Boundary Value Problems
a SpringerOpen Journal

Periodic solutions of planar Hamiltonian systems with asymmetric nonlinearities

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Abstract

In this paper, we look for periodic solutions of planar Hamiltonian systems

$$\begin{cases} x' = f(y) + p_1(t, y), \\ y' = -g(x) + p_2(t, x). \end{cases}$$

By using the Poincaré-Birkhoff twist theorem, we prove the existence and multiplicity of periodic solutions of the given system when f satisfies an asymmetric condition and the related time map satisfies an oscillating condition.

MSC: 34C11; 34C15; 34C25

Keywords: asymmetric nonlinearity; periodic solution; Poincaré-Birkhoff twist theorem

1 Introduction

In this paper, we are concerned with the existence and multiplicity of periodic solutions of planar Hamiltonian systems

$$\begin{cases} x' = f(y) + p_1(t, y), \\ y' = -g(x) + p_2(t, x), \end{cases} \tag{1.1}$$

where $f, g : \mathbf{R} \to \mathbf{R}$ are continuous, $p_i : \mathbf{R}^2 \to \mathbf{R}$ ($i = 1, 2$) are continuous and 2π-periodic with the first variable t.

In the case when $f(y) \equiv y$, $p_1(t, y) \equiv 0$ and $p_2(t, x) = p(t)$, system (1.1) becomes

$$\begin{cases} x' = y, \\ y' = -g(x) + p(t), \end{cases}$$

which is equivalent to the differential equation

$$x'' + g(x) = p(t). \tag{1.2}$$

The existence and multiplicity of periodic solutions of Eq. (1.2) have been widely studied in the literature (see [1–5] and the references therein). Recently, the periodic solutions of

planar Hamiltonian systems have been studied with an increasing interest (see [6–11]). In [11], Fonda and Sfecci studied the periodic solutions of the planar Hamiltonian systems of the type

$$x' = g_1(t, y), \qquad y' = -g_2(t, x). \tag{1.3}$$

Assume that the following conditions hold:

$$a_+ \le \liminf_{y \to +\infty} \frac{g_1(t,y)}{y} \le \limsup_{y \to +\infty} \frac{g_1(t,y)}{y} \le a^+,$$

$$a_- \le \liminf_{y \to -\infty} \frac{g_1(t,y)}{y} \le \limsup_{y \to -\infty} \frac{g_1(t,y)}{y} \le a^-,$$

and

$$b_+ \le \liminf_{x \to +\infty} \frac{g_2(t,x)}{x} \le \limsup_{x \to +\infty} \frac{g_2(t,x)}{x} \le b^+,$$

$$b_- \le \liminf_{x \to -\infty} \frac{g_2(t,x)}{x} \le \limsup_{x \to -\infty} \frac{g_2(t,x)}{x} \le b^-,$$

where $a^\pm$, $a_\pm$, $b^\pm$ and $b_\pm$ are positive constants. It was proved in [11] that system (1.3) has at least one 2π-periodic solution provided that there exists an integer $n > 0$ such that

$$\left(\frac{1}{\sqrt{a_+}} + \frac{1}{\sqrt{a_-}} \right) \left(\frac{1}{\sqrt{b_+}} + \frac{1}{\sqrt{b_-}} \right) < \frac{4}{n} \tag{1.4}$$

and

$$\left(\frac{1}{\sqrt{a^+}} + \frac{1}{\sqrt{a^-}} \right) \left(\frac{1}{\sqrt{b^+}} + \frac{1}{\sqrt{b^-}} \right) > \frac{4}{n+1}. \tag{1.5}$$

In the present paper, we shall deal with the periodic solutions of system (1.1) when the non-resonant conditions (1.4) and (1.5) do not hold. Assume the following conditions hold:

(h_1) g satisfies $\lim_{|x| \to +\infty} \mathrm{sgn}(x)g(x) = +\infty$;
(h_2) there exists a constant $L > 0$ such that, for all $x, y \in \mathbf{R}$, $|g(x) - g(y)| \le L|x - y|$;
(h_3) the limits $\lim_{|y| \to +\infty} \frac{p_i(t,y)}{y} = 0$ ($i = 1, 2$) hold uniformly with respect to $t \in [0, 2\pi]$;
(h_4) there are two positive constants a and b such that

$$\lim_{y \to +\infty} \frac{f(y)}{y} = a, \qquad \lim_{y \to -\infty} \frac{f(y)}{y} = b.$$

It is well known that the time map plays an important role in studying the periodic solutions of Eq. (1.2) (see [1, 2, 5] and the references therein). In this paper, we also use the time map to study the periodic solutions of system (1.1). Let us set

$$G(x) = \int_0^x g(s) \, ds.$$

Under condition (h_1), we can define the time map

$$\tau(c) = \int_{w(c)}^{d(c)} \frac{ds}{\sqrt{2(c - G(s))}}$$

for $c > 0$ large enough, where $w(c)$ and $d(c)$ satisfy $w(c) < 0 < d(c)$ and $G(w(c)) = G(d(c)) = c$. Assume that the time map $\tau(c)$ satisfies the condition:

(τ)   There exist a constant $\sigma > 0$, an integer $n > 0$, and two sequences $\{a_k\}$ and $\{b_k\}$ such that $\lim_{k \to \infty} a_k = +\infty$, $\lim_{k \to \infty} b_k = +\infty$; and moreover

$$\tau(a_k) < \frac{2\pi}{mn} - \sigma, \qquad \tau(b_k) > \frac{2\pi}{mn} + \sigma,$$

where $m = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$ and a, b are given in condition (h_4).

We prove the following theorem.

Theorem 1.1 *Assume that conditions (h_i) $(i = 1, \ldots, 4)$ and (τ) hold. Then system (1.1) has infinitely many 2π-periodic solutions $\{(x_k(t), y_k(t))\}_{k=1}^{\infty}$ which satisfy*

$$\lim_{k \to \infty} \left(\min_{t \in R} \left(|x_k(t)| + |y_k(t)| \right) \right) = +\infty.$$

Moreover, for each integer $k \geq 1$, both $x_k(t)$ and $y_k(t)$ have exactly $2n$ simple zeros in $[0, 2\pi)$.

From Theorem 1.1 we can obtain the following corollary.

Corollary 1.2 *Assume that a, b are two positive constants, $e, p : \mathbf{R} \to \mathbf{R}$ are continuous and conditions (h_i) $(i = 1, 2)$ and (τ) are satisfied. Then the same conclusions of Theorem 1.1 still hold for the system*

$$\begin{cases} x' = ay^+ - by^- + e(t), \\ y' = -g(x) + p(t). \end{cases}$$

Remark 1.3 From condition (h_4) we know that f can be written in the form

$$f(y) = ay^+ - by^- + h(y),$$

where $h : \mathbf{R} \to \mathbf{R}$ is continuous and satisfies

$$\lim_{|y| \to +\infty} \frac{h(y)}{y} = 0.$$

Therefore, it suffices for us to prove the main theorem for the system

$$\begin{cases} x' = ay^+ - by^- + p_1(t, y), \\ y' = -g(x) + p_2(t, x), \end{cases} \tag{1.1}'$$

where p_i $(i = 1, 2)$ satisfy condition (h_3). In the case $a = b \neq 1$, by introducing a rescaling of the time $s = at$, $u(s) = x(\frac{s}{a})$, $v(s) = y(\frac{s}{a})$, we find the equivalent system of $(1.1)'$ (having the

classical form)

$$
\begin{cases}
u' = v + q_1(s, v), \\
v' = -\tilde{g}(u) + q_2(s, u).
\end{cases}
$$

Such a rescaling cannot be easily applied in the case $a \neq b$ because we do not know when the solution will change its sign.

We finally stress the fact that the proofs of the above results will be given under the additional assumptions that f and p_i ($i = 1, 2$) are locally Lipschitz continuous with variables y or x. It is shown in Section 4 that this requirement is not restrictive and that our results are valid for any continuous functions f and p_i ($i = 1, 2$).

2 Basic lemmas

At first, we consider the auxiliary autonomous system

$$
x' = ay^+ - by^-, \qquad y' = -g(x). \tag{2.1}
$$

The orbits of system (2.1) are curves determined by the equation

$$
\Gamma_c: \quad \frac{1}{2}ay^{+2} + \frac{1}{2}by^{-2} + G(x) = c, \tag{2.2}
$$

where c is an arbitrary constant. We can easily prove the following lemma.

Lemma 2.1 *Assume that condition* (h_1) *holds. Then there exists a constant* $c_0 > 0$ *such that, for any* $c > c_0$, Γ_c *is a closed curve which is star-shaped around the origin O.*

From Lemma 2.1 we know that, for $c \geq c_0$, each Γ_c intersects with the x-axis at two points $(w(c), 0)$ and $(d(c), 0)$, where $w(c)$ and $d(c)$ are continuous and satisfy

$$
w(c) < 0 < d(c), \qquad G(w(c)) = G(d(c)) = c.
$$

Let $(x_c(t), y_c(t))$ be the solution of system (2.1) lying on the curve Γ_c with $c \geq c_0$. Obviously, $(x_c(t), y_c(t))$ is periodic. Let us denote by $T(c)$ the least period of $(x_c(t), y_c(t))$. From the first equation of (2.1) and (2.2) we have that

$$
T(c) = \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \right) \int_{w(c)}^{d(c)} \frac{ds}{\sqrt{2(c - G(s))}}.
$$

By the definition, $T(c)$ is continuous for $c \geq c_0$.

Now we perform some phase-plane analysis for system $(1.1)'$. Let $(x(t), y(t)) = (x(t, x_0, y_0), y(t, x_0, y_0))$ be the solution of system $(1.1)'$ satisfying the initial condition

$$
x(0) = x_0, \qquad y(0) = y_0.
$$

Lemma 2.2 *Assume that conditions* (h_i) ($i = 1, 2, 3$) *hold. Then each solution* $(x(t), y(t))$ *of system* $(1.1)'$ *exists uniquely on the whole t-axis.*

Proof The proof follows directly from the fact that the nonlinearities are locally Lipschitz continuous and all have at most linear growth. □

According to Lemma 2.2, the Poincaré map $P : \mathbf{R}^2 \to \mathbf{R}^2$ is well defined by

$$P : (x_0, y_0) \to (x_1, y_1) = \big(x(2\pi, x_0, y_0), y(2\pi, x_0, y_0) \big).$$

Clearly, the Poincaré map P is an area-preserving homeomorphism. The fixed points of P correspond to the 2π periodic solutions of system $(1.1)'$.

Now, we take the polar coordinates transformation $x = r \cos\theta$, $y = r \sin\theta$ to system $(1.1)'$. Under this transformation, system $(1.1)'$ becomes

$$\begin{cases} \frac{dr}{dt} = r(a \sin^+\theta - b \sin^-\theta)\cos\theta - g(r\cos\theta)\sin\theta + p_1(t, r\sin\theta)\cos\theta \\ \qquad + p_2(t, r\cos\theta)\sin\theta, \\ \frac{d\theta}{dt} = -(a\sin^+\theta - b\sin^-\theta)\sin\theta - \frac{1}{r}g(r\cos\theta)\cos\theta - \frac{1}{r}p_1(t, r\sin\theta)\sin\theta \\ \qquad + \frac{1}{r}p_2(t, r\cos\theta)\cos\theta. \end{cases} \tag{2.3}$$

Denote by $(r(t), \theta(t)) = (r(t, r_0, \theta_0), \theta(t, r_0, \theta_0))$ the solution of (2.3) with the initial value

$$r(0) = r_0, \qquad \theta(0) = \theta_0,$$

with $x_0 = r_0 \cos\theta_0$, $y_0 = r_0 \sin\theta_0$. Clearly, the Poincaré map P can be written in the polar coordinate form $P : (r_0, \theta_0) \to (r^*, \theta^*)$ with

$$r^* = r(2\pi, r_0, \theta_0), \qquad \theta^* = \theta(2\pi, r_0, \theta_0) + 2l\pi,$$

where l is an arbitrary integer.

Applying the polar coordinate transformation $x = \rho \cos\varphi$, $y = \rho \sin\varphi$ to system (2.1), we get

$$\begin{cases} \frac{d\rho}{dt} = \rho(a\sin^+\varphi - b\sin^-\varphi)\cos\varphi - g(\rho\cos\varphi)\sin\varphi, \\ \frac{d\varphi}{dt} = -(a\sin^+\varphi - b\sin^-\varphi)\sin\varphi - \frac{1}{\rho}g(\rho\cos\varphi)\cos\varphi. \end{cases} \tag{2.4}$$

Denote by $(\rho(t), \varphi(t)) = (\rho(t, \rho_0, \varphi_0), \varphi(t, \rho_0, \varphi_0))$ the solution of (2.4) satisfying the initial value

$$\rho(0) = \rho_0, \qquad \varphi(0) = \varphi_0.$$

Using conditions (h_i) $(i = 1, 2, 3)$, it is not hard to prove the following lemma.

Lemma 2.3 *Assume that conditions (h_i) $(i = 1, 2, 3)$ hold. Then there exist constants $\gamma > 1$ and $R_0 > 0$ such that*

$$\frac{r_0}{\gamma} \le r(t) \le \gamma r_0, \quad \forall t \in [0, 2\pi], r_0 \ge R_0.$$

In particular, under conditions (h_i) $(i = 1, 2)$, $\rho(t)$ satisfies the inequality

$$\frac{\rho_0}{\gamma} \le \rho(t) \le \gamma \rho_0, \quad \forall t \in [0, 2\pi], \rho_0 \ge R_0.$$

Lemma 2.4 *Assume that conditions* (h_i) $(i = 1, 2, 3)$ *hold, and let*

$$\Phi(r_0, \theta_0) = \varphi(2\pi, r_0, \theta_0) - \theta_0; \qquad \Theta(r_0, \theta_0) = \theta(2\pi, r_0, \theta_0) - \theta_0.$$

Then, for any sufficiently small ε, there exists a positive constant ζ such that

$$\left| \Phi(r_0, \theta_0) - \Theta(r_0, \theta_0) \right| \leq \varepsilon \quad \text{for } r_0 \geq \zeta.$$

Proof Let $(\bar{x}(t), \bar{y}(t)) = (\bar{x}(t, x_0, y_0), \bar{y}(t, x_0, y_0))$ be the solution of (2.1) with $(\bar{x}(0), \bar{y}(0)) = (x_0, y_0)$. It is noted that $(x(t), y(t)) = (x(t, x_0, y_0), y(t, x_0, y_0))$ is a solution of system $(1.1)'$ with $(x(0), y(0)) = (x_0, y_0)$. Set

$$u(t) = u(t, x_0, y_0) = x(t, x_0, y_0) - \bar{x}(t, x_0, y_0),$$

$$v(t) = v(t, x_0, y_0) = y(t, x_0, y_0) - \bar{y}(t, x_0, y_0).$$

Then we have

$$\frac{du(t)}{dt} = a\left[y^+(t) - \bar{y}^+(t)\right] - b\left[y^-(t) - \bar{y}^-(t)\right] + p_1\left(t, y(t)\right),$$

$$\frac{dv(t)}{dt} = -\left[g\left(x(t)\right) - g\left(\bar{x}(t)\right)\right] + p_2\left(t, x(t)\right).$$

Let $d(t) = \sqrt{u^2(t) + v^2(t)}$. Then we get

$$d'(t) \leq \delta d(t) + \left| p_1\left(t, y(t)\right) \right| + \left| p_2\left(t, x(t)\right) \right|$$

with $\delta = \frac{1}{2}(\mu + L)$, $\mu = \max\{a, b\}$. From condition (h_3) we have that, for any sufficiently small $\eta > 0$, there exists $c_\eta > 0$ such that

$$\left| p_1(t, y) \right| \leq \eta |y| + c_\eta, \quad \forall (t, y) \in \mathbf{R}^2$$

and

$$\left| p_2(t, x) \right| \leq \eta |x| + c_\eta, \quad \forall (t, x) \in \mathbf{R}^2.$$

Therefore, we obtain

$$d'(t) \leq \delta d(t) + \eta\left(\left| x(t) \right| + \left| y(t) \right|\right) + 2c_\eta.$$

Solving this inequality, we get

$$d(t) \leq \eta e^{2\pi\delta} \int_0^{2\pi} \left(\left| x(t) \right| + \left| y(t) \right|\right) dt + A_\eta \leq \sqrt{2}\eta e^{2\pi\delta} \int_0^{2\pi} \sqrt{x^2(t) + y^2(t)}\, dt + A_\eta,$$

where $A_\eta = \frac{2c_\eta e^{2\pi\delta}}{\delta}$. It follows from Lemma 2.3 that, for $t \in [0, 2\pi]$,

$$d(t) \leq \eta\beta r_0 + A_\eta,$$

where $\beta = 2\sqrt{2}\pi\gamma e^{2\pi\delta}$. Write $\psi(t) = \psi(t, r_0, \theta_0) = \varphi(t, r_0, \theta_0) - \theta(t, r_0, \theta_0)$. Clearly, if $|\psi(t)| < \pi$, then $\psi(t)$ is just the angle between the vectors $(x(t), y(t))$ and $(\bar{x}(t), \bar{y}(t))$. Hence, we have

$$\cos\psi(t) = \frac{r^2(t) + \rho^2(t) - d^2(t)}{2r(t)\rho(t)} \geq 1 - \frac{d^2(t)}{2r(t)\rho(t)}.$$

It follows that

$$\sin^2\frac{\psi(t)}{2} \leq \frac{d^2(t)}{4r(t)\rho(t)}.$$

According to Lemma 2.3, we have that if η is sufficiently small and r_0 is large enough, then

$$\left|\sin\frac{\psi(t)}{2}\right| \leq \frac{d(t)}{2\sqrt{r(t)\rho(t)}} \leq \frac{\gamma(\eta\beta r_0 + A_\eta)}{2r_0} \leq \frac{\varepsilon}{4}.$$

Since $\psi(0) = 0$ and $\psi(t)$ varies continuously as t increases from 0 to 2π, we have

$$|\psi(t)| \leq 4\left|\sin\frac{\psi(t)}{2}\right| \leq \varepsilon.$$

Consequently, we have that there exists $\zeta > 0$ such that, for $r_0 \geq \zeta$,

$$\left|\Phi(r_0, \theta_0) - \Theta(r_0, \theta_0)\right| \leq \varepsilon, \quad \forall r_0 \geq \zeta. \qquad \square$$

Lemma 2.5 *Assume that conditions (h_i) $(i = 1, 2)$ and (τ) hold. Then there exists a constant $\omega > 0$ such that, for $t \in \mathbf{R}$ and k large enough,*

$$\varphi'(t) \leq -\omega,$$

with $(\rho_0 \cos\varphi_0, \rho_0 \sin\varphi_0) \in \Gamma_{a_k}$ or $(\rho_0 \cos\varphi_0, \rho_0 \sin\varphi_0) \in \Gamma_{b_k}$.

Proof From the definition of $T(c)$ and condition (τ) we know that, for each $k \in \mathbf{N}$,

$$T(a_k) \leq \frac{2\pi}{n} - m\sigma; \qquad T(b_k) \geq \frac{2\pi}{n} + m\sigma.$$

In what follows, without loss of generality, we assume that the sequence $\{T(b_k)\}$ is bounded. Otherwise, we can replace the sequence $\{T(b_k)\}$ with a bounded one because $T(c)$ is continuous for c large enough. We shall only deal with the first case, and the second one can be proved similarly. Let us set

$$d_k = d(a_k), \qquad w_k = w(a_k).$$

Obviously, $d_k \to +\infty$, $w_k \to -\infty$ as $k \to \infty$. Next, we prove that there exist two positive constants ν_i $(i = 1, 2)$ such that

$$\liminf_{k\to\infty}\frac{g(d_k)}{d_k} = \nu_1; \qquad \liminf_{k\to\infty}\frac{g(w_k)}{w_k} = \nu_2.$$

Assume by contradiction that

$$\liminf_{k \to \infty} \frac{g(d_k)}{d_k} = 0.$$

Then there exists a subsequence of $\{d_k\}$ (we still denote it by $\{d_k\}$) such that

$$\lim_{k \to \infty} \frac{g(d_k)}{d_k} = 0.$$

Set

$$\frac{g(d_k)}{d_k} = \varepsilon_k.$$

We have that $\varepsilon_k \to 0$ as $k \to \infty$. From condition (h_2) we know that, for $0 < x \le d_k$,

$$\left| \frac{g(x)}{x} - \varepsilon_k \right| = \left| \frac{g(x)}{x} - \frac{g(d_k)}{d_k} \right| \le \frac{|g(d_k) - g(x)|}{x} + \frac{|g(d_k)(d_k - x)|}{d_k x}$$

$$\le \frac{L(d_k - x)}{x} + \left(L + \frac{|g(0)|}{d_k} \right) \frac{(d_k - x)}{x}.$$

For simplicity, we assume $g(0) = 0$. Then we get that, for $0 < x \le d_k$,

$$\left| \frac{g(x)}{x} - \varepsilon_k \right| \le \frac{2L(d_k - x)}{x}. \tag{2.5}$$

Consequently, we have that, for $0 \le x \le d_k$,

$$g(x) \le \varepsilon_k x + 2L(d_k - x).$$

It follows that, for $0 \le x \le d_k$,

$$G(d_k) - G(x) = \int_x^{d_k} g(s)\, ds \le \int_x^{d_k} \left[\varepsilon_k x + 2L(d_k - x) \right] dx$$

$$= \frac{1}{2} \varepsilon_k \left(d_k^2 - x^2 \right) + L(d_k - x)^2.$$

From the definition of T we have that

$$T(a_k) = m \int_{w_k}^{d_k} \frac{dx}{\sqrt{2(G(d_k) - G(x))}} \ge m \int_0^{d_k} \frac{dx}{\sqrt{2(G(d_k) - G(x))}}$$

$$\ge m \int_0^{d_k} \frac{dx}{\sqrt{\varepsilon_k(d_k^2 - x^2) + 2L(d_k - x)^2}}$$

$$= m \int_0^1 \frac{dt}{\sqrt{\varepsilon_k(1 - t^2) + 2L(1 - t)^2}}.$$

Since

$$\liminf_{k \to \infty} \int_0^1 \frac{dt}{\sqrt{\varepsilon_k(1 - t^2) + 2L(1 - t)^2}} \ge \frac{1}{\sqrt{2L}} \int_0^1 \frac{dt}{1 - t} = +\infty,$$

we have that

$$\lim_{k \to \infty} T(a_k) = +\infty.$$

This is a contradiction because $T(a_k)$ is a bounded sequence. Therefore, there exists a constant $v_1 > 0$ such that

$$\liminf_{k \to \infty} \frac{g(d_k)}{d_k} = v_1. \tag{2.6}$$

Similarly, there exists $v_2 > 0$ such that

$$\liminf_{k \to \infty} \frac{g(w_k)}{w_k} = v_2.$$

From condition (h_1) and (2.5), (2.6) we know that there exists sufficiently small $\varepsilon_0 > 0$ such that, for k large enough and $x \in [(1 - \varepsilon_0)d_k, d_k]$,

$$\frac{g(x)}{x} \geq \frac{g(d_k)}{d_k} - \frac{2L(d_k - x)}{x} \geq \frac{1}{2}v_1 - \frac{2\varepsilon_0 L}{1 - \varepsilon_0} \geq \frac{1}{4}v_1. \tag{2.7}$$

Therefore, if $\rho(t)\cos\varphi(t) \in [(1 - \varepsilon_0)d_k, d_k]$, then we have

$$\varphi'(t) \leq -\big(a\sin^+\varphi(t) - b\sin^-\varphi(t)\big)\sin\varphi(t) - \frac{1}{4}v_1\cos^2\varphi(t) \leq -\omega_1,$$

where $\omega_1 = \min\{a, b, \frac{1}{4}v_1\}$. Next, we deal with the case $\rho(t)\cos\varphi(t) \in [0, (1 - \varepsilon_0)d_k]$. Set $x_k = (1 - \varepsilon_0)d_k$. Assume that the line $x = x_k$ intersects with the curve Γ_{a_k} at two points (x_k, y_k^+) and (x_k, y_k^-) with $y_k^- < 0 < y_k^+$. Then we have

$$\frac{1}{2}ay_k^{+2} + G(x_k) = a_k = G(d_k), \qquad \frac{1}{2}by_k^{-2} + G(x_k) = a_k = G(d_k).$$

Therefore, we get

$$y_k^+ = \sqrt{\frac{2}{a}\big(G(d_k) - G\big((1 - \varepsilon_0)d_k\big)\big)}, \qquad y_k^- = -\sqrt{\frac{2}{b}\big(G(d_k) - G\big((1 - \varepsilon_0)d_k\big)\big)}.$$

From (2.7) we have

$$\frac{y_k^+}{x_k} = \frac{\sqrt{\frac{2}{a}(G(d_k) - G((1 - \varepsilon_0)d_k))}}{(1 - \varepsilon_0)d_k} \geq \sqrt{\frac{\varepsilon_0 v_1}{2a(1 - \varepsilon_0)}}$$

and

$$\frac{y_k^-}{x_k} = -\frac{\sqrt{\frac{2}{b}(G(d_k) - G((1 - \varepsilon_0)d_k))}}{(1 - \varepsilon_0)d_k} \leq -\sqrt{\frac{\varepsilon_0 v_1}{2b(1 - \varepsilon_0)}}.$$

Set

$$\beta^+ = \arctan\sqrt{\frac{\varepsilon_0 v_1}{2a(1 - \varepsilon_0)}}, \qquad \beta^- = \arctan\sqrt{\frac{\varepsilon_0 v_1}{2b(1 - \varepsilon_0)}}.$$

From condition (h_1) we know that there exists $A > 0$ such that $g(x) \geq 0$ for $x \geq A$. If $\rho(t)\cos\varphi(t) \in [A, (1 - \varepsilon_0)d_k]$ and $\rho(t)\sin\varphi(t) \geq 0$, then we have

$$\varphi'(t) \leq -a\sin^2\beta^+ - \frac{1}{\rho}g(\rho\cos\varphi)\cos\varphi \leq -a\sin^2\beta^+.$$

If $\rho(t)\cos\varphi(t) \in [0, A]$ and $\rho(t)\sin\varphi(t) \geq 0$, then we have that, for ρ_0 large enough,

$$\varphi'(t) \leq -a\sin^2\beta^+ - \frac{1}{\rho}g(\rho\cos\varphi)\cos\varphi \leq -\frac{1}{2}a\sin^2\beta^+.$$

Similarly, we have that, if $\rho(t)\cos\varphi(t) \in [A, (1 - \varepsilon_0)d_k]$ and $\rho(t)\sin\varphi(t) \leq 0$, then we have

$$\varphi'(t) \leq -b\sin^2\beta^- - \frac{1}{\rho}g(\rho\cos\varphi)\cos\varphi \leq -b\sin^2\beta^-.$$

If $\rho(t)\cos\varphi(t) \in [0, A]$ and $\rho(t)\sin\varphi(t) \leq 0$, then we have that, for ρ_0 large enough,

$$\varphi'(t) \leq -b\sin^2\beta^- - \frac{1}{\rho}g(\rho\cos\varphi)\cos\varphi \leq -\frac{1}{2}b\sin^2\beta^-.$$

In conclusion, we have proved that there exists $\omega_2 > 0$ such that

$$\varphi'(t) \leq -\omega_2,$$

with $(\rho_0\cos\varphi_0, \rho_0\sin\varphi_0) \in \Gamma_{a_k}$, $\rho(t)\cos\varphi(t) \geq 0$ and k large enough. Similarly, we can prove that there exists $\omega_2' > 0$ such that

$$\varphi'(t) \leq -\omega_2',$$

with $(\rho_0\cos\varphi_0, \rho_0\sin\varphi_0) \in \Gamma_{a_k}$, $\rho(t)\cos\varphi(t) \leq 0$ and k large enough. Let us set $\omega = \min\{\omega_2, \omega_2'\}$. Then we have that

$$\varphi'(t) \leq -\omega,$$

with $(\rho_0\cos\varphi_0, \rho_0\sin\varphi_0) \in \Gamma_{a_k}$ and k large enough. $\square$

Lemma 2.6 *Assume that conditions (h_i) $(i = 1, 2)$ and (τ) hold, and let $\Phi(\rho_0, \varphi_0) = \varphi(2\pi, \rho_0, \varphi_0) - \varphi_0$. Then there exist two positive constants δ and ϱ_0 such that*

$$\Phi(\rho_0, \varphi_0) < -2n\pi - \delta, \qquad (\rho_0\cos\varphi_0, \rho_0\sin\varphi_0) \in \Gamma_{a_k}, \quad a_k \geq \varrho_0;$$
$$\Phi(\rho_0, \varphi_0) > -2n\pi + \delta, \qquad (\rho_0\cos\varphi_0, \rho_0\sin\varphi_0) \in \Gamma_{b_k}, \quad b_k \geq \varrho_0.$$

Proof From Lemma 2.5 we have that there exists $\varrho_0 > 0$ such that, for $a_k \geq \varrho_0$ or $b_k \geq \varrho_0$,

$$\varphi'(t) \leq -\omega, \qquad (\rho_0\cos\varphi_0, \rho_0\sin\varphi_0) \in \Gamma_{a_k} \quad \text{or} \quad (\rho_0\cos\varphi_0, \rho_0\sin\varphi_0) \in \Gamma_{b_k}.$$

Write $\Phi(\rho_0, \varphi_0) = -2l\pi - \phi$, where $l \geq 0$ is an integer, $0 \leq \phi < 2\pi$. Let us denote by t_ϕ the time for $\varphi(t)$ to decrease from $-2l\pi$ to $-2l\pi - \phi$. If $(\rho_0\cos\varphi_0, \rho_0\sin\varphi_0) \in \Gamma_{a_k}$, then we have

$$2\pi = lT(a_k) + t_\phi = lm\tau(a_k) + t_\phi.$$

Since $t_\phi \leq T(a_k) = m\tau(a_k)$, we have

$$2\pi = lm\tau(a_k) + t_\phi \leq (l+1)m\tau(a_k) \leq (l+1)\left(\frac{2\pi}{n} - m\sigma\right).$$

It follows that $l \geq n$. If $l \geq n+1$, then we have

$$\Phi(\rho_0, \varphi_0) = -2l\pi - \phi \leq -2(n+1)\pi.$$

If $l = n$, then we have

$$t_\phi = 2\pi - nm\tau(a_k) \geq 2\pi - n\left(\frac{2\pi}{n} - m\sigma\right) \geq nm\sigma.$$

Therefore, we get

$$-\phi = \int_{nT(a_k)}^{nT(a_k)+t_\phi} \varphi'(t)\,dt \leq -nm\sigma\omega.$$

Furthermore,

$$\Phi(\rho_0, \varphi_0) \leq -2n\pi - nm\sigma\omega.$$

Set $\delta = \min\{2\pi, nm\sigma\omega\}$. Then we have

$$\Phi(\rho_0, \varphi_0) \leq -2n\pi - \delta, \qquad (\rho_0\cos\varphi_0, \rho_0\sin\varphi_0) \in \Gamma_{a_k}, \quad a_k \geq \varrho_0.$$

Similarly, we can prove

$$\Phi(\rho_0, \varphi_0) \geq -2n\pi + \delta, \qquad (\rho_0\cos\varphi_0, \rho_0\sin\varphi_0) \in \Gamma_{b_k}, \quad b_k \geq \varrho_0. \qquad \square$$

3 Proof of the main theorem

At first, we recall a generalized version of the Poincaré-Birkhoff fixed point theorem by Rebelo [12].

A generalized form of the Poincaré-Birkhoff fixed point theorem. Let $\mathcal{A} \subset \mathbf{R}^2$ be an annular region bounded by two strictly star-shaped curves around the origin, Γ_1 and Γ_2, $\Gamma_1 \subset \text{int}(\Gamma_2)$, where $\text{int}(\Gamma_2)$ denotes the interior domain bounded by Γ_2. Suppose that $F : \overline{\text{int}(\Gamma_2)} \to R^2$ is an area-preserving homeomorphism and $F|\mathcal{A}$ admits a lifting, with the standard covering projection $\Pi : (r, \theta) \to z = (r\cos\theta, r\sin\theta)$, of the form

$$\tilde{F}|\mathcal{A} : (r, \theta) \to \bigl(w(r, \theta), \theta + h(r, \theta)\bigr),$$

where w and h are continuous functions of period 2π in the first variable. Correspondingly, for $\tilde{\Gamma}_1 = \Pi^{-1}(\Gamma_1)$ and $\tilde{\Gamma}_2 = \Pi^{-1}(\Gamma_2)$, assume the twist condition

$$h(r, \theta) > 0 \quad \text{on } \tilde{\Gamma}_1; \qquad h(r, \theta) < 0 \quad \text{on } \tilde{\Gamma}_2.$$

Then, F has two fixed points z_1, z_2 in the interior of $\mathcal{A}$ such that

$$h\bigl(\Pi^{-1}(z_1)\bigr) = h\bigl(\Pi^{-1}(z_2)\bigr) = 0.$$

Remark 3.1 The assumption on the star-shaped boundaries of the annulus is a delicate hypothesis. Martins and Ureña [13] showed that the star-shapedness assumption on the interior boundary is not eliminable. Le Calvez and Wang [14] then proved that star-shapedness of the exterior boundary should also be imposed, while this assumption was not made in Ding's theorem [15].

Proof of Theorem 1.1 From Lemmas 2.4 and 2.6 we know that there exists an integer $k_0 > 0$ such that, for any $k \geq k_0$,

$$\theta(2\pi, r_0, \theta_0) - \theta_0 < -2n\pi, \qquad (r_0 \cos \theta_0, r_0 \sin \theta_0) \in \Gamma_{a_k}, \tag{3.1}$$

$$\theta(2\pi, r_0, \theta_0) - \theta_0 > -2n\pi, \qquad (r_0 \cos \theta_0, r_0 \sin \theta_0) \in \Gamma_{b_k}. \tag{3.2}$$

Without loss of generality, we assume that $a_k < b_k < a_{k+1}$ for $k \geq k_0$. Let D_k be an annular region with boundary Γ_{a_k} and Γ_{b_k}. Consider the Poincaré map $P : D_k \to \mathbf{R}^2$. Write the Poincaré map in the form

$$r^* = r(2\pi, r_0, \theta_0), \qquad \theta^* = \theta_0 + \Theta^*(r_0, \theta_0),$$

where $\Theta^*(r_0, \theta_0) = \theta(2\pi, r_0, \theta_0) - \theta_0 + 2n\pi$. From (3.1) and (3.2) we know that, for $k \geq k_0$,

$$\Theta^*(r_0, \theta_0) < 0, \qquad (r_0 \cos \theta_0, r_0 \sin \theta_0) \in \Gamma_{a_k},$$

$$\Theta^*(r_0, \theta_0) > 0, \qquad (r_0 \cos \theta_0, r_0 \sin \theta_0) \in \Gamma_{b_k}.$$

Therefore, the area-preserving homeomorphism P is twisting on the annulus D_k. On the other hand, by Lemma 2.1, we have that both Γ_{a_k} and Γ_{b_k} are star-shaped with respect to the origin O for k large enough. Consequently, all assumptions of the generalized form of the Poincaré-Birkhoff fixed point theorem are satisfied. Therefore, the Poincaré map P has at least two fixed points in annulus D_k. Furthermore, system (1.1)′ has infinitely many 2π-periodic solutions $\{(x_k(t), y_k(t))\}_{k=1}^{\infty}$ which satisfy

$$\lim_{k \to \infty} \left(\min_{t \in R} (|x_k(t)| + |y_k(t)|) \right) = +\infty.$$

Clearly, each 2π-periodic solution $(x_k(t), y_k(t))$ rotates clockwise strictly n turns around the origin in the interval $[0, 2\pi]$. It follows that both $x_k(t)$ and $y_k(t)$ have exactly $2n$ simple zeros in $[0, 2\pi)$. Hence, the conclusion of Theorem 1.1 holds. □

From Theorem 1.1 we know that the existence of periodic solutions of system (1.1) has tight relation with the asymptotic property of time map $\tau(c)$. In case g is odd, we can easily check condition (τ). Set

$$G_+ = \liminf_{x \to +\infty} \frac{2G(x)}{x^2}, \qquad G^+ = \limsup_{x \to +\infty} \frac{2G(x)}{x^2}.$$

From Theorem 3.1 in [1], we have the following lemma.

Lemma 3.2 *Assume that condition* (h_1) *holds, g is odd and* $G_+ < G^+$. *Then*

$$\left[\frac{\pi}{\sqrt{G^+}}, \frac{\pi}{\sqrt{G_+}}\right] \subset [\tau_+, \tau^+],$$

where $\tau_+ = \liminf_{c \to +\infty} \tau(c)$, $\tau^+ = \limsup_{c \to +\infty} \tau(c)$.

Applying Theorem 1.1 and Lemma 3.2, we can obtain the following corollary.

Corollary 3.3 *Assume that conditions* (h_i) $(i = 1, \ldots, 4)$ *hold. Let* $g(x)$ *be an odd function and* $G_+ < G^+$. *If*

$$\exists n \in \mathbf{N}, \quad \frac{m^2 n^2}{4} \in (G_+, G^+) \quad \left(m = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right),$$

then system (1.1) *has infinitely many* 2π-*periodic solutions* $\{(x_k(t), y_k(t))\}_{k=1}^{\infty}$ *which satisfy*

$$\lim_{k \to \infty} \left(\min_{t \in R}\left(|x_k(t)| + |y_k(t)|\right)\right) = +\infty.$$

Moreover, for each integer $k \geq 1$, *both* $x_k(t)$ *and* $y_k(t)$ *have exactly* $2n$ *simple zeros in* $[0, 2\pi)$.

4 Concluding remarks

The restrictions on the local Lipschitz conditions of f and p_i $(i = 1, 2)$ made in the proofs of the above sections can be removed. Indeed, Lemmas 2.4 and 2.6 guarantee the applicability of the following non-uniqueness version of the Poincaré-Birkhoff theorem which was proved by Fonda and Ureña in [16]. We now state this theorem for a general Hamiltonian system in $\mathbf{R}^{2N}$. Let us consider the (time-dependent) Hamiltonian system

$$\begin{cases} x' = \nabla_y H(t, x, y), \\ y' = -\nabla_x H(t, x, y), \end{cases} \tag{4.1}$$

where the continuous function $H : \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$, $H = H(t, x, y)$ is T-periodic in its first variable t and continuously differentiable with respect to (x, y), $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N)$.

We next introduce the definition of rotation number of a continuous path in $\mathbf{R}^2$. Let $w : [t_1, t_2] \to \mathbf{R}^2$ be a continuous path such that $w(t) \neq (0, 0)$ for every $t \in [t_1, t_2]$. The rotation number of w around the origin is defined as

$$\text{Rot}\big(w(t); [t_1, t_2]\big) = \frac{\theta(t_2) - \theta(t_1)}{2\pi},$$

where $\theta : [t_1, t_2] \to \mathbf{R}$ is a continuous determination of the argument along w, i.e., $w(t) = |w(t)|(\cos\theta(t), \sin\theta(t))$.

Assume that for each $i = 1, \ldots, N$, there are two strictly star-shaped Jordan closed curves around the origin $\Gamma_1^i, \Gamma_2^i \subset \mathbf{R}^2$ such that

$$o \in \mathcal{D}\big(\Gamma_1^i\big) \subset \overline{\mathcal{D}\big(\Gamma_1^i\big)} \subset \mathcal{D}\big(\Gamma_2^i\big).$$

Here we denote by $\mathcal{D}(\Gamma)$ the open bounded region bounded by the Jordan closed curve Γ. Consider the generalized annular region

$$\mathcal{A} = \left[\overline{\mathcal{D}(\Gamma_2^1)} \setminus \mathcal{D}(\Gamma_1^1)\right] \times \cdots \times \left[\overline{\mathcal{D}(\Gamma_2^N)} \setminus \mathcal{D}(\Gamma_1^N)\right] \subset \mathbf{R}^{2N}.$$

Theorem 4.1 ([16]) *Under the framework above, denoting $z_i(t) = (x_i(t), y_i(t))$, assume that every solution $z(t) = (z_1(t), \ldots, z_N(t))$ of (4.1) departing from $z(0) \in \partial \mathcal{A}$ is defined on $[0, T]$ and satisfies*

$$z_i(t) \neq (0,0) \quad \text{for every } t \in [0, T] \text{ and } i = 1, \ldots, N.$$

Assume that there are integer numbers $v_1, \ldots, v_N \in \mathbf{Z}$ such that, for each $i = 1, \ldots, N$, either

$$\mathrm{Rot}\big(z_i(t); [0, T]\big) \begin{cases} < v_i, & \text{if } z_i(0) \in \Gamma_1^i, \\ > v_i, & \text{if } z_i(0) \in \Gamma_2^i, \end{cases}$$

or

$$\mathrm{Rot}\big(z_i(t); [0, T]\big) \begin{cases} > v_i, & \text{if } z_i(0) \in \Gamma_1^i, \\ < v_i, & \text{if } z_i(0) \in \Gamma_2^i. \end{cases}$$

Then Hamiltonian system (4.1) has at least $N + 1$ distinct T-periodic solutions $z^0(t), \ldots, z^N(t)$, with $z^0(0), \ldots, z^N(0) \in \mathcal{A}$, such that

$$\mathrm{Rot}\big(z_i^k(t); [0, T]\big) = v_i \quad \text{for every } k = 0, \ldots, N \text{ and } i = 1, \ldots, N.$$

Remark 4.2 It is noted that there is no requirement of uniqueness of Cauchy problems associated to system (4.1) in this higher dimensional Poincaré-Birkhoff theorem for Hamiltonian flows. In [17, 18], Theorem 4.1 is applied to prove the multiplicity of periodic solutions of weakly coupled Hamiltonian systems. Theorem 1.1 in the present paper can also be extended to a weakly coupled system of the type

$$\begin{cases} x_i' = f_i(y_i) + p_{1i}(t, x, y), \\ y_i' = -g_i(x_i) + p_{2i}(t, x, y) \end{cases} \quad (i = 1, \ldots, N), \tag{4.2}$$

where $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N)$, $f_i, g_i : \mathbf{R} \to \mathbf{R}$ $(i = 1, \ldots, N)$ are continuous, $p_{ji} : \mathbf{R}^{2N+1} \to \mathbf{R}$ $(j = 1, 2, i = 1, \ldots, N)$ are continuous and 2π-periodic with the variable t. Assume that there is a function $U : \mathbf{R}^{2N+1} \to \mathbf{R}$ such that

$$\frac{\partial U}{\partial y_i}(t, x, y) = p_{1i}(t, x, y), \qquad \frac{\partial U}{\partial x_i}(t, x, y) = -p_{2i}(t, x, y) \quad (i = 1, \ldots, N).$$

In this case system (4.2) is a Hamiltonian system. Simple examples of such functions can be given. For example,

$$U(t, x, y) = p(t) \prod_{i=1}^{i=N} \sin x_i \sin y_i.$$

Assume that the following conditions hold:

(h'_1) g_i satisfies $\lim_{|u| \to +\infty} \mathrm{sgn}(u) g_i(u) = +\infty$, $i = 1, \ldots, N$;

(h'_2) there exist constants $L_i > 0$ such that, for all $u, v \in \mathbf{R}$, $|g_i(u) - g_i(v)| \leq L_i |u - v|$, $i = 1, \ldots, N$;

(h'_3) there are constants $M_1 > 0$, $M_2 > 0$ and $0 < \gamma_i < 1$ such that $|p_{ji}(t, x, y)| \leq M_1(|x_i| + |y_i|)^{\gamma_i} + M_2$ for all $t \in [0, 2\pi]$, $(x, y) \in \mathbf{R}^{2N}$, $j = 1, 2$, $i = 1, \ldots, N$;

(h'_4) there are positive constants a_i and b_i such that

$$\lim_{v \to +\infty} \frac{f_i(v)}{v} = a_i, \qquad \lim_{v \to -\infty} \frac{f_i(v)}{v} = b_i.$$

Set $G_i(u) = \int_0^u g_i(s)\, ds$, $i = 1, \ldots, N$. Let us define the time maps

$$\tau_i(c) = \int_{w_i(c)}^{d_i(c)} \frac{ds}{\sqrt{2(c - G_i(s))}}$$

for $c > 0$ large enough, where $w_i(c)$ and $d_i(c)$ satisfy $w_i(c) < 0 < d_i(c)$ and $G_i(w_i(c)) = G_i(d_i(c)) = c$. Assume that the time map $\tau_i(c)$ satisfies the condition:

(τ') There exist constants $\sigma_i > 0$, integers $n_i > 0$, and sequences $\{a_k^i\}$ and $\{b_k^i\}$ such that $\lim_{k \to \infty} a_k^i = +\infty$, $\lim_{k \to \infty} b_k^i = +\infty$; and moreover

$$\tau_i(a_k^i) < \frac{2\pi}{m_i n_i} - \sigma_i, \qquad \tau_i(b_k^i) > \frac{2\pi}{m_i n_i} + \sigma_i, \quad i = 1, \ldots, N,$$

where $m_i = \frac{1}{\sqrt{a_i}} + \frac{1}{\sqrt{b_i}}$ and a_i, b_i are given in condition (h'_4).

With a slight modification of the proof of Theorem 1.1 and using the higher dimensional Poincaré-Birkhoff Theorem 4.1, we can prove the result.

Theorem 4.3 *Assume that conditions* (h'_i) ($i = 1, \ldots, 4$) *and* (τ') *hold. Then Hamiltonian system* (4.2) *has infinitely many* 2π-*periodic solutions* $\{(x^k(t), y^k(t))\}_{k=1}^\infty$ *which satisfy*

$$\lim_{k \to \infty} \left(\min_{t \in \mathbf{R}} (|x^k(t)| + |y^k(t)|) \right) = +\infty,$$

where $|x^k(t)| = \sum_{i=1}^{i=N} |x_i^k(t)|$, $|y^k(t)| = \sum_{i=1}^{i=N} |y_i^k(t)|$. *Moreover, for each index* i, *both* $x_i^k(t)$ *and* $y_i^k(t)$ *have exactly* $2n_i$ *simple zeros in the interval* $[0, 2\pi)$.

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