval $[\sigma,t]$, we get

$$\begin{aligned}\lambda\int_{u(\sigma)}^{u(t)}g_0(u)\,du&=\lambda\int_\sigma^t g_0\big(u(t)\big)u'(t)\,dt\\&=-\int_\sigma^t v^{(m)}(t+\tau)u'(t)\,dt-\lambda\int_\sigma^t f\big(u(t+\tau)\big)u'(t+\tau)u'(t)\,dt\\&\quad-\lambda\int_\sigma^t g_1\big(t+\tau,u(t)\big)u'(t)\,dt+\lambda\int_\sigma^t e(t+\tau)u'(t)\,dt,\end{aligned}$$

which together with (3.20) yields

$$
\begin{aligned}
&\lambda\left|\int_{u(\tau)}^{u(t)} g_0(u)\,du\right| \\
&\quad = \lambda\left|\int_{\tau}^{t} g_0\big(u(t)\big)u'(t)\,dt\right| \\
&\quad \le \int_0^T \left|v^{(m)}(t+\tau)\right|\left|u'(t)\right|dt + \lambda\int_0^T \left|f\big(u(t+\tau)\big)\right|\left|u'(t+\tau)\right|\left|u'(t)\right|dt \\
&\qquad + \lambda\int_0^T \left|g_1\big(t+\tau,u(t)\big)\right|\left|u'(t)\right|dt + \lambda\int_0^T \left|e(t+\tau)\right|\left|u'(t)\right|dt \\
&\quad \le \lambda\left|u'\right|_0\left[\alpha TM_4^{p-2}M_1(m)M_3 + \beta TM_1(m)M_3 + 2m_0TM_4^{p-1} + 2Tm_1 + |e|_0T\right] \\
&\qquad + \lambda\left|u'\right|_0^2\int_0^T \left|f\big(u(t+\tau)\big)\right|dt + \lambda\left|u'\right|_0\int_0^T \left|g_1\big(t+\tau,u(t)\big)\right|dt \\
&\qquad + \lambda\left|u'\right|_0\int_0^T \left|e(t+\tau)\right|dt.
\end{aligned}
$$

Furthermore, set

$$
F_{M_4} = \max_{|u|\le M_4}\left|f(u)\right| \quad \text{and} \quad G_{M_4} = \max_{t\in[0,T],|u|\le M_4}\left|g_1(t,u)\right|,
$$

then we have

$$
\begin{aligned}
\lambda\left|\int_{u(\tau)}^{u(t)} g_0(u)\,du\right| &\le \lambda\left|u'\right|_0\left[\alpha TM_4^{p-2}M_1(m)M_3 + \beta TM_1(m)M_3\right] \\
&\quad + \lambda\left|u'\right|_0\left[2m_0TM_4^{p-1} + 2Tm_1 + |e|_0T\right] \\
&\quad + \lambda\left|u'\right|_0^2 TF_{M_4} + \lambda\left|u'\right|_0 TG_{M_4} + \lambda\left|u'\right|_0 T|e|_0,
\end{aligned}
$$

by (3.17) and Lemma 2.2, we obtain

$$
\begin{aligned}
\left|\int_{u(\tau)}^{u(t)} g_0(u)\,du\right| &\le M_1(m)M_3\left[\alpha TM_4^{p-2}M_1(m)M_3 + \beta TM_1(m)M_3\right] \\
&\quad + M_1(m)M_3\left[2m_0TM_4^{p-1} + 2Tm_1 + |e|_0T\right] \\
&\quad + \left[M_1(m)M_3\right]^2 TF_{M_4} + M_1(m)M_3TG_{M_4} \\
&\quad + M_1(m)M_3T|e|_0 \\
&< +\infty.
\end{aligned}
\tag{3.24}
$$

According to condition (2) in $[H_2]$, we see that there exists a constant $M_5 > 0$ such that, for $t \in [\sigma, T]$,

$$
u(t) \ge M_5. \tag{3.25}
$$

In a similar way, we can handle the case of $t \in [0, \sigma]$.

Let us define

$$0 < A_1 = \min\{D_1, M_5\}, \quad \text{and} \quad A_2 = \max\{D_2, M_4\}.$$

Then by (3.4), (3.10), (3.18) and (3.25), we can obtain

$$A_1 \le u(t) \le A_2. \tag{3.26}$$

Clearly, A_1 and A_2 are independent of λ. Therefore, the proof of Theorem 3.1 is complete. □

Theorem 3.2 *Assume that all the conditions in Theorem* 3.1 *hold, then system* (1.5) *has at least one positive T-periodic solution.*

Proof Set

$$\Omega = \left\{x = (u,v)^\top \in X : \frac{A_1}{2} < u(t) < A_2 + 1, |v|_0 < A_3 + 1\right\}.$$

From (2.3) and (2.4), one can easily see that Ω is an open bounded subset of X and N is L-compact on $\overline{\Omega}$. Then the conditions (1) and (2) of Lemma 2.3 are satisfied.

In the following, we prove that condition (3) of Lemma 2.3 also holds.

Now, we let

$$\omega = Bx = B\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u - \frac{A_1+A_2}{2} \\ v \end{pmatrix}.$$

Define a linear isomorphism

$$J: \quad \operatorname{Im} Q \to \ker L, \qquad J(u,v) = \begin{pmatrix} v \\ -u \end{pmatrix},$$

and define

$$H(\mu,x) = \mu Bx + (1-\mu)JQNx, \quad \forall (x,\mu) \in (\Omega \cap \ker L) \times [0,1].$$

Then we can get

$$K(\mu,x) = \begin{pmatrix} \mu u - \frac{\mu(A_1+A_2)}{2} \\ \mu v \end{pmatrix} + \frac{1-\mu}{T}\begin{pmatrix} \int_0^T [f(u)u' + g(t,u)]\,dt \\ (1-|c|)^{-1}\int_0^T \varphi_q(v)\,dt \end{pmatrix}. \tag{3.27}$$

In order to prove the condition (3) of Lemma 2.3 is also satisfied, firstly, we prove that $K(\mu,x)$ is a homotopic mapping. By way of contradiction, i.e., suppose that there exist $\mu_0 \in [0,1]$ and $x_0 = \binom{u_0}{v_0} \in \partial(\Omega \cap \ker L)$ such that $K(\mu_0,x_0) = 0$. Then, substituting μ_0 and x_0 into (3.27), we have

$$K(\mu_0,x_0) = \begin{pmatrix} \mu_0 u_0 - \frac{\mu_0(A_1+A_2)}{2} + (1-\mu_0)\overline{g}(u_0) \\ \mu_0 v_0 + \frac{1-\mu_0}{1-|c|}\varphi_q(v_0) \end{pmatrix}. \tag{3.28}$$

It follows from $K(\mu_0, x_0) = 0$ that

$$\mu_0 \nu_0 + \frac{1-\mu_0}{1-|c|}\varphi_q(\nu_0) = 0,$$

which together with $\mu_0 \in [0,1]$ gives $\nu_0 = 0$. Thus, we can get $u_0 = \frac{A_1}{2}$ or $A_2 + 1$. Furthermore, it follows from $[H_1](2)$ that $\overline{g}(\frac{A_1}{2}) < 0$ and $\overline{g}(A_2+1) > 0$, substituting $u_0 = \frac{A_1}{2}$ or A_2+1 into (3.28), we can obtain

$$\mu_0 u_0 - \frac{\mu_0(A_1+A_2)}{2} + (1-\mu_0)\overline{g}(u_0) < 0 \tag{3.29}$$

and

$$\mu_0 u_0 - \frac{\mu_0(A_1+A_2)}{2} + (1-\mu_0)\overline{g}(u_0) > 0. \tag{3.30}$$

From (3.29) and (3.30), we have $K(\mu_0, x_0) \neq 0$, which is a contradiction. Therefore $K(\mu, x)$ is a homotopic mapping and $x^\top K(\mu, x) \neq 0$. For all $(x,\mu) \in (\partial\Omega \cap \ker L) \times [0,1]$, we get

$$\begin{aligned}
\deg(JQN, \Omega \cap \ker L, 0) &= \deg\bigl(K(0,x), \Omega \cap \ker L, 0\bigr) \\
&= \deg\bigl(K(1,x), \Omega \cap \ker L, 0\bigr) \\
&= \deg(Bx, \Omega \cap \ker L, 0) \\
&= \sum_{x \in B^{-1}(0)} \operatorname{sgn}\bigl(\det B'(x)\bigr) \\
&= 1 \neq 0.
\end{aligned}$$

Thus, the condition (3) of Lemma 2.3 is also satisfied. Therefore, we can conclude that equation (1.5) has at least one positive T-periodic solution. □

4 Examples

In this section, we provide two examples to illustrate our main result.

Example 4.1 Consider the following third-order p-Laplacian neutral functional differential equation:

$$\begin{aligned}
&\left(\varphi_4\left(\left(x(t) - \frac{1}{3}x\left(t - \frac{\pi}{4}\right)\right)^{(3)}\right)\right)^{(3)} + \left(\frac{u^4(t)}{6+u^2(t)} + 3\right)u'(t) \\
&\quad + \frac{1}{64}(1+\sin 4t)u^3(t-\tau) - \frac{1}{u^3(t-\tau)} = \frac{1}{16}\sin 4t.
\end{aligned} \tag{4.1}$$

Conclusion *Problem* (4.1) *has at least one positive* $\pi/2$*-periodic solution.*

Proof Corresponding to (1.5), we have

$$\begin{aligned}
&f\bigl(u(t)\bigr) = \frac{u^4}{6+u^2} + 3, \quad u \in (0,+\infty), \qquad e(t) = \frac{1}{16}\sin 4t, \\
&g\bigl(t, u(t-\tau)\bigr) = \frac{1}{64}(1+\sin 4t)u^3(t-\tau) - \frac{1}{u^3(t-\tau)}, \quad u \in (0,+\infty).
\end{aligned}$$

Then we can have and choose

$$m = 3, \qquad p = 4, \qquad \gamma = \frac{\pi}{4}, \qquad c = \frac{1}{3}, \qquad T = \frac{\pi}{2},$$
$$\alpha = \frac{1}{6}, \qquad \beta = 3, \qquad m_0 = \frac{1}{32}, \qquad D_1 = 1, \qquad D_2 = 9.$$

Moreover, by Lemma 2.2 we have $M_1(3) = T/12$. Thus, the conditions $[H_1]$-$[H_3]$ are satisfied. Meanwhile, we also have

$$\frac{M_1^p(m)T^{2p-1}(1+|c|)(2m_0T+\alpha)}{(1-|c|)^p} \approx 0.026 < 1.$$

Hence, by applying Theorem 3.1-3.2, we can see that equation (4.1) has at least one positive $\pi/2$-periodic solution. □

Example 4.2 Consider the following fourth-order p-Laplacian neutral functional differential equation:

$$\left(\varphi_4\left(\left(x(t) - \frac{1}{10}x\left(t - \frac{\pi}{8}\right)\right)^{(4)}\right)\right)^{(4)} + \left(\frac{u^4(t)}{8+u^2(t)} + 9\right)u'(t)$$
$$+ \frac{1}{64}(1+\sin 2t)u^3(t-\tau) - \frac{1}{u^3(t-\tau)} = \frac{1}{32}\sin 2t. \tag{4.2}$$

Conclusion *Problem* (4.2) *has at least one positive* π*-periodic solution.*

Proof Corresponding to (1.5), we have

$$f\big(u(t)\big) = \frac{u^4}{8+u^2} + 9, \quad u \in (0,+\infty), \qquad e(t) = \frac{1}{32}\sin 2t,$$
$$g\big(t, u(t-\tau)\big) = \frac{1}{64}(1+\sin 2t)u^3(t-\tau) - \frac{1}{u^3(t-\tau)}, \quad u \in (0,+\infty).$$

Then we can choose

$$m = 4, \qquad p = 4, \qquad \gamma = \frac{\pi}{8}, \qquad c = \frac{1}{10}, \qquad T = \pi,$$
$$\alpha = \frac{1}{8}, \qquad \beta = 9, \qquad m_0 = \frac{1}{32}, \qquad D_1 = 1, \qquad D_2 = 9.$$

Moreover, by Lemma 2.2, we get $M_1(4) = \frac{T^2}{24\sqrt{15}}$. Thus, the conditions $[H_1]$-$[H_3]$ are satisfied. Meanwhile, we also have

$$\frac{M_1^p(m)T^{2p-1}(1+|c|)(2m_0T+\alpha)}{(1-|c|)^p} \approx 0.207 < 1.$$

Hence, by applying Theorem 3.1-3.2, we can see that equation (4.2) has at least one positive π-periodic solution. □

Example 4.3 Consider the following fourth-order p-Laplacian neutral functional differential equation:

$$\begin{aligned}&\left(\varphi_2\left(\left(x(t)-\frac{1}{10}x\left(t-\frac{\pi}{8}\right)\right)^{(4)}\right)\right)^{(4)}+9u'(t)\\&\quad+\frac{1}{64}(1+\sin 4t)u(t-\tau)-\frac{1}{u(t-\tau)}=\frac{1}{32}\sin 4t.\end{aligned}\tag{4.3}$$

Conclusion *Problem* (4.3) *has at least one positive* $\frac{\pi}{2}$*-periodic solution.*

Proof Corresponding to (1.5), we have

$$\begin{aligned}&f\big(u(t)\big)=9,\quad u\in(0,+\infty),\qquad e(t)=\frac{1}{32}\sin 4t,\\&g\big(t,u(t-\tau)\big)=\frac{1}{64}(1+\sin 4t)u(t-\tau)-\frac{1}{u(t-\tau)},\quad u\in(0,+\infty).\end{aligned}$$

Then we can choose

$$\begin{aligned}&m=4,\qquad p=2,\qquad \gamma=\frac{\pi}{8},\qquad c=\frac{1}{10},\qquad T=\frac{\pi}{2},\\&\alpha=0,\qquad \beta=9,\qquad m_0=\frac{1}{32},\qquad D_1=1,\qquad D_2=9.\end{aligned}$$

Moreover, by Lemma 2.2, we get $M_1(4)=\frac{T^2}{24\sqrt{15}}$. Thus, the conditions $[H_1]$-$[H_3]$ are satisfied. Meanwhile, we also have

$$\frac{M_1^p(m)T^{2p-1}(1+|c|)(2m_0T+\alpha)}{(1-|c|)^p}\approx 0.042<1.$$

Hence, by applying Theorem 3.1-3.2, we can see that equation (4.3) has at least one positive $\frac{\pi}{2}$-periodic solution. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have equally contributed to obtaining new results in this article and also read and approved the final manuscript.

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