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Multiplicity of positive radial solutions of p-Laplacian problems with nonlinear gradient term

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Abstract

In the present paper, we prove the existence of at least three radial solutions of the p-Laplacian problem with nonlinear gradient term

$$\begin{cases} \Delta_p v + f(|x|, v, |\nabla v|) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

and the corresponding one-parameter problem. Here Ω is a unit ball in $\mathbb{R}^N$. Our approach relies on the Avery-Peterson fixed point theorem. In contrast with the usual hypotheses, no asymptotic behavior is assumed on the nonlinearity f with respect to $\phi_p(\cdot)$.

MSC: 35J92; 35J62; 35A09

Keywords: p-Laplacian; nonlinear gradient term; radial solution; Avery-Peterson fixed point theorem

1 Introduction

In the present paper, we are concerned with the multiplicity of positive radial solutions to the quasilinear elliptic p-Laplacian problem with nonlinear gradient term

$$\begin{cases} \Delta_p v + f(|x|, v, |\nabla v|) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and the corresponding one-parameter problem

$$\begin{cases} \Delta_p v + \lambda f(|x|, v, |\nabla v|) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a unit ball in $\mathbb{R}^N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $p > 1$, and $f : [0, +\infty) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous with $f(r, s, t) > 0$ for all $(r, s, t) \in (0, 1] \times (0, +\infty) \times [0, +\infty)$.

In recent years, the elliptic p-Laplacian problems with nonlinear gradient term have been extensively studied via different methods [1–6], for example, critical point theory, Schauder's fixed point theorem, Schaefer's fixed point theorem, sub- and supersolutions, and so on. However, most of these results are concerned with the existence of one or two solutions, and a few works refer to the existence of three solutions for problems (1.1) and (1.2). In 2012, Bueno et al. [1] considered the p-Laplacian problem with dependence on the gradient

$$\begin{cases} -\Delta_p v = \omega(x) f(v, |\nabla v|) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is a smooth bounded domain, $\omega : \Omega \to \mathbb{R}$ is a continuous nonnegative function with isolated zeros, and the C^1-nonlinearity $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfies some local hypotheses. By applying the Schauder fixed point theorem and sub- and supersolutions, the authors showed that problem (1.3) has a positive solution. Moreover, as an application, the authors obtained that there exits $\lambda^* > 0$ such that the p-growth one-parameter problem

$$\begin{cases} -\Delta_p u = \lambda u^{q-1}(1 + |\nabla u|^p) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

with $1 < q < p$ has a positive solution for each $\lambda \in (0, \lambda^*]$.

When the nonlinearity f does not depend on the gradient, He [7] considered the p-Laplacian problem

$$\begin{cases} \Delta_p v + f(v) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

and using the Leggett-Williams fixed point theorem, established the existence of at least three radial solutions. For other works concerned with p-Laplacian problems, we refer the reader to [8–18, 20, 21].

Motivated by the above works, the aim of this paper is to study the multiplicity of positive radial solutions of problems (1.1) and (1.2). Under the hypothesis that f has a local behavior and need not satisfy superlinear condition at the origin and sublinear condition at $+\infty$ with respect to $\phi_p(s) := |s|^{p-2}s$, $s \in \mathbb{R}$, by using the Avery-Peterson fixed point theorem we obtain the existence of triple radial solutions of the above problems. To the best of our knowledge, problems (1.1) and (1.2) have not been studied via this fixed point theorem.

2 Main results

In order to present existence results of positive radial solutions for problems (1.1) and (1.2), setting $r = |x|$ and $v(x) = u(r)$, problems (1.1) and (1.2) become respectively

$$\begin{cases} (r^{N-1}\phi_p(u'))' + r^{N-1} f(r, u, |u'|) = 0, & r \in (0,1), \\ u'(0) = 0, \qquad u(1) = 0, \end{cases} \tag{2.1}$$

and

$$\begin{cases} (r^{N-1}\phi_p(u'))' + \lambda r^{N-1} f(r,u,|u'|) = 0, & r \in (0,1), \\ u'(0) = 0, \qquad u(1) = 0. \end{cases} \tag{2.2}$$

Our approach on problem (2.1) relies upon the Avery-Peterson fixed point theorem, which we recall here for the convenience of the reader.

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then for positive real numbers a, b, c, and d, we define the convex sets

$$P(\gamma,d) = \{x \in P : \gamma(x) < d\},$$
$$P(\gamma,\alpha,b,d) = \{x \in P : b \le \alpha(x), \gamma(x) \le d\} \quad \text{and}$$
$$P(\gamma,\theta,\alpha,b,c,d) = \{x \in P : b \le \alpha(x), \theta(x) \le c, \gamma(x) \le d\}$$

and the closed set

$$R(\gamma,\psi,a,d) = \{x \in P : a \le \psi(x), \gamma(x) \le d\}.$$

The following fixed point theorem due to Avery and Peterson is fundamental in the proofs of our main results.

Lemma 2.1 ([19]) *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \le \lambda\psi(x)$ for $0 \le \lambda \le 1$ such that, for some positive numbers M and d,*

$$\alpha(x) \le \psi(x) \quad \textit{and} \quad \|x\| \le M\gamma(x)$$

for all $x \in \overline{P(\gamma,d)}$. Suppose that $A : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$ is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that

(i) *$\{x \in P(\gamma,\theta,\alpha,b,c,d) : \alpha(x) > b\} \ne \emptyset$ and $\alpha(Ax) > b$ for $x \in P(\gamma,\theta,\alpha,b,c,d)$;*

(ii) *$\alpha(Ax) > b$ for $x \in P(\gamma,\alpha,b,d)$ with $\theta(Ax) > c$;*

(iii) *$0 \notin R(\gamma,\psi,a,d)$ and $\psi(Ax) < a$ for $x \in R(\gamma,\psi,a,d)$ with $\psi(x) = a$.*

Then, A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma,d)}$ such that

$$\gamma(x_i) \le d \quad \textit{for } i = 1,2,3, \qquad b < \alpha(x_1);$$
$$a < \psi(x_2) \quad \textit{with } \alpha(x_2) < b, \qquad \psi(x_3) < a.$$

Remark 2.1 In Lemma 2.1, if $\gamma(u) \le d$ and $u \in P$ imply that $\theta(u) \le c$ and $u \in P$, then assumption (i) implies assumption (ii).

We further take $E = (C^1[0,1], \|\cdot\|)$ with the maximum norm

$$\|x\| = \max\Big\{\max_{0\le r\le 1}|u(r)|, \max_{0\le r\le 1}|u'(r)|\Big\}$$

and define the cone $P \subset E$ by

$$P = \{u \in E : u(r) \text{ is nonnegative and nonincreasing on } [0,1], u'(0) = u(1) = 0\}.$$

Now we define the nonlinear operator A on P as follows:

$$(Au)(r) = \int_r^1 \phi_q\left(\frac{1}{t^{N-1}} \int_0^t \tau^{N-1} f(\tau, u(\tau), |u'(\tau)|)\,d\tau\right) dt, \quad u \in P.$$

Then $(Au)(r) \geq 0$ for all $r \in [0,1]$, and $(Au)'(0) = (Au)(1) = 0$, which implies $A(P) \subset P$. Moreover, by a standard argument it is easy to show that $A : P \to P$ is completely continuous. In addition, it can be easily proved that u is a solution of problem (2.1) if $u \in P$ is a fixed point of the nonlinear operator A.

Define the nonnegative continuous concave functional α, the nonnegative continuous convex functionals θ, γ, and the nonnegative continuous functional ψ on the cone P by

$$\gamma(u) = \max_{0 \leq r \leq 1} |u'(r)|, \qquad \psi(u) = \theta(u) = \max_{0 \leq r \leq 1} |u(r)|, \qquad \alpha(u) = \min_{0 \leq r \leq 1-\eta} |u(r)|,$$

where $\eta \in (0,1)$. Then it is easy to see that $\alpha(u) \leq \psi(u)$ and $\|u\| \leq \gamma(u)$ for $u \in P$.

Theorem 2.1 *Assume that there exist constants a, b, d, and η with $0 < a < b \leq \eta d$ such that*

(H_1) $f(r,s,t) \leq N\phi_p(d)$ *for all* $(r,s,t) \in [0,1] \times [0,d] \times [0,d]$;
(H_2) $f(r,s,t) \geq \frac{N}{(1-\eta)^N}\phi_p(\frac{b}{\eta})$ *for all* $(r,s,t) \in [0,1-\eta] \times [b,d] \times [0,d]$;
(H_3) $f(r,s,t) \leq N\phi_p(a)$ *for all* $(r,s,t) \in [0,1] \times [0,a] \times [0,d]$.

Then, problem (1.1) *has at least three radial solutions* u_1, u_2, u_3 *satisfying*

$$\begin{aligned} &\max_{0 \leq r \leq 1} |u_i'(r)| \leq d \quad \text{for } i = 1,2,3, \qquad b < \min_{0 \leq r \leq 1-\eta} |u_1(r)|; \\ &a < \max_{0 \leq r \leq 1} |u_2(r)| \quad \text{with} \min_{0 \leq r \leq 1-\eta} |u_2(r)| < b, \qquad \max_{0 \leq r \leq 1} |u_3(r)| < a. \end{aligned} \tag{2.3}$$

Proof Choosing $c = d$, we divide the proof into three steps.

Step 1. We show that $A : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$. To do this, let $u \in \overline{P(\gamma,d)}$. Then $-d \leq u'(r) \leq 0$ for $r \in [0,1]$, and thus $0 \leq u(r) = \int_1^r u'(s)\,ds \leq \int_0^1 |u'(s)|\,ds \leq d$ for $r \in [0,1]$. Hence, from assumption (H_1) it follows that

$$\begin{aligned} \gamma(Au) &= \max_{0 \leq r \leq 1} \phi_q\left(\frac{1}{r^{N-1}} \int_0^r \tau^{N-1} f(\tau, u(\tau), |u'(\tau)|)\,d\tau\right) \\ &\leq \max_{0 \leq r \leq 1} \phi_q\left(\frac{1}{r^{N-1}} \int_0^r \tau^{N-1} N\phi_p(d)\,d\tau\right) \\ &= \max_{0 \leq r \leq 1} \phi_q(\phi_p(d)r) \leq d, \quad \forall u \in \overline{P(\gamma,d)}. \end{aligned}$$

Therefore, $A : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$.

Step 2. We check assumption (i) of Lemma 2.1. To do this, let $u(r) \equiv b/\eta$ on $[0,1]$. Then $\gamma(u) = 0 < d$, $\theta(u) = b/\eta \leq d = c$, $\alpha(u) = b/\eta > b$. Hence, $\{x \in P(\gamma,\theta,\alpha,b,c,d) : \alpha(x) > b\} \neq \emptyset$.

Let $u \in P(\gamma,\theta,\alpha,b,c,d)$. Then $\gamma(u) \leq d$, $\theta(u) \leq c = d$, $\alpha(u) \geq b$, and thus

$$-d \leq u'(r) \leq 0, \quad \forall r \in [0,1], \qquad b \leq u(r) \leq d, \quad \forall r \in [0,1-\eta].$$

So from (H_2) we have

$$\begin{aligned}
\alpha(Au) &= \int_{1-\eta}^{1} \phi_q\left(\frac{1}{t^{N-1}}\int_0^t \tau^{N-1} f\big(\tau,u(\tau),\left|u'(\tau)\right|\big)\,\mathrm{d}\tau\right)\mathrm{d}t \\
&\geq \int_{1-\eta}^{1} \phi_q\left(\frac{1}{t^{N-1}}\int_0^{1-\eta} \tau^{N-1} f\big(\tau,u(\tau),\left|u'(\tau)\right|\big)\,\mathrm{d}\tau\right)\mathrm{d}t \\
&> \eta\phi_q\left(\int_0^{1-\eta} \tau^{N-1}\frac{N}{(1-\eta)^N}\phi_p\left(\frac{b}{\eta}\right)\mathrm{d}\tau\right) \\
&= \eta\phi_q\left(\phi_p\left(\frac{b}{\eta}\right)\right) = b, \quad \forall u \in P(\gamma,\theta,\alpha,b,c,d).
\end{aligned}$$

Step 3. We check assumption (iii) of Lemma 2.1. Notice that $\psi(0) = 0 < a$, and thus $0 \notin R(\gamma,\psi,a,d)$. Let $u \in R(\gamma,\psi,a,d)$ with $\psi(u) = a$. Then $\gamma(u) \leq d$ and $\psi(u) = a$, and hence $-d \leq u'(r) \leq 0$ and $0 \leq u(r) \leq a$ for all $r \in [0,1]$. It follows from (H_3) that

$$\begin{aligned}
\psi(Au) &= \int_0^1 \phi_q\left(\frac{1}{s^{N-1}}\int_0^s \tau^{N-1} f\big(\tau,u(\tau),\left|u'(\tau)\right|\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\leq \int_0^1 \phi_q\left(\frac{1}{s^{N-1}}\int_0^s \tau^{N-1} N\phi_p(a)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&= \int_0^1 \phi_q\big(\phi_p(a)s\big)\,\mathrm{d}s \\
&< a \quad \text{for } u \in R(\gamma,\psi,a,d) \text{ with } \psi(u) = a.
\end{aligned}$$

In summary, by Remark 2.1 A has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma,d)}$, which are radial solutions of problem (1.1) satisfying (2.3). This completes the proof of the theorem. □

Remark 2.2 In Theorem 2.1, assumptions (H_1) and (H_3) can be replaced by

(H_1') $f^\infty := \overline{\lim}_{s+t\to+\infty} \max_{r\in[0,1]} \frac{f(r,s,t)}{\phi_p(s+t)} < N/\phi_p(2)$

and

(H_3') $f^0 := \overline{\lim}_{s\to 0^+} \max_{(r,t)\in[0,1]\times[0,d]} \frac{f(r,s,t)}{\phi_p(s)} < N$,

respectively.

From Theorem 2.1 we can easily get the existence of three radial solutions of one-parameter problem (1.2).

Theorem 2.2 *Assume that there exist constants a, b, d, and η with* $0 < a < b < \eta d < d$ *such that*

$$\frac{\phi_p(b/\eta)}{(1-\eta)^N \min_{[0,1-\eta]\times[b,d]\times[0,d]} f(r,s,t)} \le \min\left\{\frac{\phi_p(a)}{\max_{[0,1]\times[0,a]\times[0,d]} f(r,s,t)}, \frac{\phi_p(d)}{\max_{[0,1]\times[0,d]\times[0,d]} f(r,s,t)}\right\}.$$

Then, one-parameter problem (1.2) *has at least three radial solutions* u_1, u_2, u_3 *satisfying* (2.3), *provided that*

$$\frac{N\phi_p(b/\eta)}{(1-\eta)^N \min_{[0,1-\eta]\times[b,d]\times[0,d]} f(r,s,t)} \le \lambda \le \min\left\{\frac{N\phi_p(a)}{\max_{[0,1]\times[0,a]\times[0,d]} f(r,s,t)}, \frac{N\phi_p(d)}{\max_{[0,1]\times[0,d]\times[0,d]} f(r,s,t)}\right\}.$$

To illustrate our main results, we present the following example.

Example 2.1 Consider the Dirichlet problem

$$\begin{cases} \Delta_p v + f(|x|, v, |\nabla v|) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where Ω is a unit ball in $\mathbb{R}^2$, $p = \frac{3}{2}$, and

$$f(r,s,t) = \frac{1}{2}(1-r) + \min\{s^4, 16\} + \frac{1}{2}\left(\frac{t}{100}\right)^2.$$

Choose $a = 1$, $b = 2$, $d = 100$, and $\eta = 1/2$. Since $p = 3/2$ and $N = 2$, it follows that

$$N\phi_p(d) = 20, \qquad \frac{N}{(1-\eta)^N}\phi_p\left(\frac{b}{\eta}\right) = 16, \qquad N\phi_p(a) = 2.$$

So, $f(r,s,t)$ satisfies

(i) $f(r,s,t) \le 17 < N\phi_p(d)$, $\forall (r,s,t) \in [0,1] \times [0,100] \times [0,100]$;
(ii) $f(r,s,t) \ge 16.25 > \frac{N}{(1-\eta)^N}\phi_p(\frac{b}{\eta})$, $\forall (r,s,t) \in [0,\frac{1}{2}] \times [2,100] \times [0,100]$;
(iii) $f(r,s,t) \le 2 = N\phi_p(a)$, $\forall (r,s,t) \in [0,1] \times [0,1] \times [0,100]$.

Hence, by Theorem 2.1 the Dirichlet problem (2.4) has at least three radial solutions u_1, u_2, u_3 satisfying

$$\max_{0\le r\le 1}\left|u_i'(r)\right| \le 100 \quad \text{for } i = 1,2,3, \qquad 2 < \min_{0\le r\le 1/2}\left|u_1(r)\right|;$$

$$1 < \max_{0\le r\le 1}\left|u_2(r)\right| \quad \text{with } \min_{0\le r\le 1/2}\left|u_2(r)\right| < 2, \qquad \max_{0\le r\le 1}\left|u_3(r)\right| < 1.$$

Noticing that $f(r,0,0) \not\equiv 0$ on $[0,1]$, we have that the three radial solutions u_1, u_2, u_3 are positive.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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