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# Proving uniqueness for the solution of the problem of homogeneous and anisotropic micropolar thermoelasticity

Adina Chirilă[1], Ravi P Agarwal[2] and Marin Marin[1*]

*Correspondence:
m.marin@unitbv.ro
[1]Department of Mathematics, Transilvania University of Braşov, Braşov, Romania
Full list of author information is available at the end of the article

**Abstract**
In this paper we derive some identities for the solution of the problem of homogeneous and anisotropic micropolar thermoelasticity. These can be applied to proving uniqueness of the solution of the corresponding boundary initial value problem.

## 1 Introduction

In [1], the author derives some uniqueness criteria for solutions of the Cauchy problem for the standard equations of dynamical linear thermoelasticity backward in time. Lagrange-Brun identities are combined with some differential inequalities in order to show that the final boundary value problem associated with the linear thermoelasticity backward in time has at most one solution in appropriate classes of displacement-temperature fields. The uniqueness results are obtained under the assumptions that the density mass and the specific heat are strictly positive and the conductivity tensor is positive definite.

According to [2], in a micropolar continuum the deformation is described not only by the displacement vector, but also by an independent rotation vector. This rotation vector specifies the orientation of a triad of director vectors attached to each material particle. A material point can experience a microrotation without undergoing a macrodisplacement.

The results obtained in this paper are extensions of those for the solutions of the Cauchy problem for the standard equations of dynamical linear thermoelasticity. The identities established in this paper lay the foundations for a uniqueness result.

In [3], Eringen establishes a uniqueness theorem for the boundary initial value problem of linear micropolar elastodynamics by means of some relations between the kinetic density, the strain energy density and the total power of the applied forces.

The spatial and the time arguments of a function will be omitted when there is no likelihood of confusion. A superposed dot denotes differentiation with respect to time $t$ and a subscript preceded by a comma denotes differentiation with respect to the corresponding spatial variable. The subscripts $i, j, k, m, n$ take values $1, 2, 3$ and summation is implied by index repetition.

Let $\bar{B}$ denote a regular region of the three dimensional Euclidean space occupied by a homogeneous micropolar body whose boundary is $\partial B$. The interior of $\bar{B}$ is denoted by $B$.

The governing equations of the theory of anisotropic and homogeneous micropolar thermoelasticity, as described in [4] and also in [5–8] and [9], are the equations of motion

$$t_{ij,j} + \rho F_i = \rho \ddot{u}_i, \qquad m_{ij,j} + \varepsilon_{ijk} t_{jk} + \rho M_i = I_{ij} \ddot{\varphi}_j, \tag{1}$$

and the equation of energy

$$\rho \dot{\eta} = \frac{\rho}{\theta_0} r - q_{i,i}. \tag{2}$$

Equations (1) and (2) are defined for $(x,t) \in B \times [0,\infty)$.

When the reference solid has a center of symmetry at each point, but is otherwise non-isotropic, then the constitutive equations, defined for $(x,t) \in \bar{B} \times [0,\infty)$, are

$$\begin{aligned} t_{ij} &= A_{ijmn} \varepsilon_{mn} + B_{ijmn} \gamma_{mn} - D_{ij} \theta, \\ m_{ij} &= B_{mnij} \varepsilon_{mn} + C_{ijmn} \gamma_{mn} - E_{ij} \theta, \\ \rho \eta &= D_{ij} \varepsilon_{ij} + E_{ij} \gamma_{ij} + \frac{c}{\theta_0} \theta, \\ q_i &= -\frac{1}{\theta_0} K_{ij} \beta_j. \end{aligned} \tag{3}$$

The deformation tensors $\varepsilon_{ij}$ and $\gamma_{ij}$ used in equations (3) are defined in $\bar{B} \times [0,\infty)$ by means of the geometric equations

$$\varepsilon_{ij} = u_{j,i} + \varepsilon_{jik} \varphi_k, \qquad \gamma_{ij} = \varphi_{j,i}. \tag{4}$$

The system of equations is complete if we add the law of heat flow

$$\beta_i = \theta_{,i} \tag{5}$$

for all $(x,t) \in \bar{B} \times [0,\infty)$.

In the equations above we used the following notations: $u_i$ are the components of the displacement vector, $\varphi_i$ are the components of the microrotation vector, $t_{ij}$ are the components of the stress tensor, $m_{ij}$ are the components of the couple stress tensor, $q_i$ are the components of the heat conduction vector, $\eta$ is the specific entropy per unit mass, $\rho$ is the constant reference density, $\theta_0$ is the constant reference temperature, $\theta$ is the temperature measured from the temperature $\theta_0$, $c$ is the specific heat, $I_{ij}$ are the components of the inertia, $\beta_i$ are the components of the thermal displacement gradient vector, $F_i$ are the components of the external body force vector, $M_i$ are the components of the external body couple vector, $r$ is the external rate of heat supply per unit mass and $\varepsilon_{ijk}$ is the alternating symbol.

The coefficients from (1) and (3), that is, $A_{ijmn}$, $B_{ijmn}$, $C_{ijmn}$, $D_{ij}$, $E_{ij}$, $I_{ij}$, $K_{ij}$, and $c$ are constant constitutive coefficients subject to the following symmetry conditions:

$$A_{ijmn} = A_{mnij}, \qquad C_{ijmn} = C_{mnij}, \qquad I_{ij} = I_{ji}, \qquad K_{ij} = K_{ji}. \tag{6}$$

We further assume that $c_0 \geq 0$ and $K_{ij}\theta_{,i}\theta_{,j} \geq 0$.

The free energy $\Psi$, used to obtain the constitutive equations, is given by

$$\begin{aligned} \rho\Psi = {} & \frac{1}{2}A_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + B_{ijmn}\varepsilon_{ij}\gamma_{mn} + \frac{1}{2}C_{ijmn}\gamma_{ij}\gamma_{mn} \\ & - D_{ij}\varepsilon_{ij}\theta - E_{ij}\gamma_{ij}\theta - \frac{c}{2\theta_0}\theta^2 + \frac{c}{2\theta_0}K_{ij}\tau_{,i}\tau_{,j}. \end{aligned} \tag{7}$$

We denoted by $\tau$ the thermal displacement related to the temperature variation. The relationship between $\tau$ and $\theta$ is given by

$$\dot{\tau} = \theta. \tag{8}$$

We consider the initial conditions

$$\begin{aligned} & u_i(x,0) = u_i^0(x), \\ & \dot{u}_i(x,0) = \dot{u}_i^0(x), \\ & \theta(x,0) = \theta^0(x), \quad x \in \bar{B}, \\ & \varphi_j(x,0) = \varphi_j^0(x), \quad x \in \bar{B}, \\ & \dot{\varphi}_j(x,0) = \dot{\varphi}_j^0(x), \quad x \in \bar{B}, \end{aligned} \tag{9}$$

and the boundary conditions

$$\begin{aligned} & u_i(x,t) = \tilde{u}_i(x,t) \quad \text{on } \bar{\Sigma}_1 \times [0,\infty), \\ & t_i(x,t) = \tilde{t}_i(x,t) \quad \text{on } \Sigma_2 \times [0,\infty), \\ & \theta(x,t) = \tilde{\theta}(x,t) \quad \text{on } \bar{\Sigma}_3 \times [0,\infty), \\ & q(x,t) = \tilde{q}(x,t) \quad \text{on } \Sigma_4 \times [0,\infty), \\ & \varphi_j(x,t) = \tilde{\varphi}_j(x,t) \quad \text{on } \bar{\Sigma}_5 \times [0,\infty), \\ & m_i(x,t) = \tilde{m}_i(x,t) \quad \text{on } \Sigma_6 \times [0,\infty), \end{aligned} \tag{10}$$

where $u_i^0, \dot{u}_i^0, \theta^0, \dot{\varphi}_j^0, \tilde{u}_i, \tilde{t}_i, \tilde{\theta}, \tilde{q}, \tilde{\varphi}_j$, and $\tilde{m}_i$ are prescribed functions. We have

$$t_i(x,s) := t_{ij}(x,s)n_j(x), \tag{11}$$

$$q(x,s) := q_i(x,s)n_i(x), \tag{12}$$

$$m_i(x,s) := m_{ij}(x,s)n_j(x), \tag{13}$$

where $n_i$ are the components of the outward unit normal vector to the boundary surface and $\Sigma_1$, $\Sigma_2$, $\Sigma_3$, $\Sigma_4$, $\Sigma_5$, and $\Sigma_6$ are subsurfaces of $\partial B$ such that $\bar{\Sigma}_1 \cup \Sigma_2 = \bar{\Sigma}_3 \cup \Sigma_4 = \bar{\Sigma}_5 \cup \Sigma_6 = \partial B$ and $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \Sigma_5 \cap \Sigma_6 = \varnothing$.

Following [3], we impose some continuity conditions

$$\begin{aligned}
&\{u,\varphi\}\in C^{1,2},\\
&\{\varepsilon,\gamma,F_i,M_i,r\}\in C^{0,0},\\
&\{t,m,q\}\in C^{1,0},\\
&\theta\in C^{1,0},\qquad \eta\in C^{0,1},\\
&\{\tilde{u}_i,\tilde{t}_i,\tilde{\varphi}_j,\tilde{m}_i,\tilde{\theta},\tilde{q}\}\in C^{0,0}\quad\text{on } \partial B\times[0,\infty),\\
&\{u_i^0,\dot{u}_i^0,\varphi_j^0,\dot{\varphi}_j^0,\theta^0\}\in C^0\quad\text{on } \bar{B},\\
&\{D_{ij},E_{ij},A_{ijmn},B_{ijmn},C_{ijmn},c,K_{ij},\rho,I_{ij}\}\in C^0.
\end{aligned}\tag{14}$$

We use the symbol $C^{i,j}$ to denote the class of functions whose space partial derivatives of order up to and including $i$ and whose time derivatives of order up to and including $j$ are continuous. Such formal continuity requirements exist to any order as demanded by the existing expressions, unless otherwise stated. Furthermore, we need

$$\begin{aligned}
&\theta\in C^{1,1}\quad\text{on } \bar{B}\times[0,\infty),\\
&E_{ij},D_{ij}\in C^1(\bar{B}).
\end{aligned}\tag{15}$$

Following [3], we define an admissible state to be the collection $S\{u,\varphi;\varepsilon,\gamma;t,m;q,\theta\}$ of the ordered set of functions $u,\varphi,\varepsilon,\gamma,t,m,q$, and $\theta$ whose continuity requirements are described above. If $S$ meets the constitutive equations (3) and the strain displacement requirements (4) then we say that $S$ is kinematically admissible. If a kinematically admissible state meets the boundary and initial conditions and satisfies the equations of motion (1) and the equation of energy (2), we call it the solution of the mixed problem.

Introducing the constitutive equations (3) and the geometric equations (4) in the equations of motion (1) and the equation of energy (2), we obtain a system of coupled partial differential equations in terms of the displacements $u_i$, the microrotations $\varphi_i$ and the thermal displacements $\theta$

$$\begin{aligned}
&\big[A_{ijmn}(u_{n,m}+\varepsilon_{nmk}\varphi_k)+B_{ijmn}\varphi_{n,m}-D_{ij}\theta\big]_{,j}+\rho F_i=\rho\ddot{u}_i,\\
&\big[B_{mnij}(u_{n,m}+\varepsilon_{nmk}\varphi_k)+C_{ijmn}\varphi_{n,m}-E_{ij}\theta\big]_{,j}\\
&\quad+\varepsilon_{ijk}\big[A_{jkmn}(u_{n,m}+\varepsilon_{nmk}\varphi_k)+B_{jkmn}\varphi_{n,m}-D_{jk}\theta\big]+\rho M_i=I_{ij}\ddot{\varphi}_j,\\
&D_{ij}(\dot{u}_{j,i}+\varepsilon_{jik}\dot{\varphi}_k)+E_{ij}\dot{\varphi}_{j,i}+\frac{c}{\theta_0}\dot{\theta}=\frac{\rho}{\theta_0}r+\frac{1}{\theta_0}(K_{ij}\theta_{,j})_{,i},
\end{aligned}\tag{16}$$

for any $(x,t)\in B\times[0,\infty)$.

By a solution of the initial boundary value problem of the micropolar thermoelasticity in the cylinder $B\times[0,T)$ we mean an ordered array $\{u_i,\varphi_i,\theta\}$ which satisfies system (16) for all $(x,t)\in B\times[0,T)$, the initial conditions (9) and the boundary conditions (10).

## 2 Some useful identities

Throughout this paper it is assumed that a solution $\{u_i,\varphi_i,\theta\}$ exists. Following [10] and [1], we establish some auxiliary identities for the solution of the initial boundary value problem. We consider the external data

$$\mathcal{D} = \{F_i, M_i, r; u_i^0, \dot{u}_i^0, \theta^0, \varphi_j^0, \dot{\varphi}_j^0; \tilde{u}_i, \tilde{t}_i, \tilde{\theta}, \tilde{q}, \tilde{\varphi}_j, \tilde{m}_i\}. \tag{17}$$

**Lemma 2.1** *Let us consider a solution of the initial boundary value problem corresponding to the external data $\mathcal{D}$. Then for all $t \in [0, T)$ we have*

$$\begin{aligned}
&\int_B \rho \dot{u}_i(t)\dot{u}_i(t)\,\mathrm{d}v \\
&\quad = \int_B \rho \dot{u}_i(0)\dot{u}_i(0)\,\mathrm{d}v + 2\int_0^t \int_B \rho F_i(s)\dot{u}_i(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad + 2\int_0^t \int_{\partial B} \dot{u}_i(s)t_i(s)\,\mathrm{d}a\,\mathrm{d}s + 2\int_0^t \int_B D_{ij}\theta(s)\dot{u}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad - 2\int_0^t \int_B A_{ijmn}\bigl[u_{n,m}(s) + \varepsilon_{nmk}\varphi_k(s)\bigr]\dot{u}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad - 2\int_0^t \int_B B_{ijmn}\varphi_{n,m}(s)\dot{u}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\int_B \frac{1}{\theta_0} c\theta^2(t)\,\mathrm{d}v + 2\int_0^t \int_B \frac{1}{\theta_0} K_{ij}\beta_i(s)\beta_j(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\quad = 2\int_0^t \int_B \theta(s)\frac{\rho}{\theta_0} r(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad - 2\int_0^t \int_{\partial B} \theta(s)q(s)\,\mathrm{d}a\,\mathrm{d}s - 2\int_0^t \int_B \theta(s)D_{ij}\bigl[\dot{u}_{j,i}(s) + \varepsilon_{jik}\dot{\varphi}_k(s)\bigr]\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad - 2\int_0^t \int_B \theta(s)E_{ij}\dot{\varphi}_{j,i}(s)\,\mathrm{d}v\,\mathrm{d}s + \int_B \frac{1}{\theta_0} c\theta^2(0)\,\mathrm{d}v,
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\int_B I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t)\,\mathrm{d}v \\
&\quad = \int_B I_{ij}\dot{\varphi}_i(0)\dot{\varphi}_j(0)\,\mathrm{d}v + 2\int_0^t \int_{\partial B} \dot{\varphi}_i(s)m_{ij}(s)n_j\,\mathrm{d}a\,\mathrm{d}s \\
&\qquad - 2\int_0^t \int_B B_{mnij}\bigl[u_{n,m}(s) + \varepsilon_{nmk}\varphi_k(s)\bigr]\dot{\varphi}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad + 2\int_0^t \int_B C_{ijmn}\varphi_{n,m}(s)\dot{\varphi}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad - 2\int_0^t \int_B E_{ij}\theta(s)\dot{\varphi}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s + 2\int_0^t \int_B \varepsilon_{ijk}A_{jkmn}\bigl[u_{n,m}(s) + \varepsilon_{nmk}\varphi_k(s)\bigr]\dot{\varphi}_i(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad + 2\int_0^t \int_B \varepsilon_{ijk}B_{jkmn}\varphi_{n,m}(s)\dot{\varphi}_i(s)\,\mathrm{d}v\,\mathrm{d}s - 2\int_0^t \int_B \varepsilon_{ijk}D_{jk}\theta(s)\dot{\varphi}_i(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad + 2\int_0^t \int_B \rho M_i\dot{\varphi}_i(s)\,\mathrm{d}v\,\mathrm{d}s.
\end{aligned} \tag{20}$$

*Proof* We multiply the first equation from (1) by $\dot{u}_i$ to obtain

$$\rho\dot{u}_i(s)\ddot{u}_i(s) = \rho F_i(s)\dot{u}_i(s) + \bigl[t_{ij}(s)\dot{u}_i(s)\bigr]_{,j} - t_{ij}(s)\dot{u}_{i,j}(s). \tag{21}$$

We further have

$$\begin{aligned}&\frac{\partial}{\partial s}\left\{\frac{1}{2}\left[\rho\dot{u}_i(s)\dot{u}_i(s)\right]\right\}\\&\quad=\rho F_i(s)\dot{u}_i(s)+\left[t_{ij}(s)\dot{u}_i(s)\right]_{,j}\\&\qquad-A_{ijmn}\left[u_{n,m}(s)+\varepsilon_{nmk}\varphi_k(s)\right]\dot{u}_{i,j}(s)-B_{ijmn}\varphi_{n,m}(s)\dot{u}_{i,j}(s)+D_{ij}\theta(s)\dot{u}_{i,j}(s).\end{aligned}\tag{22}$$

We integrate this relation over $B\times[0,t]$, $t\in[0,T)$, and use the divergence theorem and equation (11) to obtain equation (18).

We have

$$\frac{\partial}{\partial s}\left\{\frac{1}{2}\frac{1}{\theta_0}c\theta^2(s)\right\}=\theta(s)\frac{1}{\theta_0}c\dot{\theta}(s).\tag{23}$$

We further have by the third relation in (3) and by (2)

$$\frac{\partial}{\partial s}\left\{\frac{1}{2}\frac{1}{\theta_0}c\theta^2(s)\right\}=\theta(s)\frac{\rho}{\theta_0}r(s)-\theta(s)q_{i,i}(s)-\theta(s)D_{ij}\dot{\varepsilon}_{ij}(s)-\theta(s)E_{ij}\dot{\gamma}_{ij}(s),\tag{24}$$

which by $\theta(s)q_{i,i}(s)=[\theta(s)q_i(s)]_{,i}-\theta_{,i}(s)q_i(s)$, by the fourth relation in (3), and by (5) becomes

$$\begin{aligned}&\frac{\partial}{\partial s}\left\{\frac{1}{2}\frac{1}{\theta_0}c\theta^2(s)\right\}\\&\quad=\theta(s)\frac{\rho}{\theta_0}r(s)-\left[\theta(s)q_i(s)\right]_{,i}-\beta_i(s)\frac{1}{\theta_0}K_{ij}\beta_j(s)-\theta(s)D_{ij}\dot{\varepsilon}_{ij}(s)-\theta(s)E_{ij}\dot{\gamma}_{ij}(s).\end{aligned}\tag{25}$$

We integrate this relation over $B\times[0,t]$, $t\in[0,T)$, and use the divergence theorem and equation (12) to obtain equation (19).

We multiply the second equation from (1) by $\dot{\varphi}_i$ to obtain

$$I_{ij}\ddot{\varphi}_j(s)\dot{\varphi}_i(s)=\left[m_{ij}(s)\dot{\varphi}_i(s)\right]_{,j}-m_{ij}(s)\dot{\varphi}_{i,j}(s)+\varepsilon_{ijk}t_{jk}(s)\dot{\varphi}_i(s)+\rho M_i\dot{\varphi}_i(s).\tag{26}$$

We further have

$$\begin{aligned}&\frac{\partial}{\partial s}\left\{\frac{1}{2}\left[I_{ij}\dot{\varphi}_i(s)\dot{\varphi}_j(s)\right]\right\}\\&\quad=\left[m_{ij}(s)\dot{\varphi}_i(s)\right]_{,j}-\left[B_{mnij}\varepsilon_{mn}(s)+C_{ijmn}\gamma_{mn}(s)-E_{ij}\theta(s)\right]\dot{\varphi}_{i,j}(s)\\&\qquad+\varepsilon_{ijk}\left[A_{jkmn}\varepsilon_{mn}(s)+B_{jkmn}\gamma_{mn}(s)-D_{jk}\theta(s)\right]\dot{\varphi}_i(s)+\rho M_i\dot{\varphi}_i(s)\\&\quad=\left[m_{ij}(s)\dot{\varphi}_i(s)\right]_{,j}-B_{mnij}\left[u_{n,m}(s)+\varepsilon_{nmk}\varphi_k(s)\right]\dot{\varphi}_{i,j}(s)+C_{ijmn}\varphi_{n,m}(s)\dot{\varphi}_{i,j}(s)\\&\qquad-E_{ij}\theta(s)\dot{\varphi}_{i,j}(s)+\varepsilon_{ijk}A_{jkmn}\left[u_{n,m}(s)+\varepsilon_{nmk}\varphi_k(s)\right]\dot{\varphi}_i(s)\\&\qquad+\varepsilon_{ijk}B_{jkmn}\varphi_{n,m}(s)\dot{\varphi}_i(s)-\varepsilon_{ijk}D_{jk}\theta(s)\dot{\varphi}_i(s)+\rho M_i\dot{\varphi}_i(s).\end{aligned}\tag{27}$$

We integrate this relation over $B\times[0,t]$, $t\in[0,T)$, and use the divergence theorem and equation (13) to obtain equation (20). □

**Lemma 2.2** *Let us consider a solution of the initial boundary value problem corresponding to the external data $\mathcal{D}$. Then for all $t \in [0, \frac{T}{2})$ we have*

$$\begin{aligned}
&\int_B \rho \dot{u}_i(t)\dot{u}_i(t)\,\mathrm{d}v \\
&\quad = \int_B \rho \dot{u}_i(0)\dot{u}_i(2t)\,\mathrm{d}v \\
&\qquad + \int_0^t \int_B \big[\rho F_i(t-s)\dot{u}_i(t+s) - \rho F_i(t+s)\dot{u}_i(t-s)\big]\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad + \int_0^t \int_{\partial B} \big[t_i(t-s)\dot{u}_i(t+s) - t_i(t+s)\dot{u}_i(t-s)\big]\,\mathrm{d}a\,\mathrm{d}s \\
&\qquad + \int_0^t \int_B \big\{\dot{u}_{i,j}(t-s)A_{ijmn}[u_{n,m} + \varepsilon_{nmk}\varphi_k](t+s) \\
&\qquad - \dot{u}_{i,j}(t+s)A_{ijmn}[u_{n,m} + \varepsilon_{nmk}\varphi_k](t-s)\big\}\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad + \int_0^t \int_B \big[\dot{u}_{i,j}(t-s)B_{ijmn}\varphi_{n,m}(t+s) - \dot{u}_{i,j}(t+s)B_{ijmn}\,\varphi_{n,m}(t-s)\big]\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad + \int_0^t \int_B \big[\dot{u}_{i,j}(t+s)D_{ij}\theta(t-s) - \dot{u}_{i,j}(t-s)D_{ij}\theta(t+s)\big]\,\mathrm{d}v\,\mathrm{d}s. \qquad (28)
\end{aligned}$$

*Proof* Let us consider $s \in [0, t]$, $t \in [0, \frac{T}{2})$. Then, using the identity

$$-\frac{\partial}{\partial s}\big[\rho \dot{u}_i(t-s)\dot{u}_i(t+s)\big] = \rho \dot{u}_i(t+s)\ddot{u}_i(t-s) - \rho \dot{u}_i(t-s)\ddot{u}_i(t+s), \qquad (29)$$

we obtain

$$\rho \dot{u}_i(t)\dot{u}_i(t) = \rho \dot{u}_i(0)\dot{u}_i(2t) + \int_0^t \rho\big[\dot{u}_i(t+s)\ddot{u}_i(t-s) - \dot{u}_i(t-s)\ddot{u}_i(t+s)\big]\,\mathrm{d}s. \qquad (30)$$

By the first relation from (1), we have

$$\begin{aligned}
&\rho\big[\dot{u}_i(t+s)\ddot{u}_i(t-s) - \dot{u}_i(t-s)\ddot{u}_i(t+s)\big] \\
&\quad = \rho F_i(t-s)\dot{u}_i(t+s) \\
&\qquad - \rho F_i(t+s)\dot{u}_i(t-s) + \big[t_{ij}(t-s)\dot{u}_i(t+s) - t_{ij}(t+s)\dot{u}_i(t-s)\big]_{,j} \\
&\qquad - t_{ij}(t-s)\dot{u}_{i,j}(t+s) + t_{ij}(t+s)\dot{u}_{i,j}(t-s). \qquad (31)
\end{aligned}$$

By the first relation in (3) and by (4) we have

$$\begin{aligned}
&t_{ij}(t+s)\dot{u}_{i,j}(t-s) - t_{ij}(t-s)\dot{u}_{i,j}(t+s) \\
&\quad = \dot{u}_{i,j}(t-s)A_{ijmn}[u_{n,m} + \varepsilon_{nmk}\varphi_k](t+s) \\
&\qquad - \dot{u}_{i,j}(t+s)A_{ijmn}[u_{n,m} + \varepsilon_{nmk}\varphi_k](t-s) \\
&\qquad + \dot{u}_{i,j}(t-s)B_{ijmn}\varphi_{n,m}(t+s) - \dot{u}_{i,j}(t+s)B_{ijmn}\varphi_{n,m}(t-s) \\
&\qquad - \dot{u}_{i,j}(t-s)D_{ij}\theta(t+s) + \dot{u}_{i,j}(t+s)D_{ij}\theta(t-s). \qquad (32)
\end{aligned}$$

Now we substitute equation (32) in equation (31) and this in equation (30) to obtain

$$\begin{aligned}
&\rho\dot{u}_i(t)\dot{u}_i(t)\\
&\quad= \rho\dot{u}_i(0)\dot{u}_i(2t) + \int_0^t \big[\rho F_i(t-s)\dot{u}_i(t+s) - \rho F_i(t+s)\dot{u}_i(t-s)\big]\,\mathrm{d}s\\
&\qquad+ \int_0^t \big[t_{ij}(t-s)\dot{u}_i(t+s) - t_{ij}(t+s)\dot{u}_i(t-s)\big]_{,j}\,\mathrm{d}s\\
&\qquad+ \int_0^t \big\{\dot{u}_{i,j}(t-s)A_{ijmn}[u_{n,m} + \varepsilon_{nmk}\varphi_k](t+s)\\
&\qquad- \dot{u}_{i,j}(t+s)A_{ijmn}[u_{n,m} + \varepsilon_{nmk}\varphi_k](t-s)\big\}\,\mathrm{d}s\\
&\qquad+ \int_0^t \big[\dot{u}_{i,j}(t-s)B_{ijmn}\varphi_{n,m}(t+s) - \dot{u}_{i,j}(t+s)B_{ijmn}\varphi_{n,m}(t-s)\big]\,\mathrm{d}s\\
&\qquad+ \int_0^t \big[\dot{u}_{i,j}(t+s)D_{ij}\theta(t-s) - \dot{u}_{i,j}(t-s)D_{ij}\theta(t+s)\big]\,\mathrm{d}s. \qquad (33)
\end{aligned}$$

We integrate equation (33) over $B$, use the divergence theorem and equation (11) to obtain the final result. □

## 3 Zero external data

Suppose that the boundary initial value problem of linear micropolar elastodynamics has two solutions $u^{(\alpha)}$, $\varphi^{(\alpha)}$, $\theta^{(\alpha)}$, $\alpha = 1,2$. Let $u = u^{(1)} - u^{(2)}$, $\varphi = \varphi^{(1)} - \varphi^{(2)}$, $\theta = \theta^{(1)} - \theta^{(2)}$. Then $u$, $\varphi$, and $\theta$ satisfy (1)-(4), (9), and (10) with $F_i = M_i = 0$, $r = 0$, $\tilde{u}_i = \tilde{t}_i = \tilde{\theta} = \tilde{q} = \tilde{\varphi}_j = \tilde{m}_i = 0$, $u_i^0 = \dot{u}_i^0 = \theta^0 = \varphi_j^0 = \dot{\varphi}_j^0 = 0$, *i.e.* homogeneous equations and boundary and initial conditions.

**Lemma 3.1** *Let us consider a solution of the initial boundary value problem corresponding to zero external data $\mathcal{D} = 0$. Then for all $t \in [0, T)$ we have*

$$\begin{aligned}
&\int_B \rho\dot{u}_i(t)\dot{u}_i(t)\,\mathrm{d}v\\
&\quad= 2\int_0^t \int_B D_{ij}\theta(s)\dot{u}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s\\
&\qquad- 2\int_0^t \int_B A_{ijmn}\big[u_{n,m}(s) + \varepsilon_{nmk}\varphi_k(s)\big]\dot{u}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s\\
&\qquad- 2\int_0^t \int_B B_{ijmn}\varphi_{n,m}(s)\dot{u}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s, \qquad (34)
\end{aligned}$$

$$\begin{aligned}
&\int_B \frac{1}{\theta_0}c\theta^2(t)\,\mathrm{d}v + 2\int_0^t \int_B \frac{1}{\theta_0}K_{ij}\beta_i(s)\beta_j(s)\,\mathrm{d}v\,\mathrm{d}s\\
&\quad= -2\int_0^t \int_B \theta(s)D_{ij}\big[\dot{u}_{j,i}(s) + \varepsilon_{jik}\dot{\varphi}_k(s)\big]\,\mathrm{d}v\,\mathrm{d}s - 2\int_0^t \int_B \theta(s)E_{ij}\dot{\varphi}_{j,i}(s)\,\mathrm{d}v\,\mathrm{d}s, \qquad (35)
\end{aligned}$$

$$\begin{aligned}
&\int_B I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t)\,\mathrm{d}v\\
&\quad= -2\int_0^t \int_B \big\{B_{mnij}\big[u_{n,m}(s) + \varepsilon_{nmk}\varphi_k(s)\big]
\end{aligned}$$

$$+ C_{ijmn}\varphi_{n,m}(s) - E_{ij}\theta(s)\big\}\dot{\varphi}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s$$
$$+ 2\int_0^t \int_B \varepsilon_{ijk}\big\{A_{jkmn}\big[u_{n,m}(s) + \varepsilon_{nmk}\varphi_k(s)\big]$$
$$+ B_{jkmn}\varphi_{n,m}(s) - D_{jk}\theta(s)\big\}\dot{\varphi}_i(s)\,\mathrm{d}v\,\mathrm{d}s, \tag{36}$$

$$\int_B \rho\dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t) + \frac{1}{\theta_0}c\theta^2(t)\,\mathrm{d}v$$
$$= -2\int_0^t \int_B \big[u_{n,m}(s) + \varepsilon_{nmk}\varphi_k(s)\big]\big[A_{ijmn}\dot{u}_{i,j}(s) + B_{mnij}\dot{\varphi}_{i,j}(s)$$
$$- \varepsilon_{ijk}A_{jkmn}\dot{\varphi}_i(s)\big]\,\mathrm{d}v\,\mathrm{d}s - 2\int_0^t \int_B \varphi_{n,m}(s)\big[B_{ijmn}\dot{u}_{i,j}(s) + C_{ijmn}\dot{\varphi}_{i,j}(s)$$
$$- \varepsilon_{ijk}B_{jkmn}\dot{\varphi}_i(s)\big]\,\mathrm{d}v\,\mathrm{d}s + 2\int_0^t \int_B D_{ij}\theta(s)\dot{u}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s + 2\int_0^t \int_B E_{ij}\theta(s)\dot{\varphi}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s$$
$$- 2\int_0^t \int_B \varepsilon_{ijk}D_{jk}\theta(s)\dot{\varphi}_i(s)\,\mathrm{d}v\,\mathrm{d}s - 2\int_0^t \int_B \theta(s)D_{ij}\big[\dot{u}_{j,i}(s) + \varepsilon_{jik}\dot{\varphi}_k(s)\big]\,\mathrm{d}v\,\mathrm{d}s$$
$$- 2\int_0^t \int_B \theta(s)E_{ij}\dot{\varphi}_{j,i}(s)\,\mathrm{d}v\,\mathrm{d}s - 2\int_0^t \int_B \frac{1}{\theta_0}K_{ij}\beta_i(s)\beta_j(s)\,\mathrm{d}v\,\mathrm{d}s. \tag{37}$$

*Proof* We choose $\dot{u}_i^0(x) = 0, x \in \bar{B}$, $\tilde{u}_i(x,t) = 0$ on $\bar{\Sigma}_1 \times [0,\infty)$, $\tilde{t}_i(x,t) = 0$ on $\Sigma_2 \times [0,\infty)$ and $F_i(x,t) = 0$ on $\bar{B} \times [0,\infty)$ in (18) to obtain equation (34).

We choose $\theta^0(x) = 0, x \in \bar{B}$, $\tilde{\theta}(x,t) = 0$ on $\bar{\Sigma}_3 \times [0,\infty)$, $\tilde{q}(x,t) = 0$ on $\Sigma_4 \times [0,\infty)$ and $r(x,t) = 0$ on $\bar{B} \times [0,\infty)$ in (19) to obtain equation (35).

We choose $\dot{\varphi}_j^0(x) = 0, x \in \bar{B}$, $\tilde{\varphi}_j(x,t) = 0$ on $\bar{\Sigma}_5 \times [0,\infty)$, $\tilde{m}_i(x,t) = 0$ on $\Sigma_6 \times [0,\infty)$ and $M_i(x,t) = 0$ on $\bar{B} \times [0,\infty)$ in (20) to obtain equation (36).

We add up the previous three formulas. □

**Lemma 3.2** *Let us consider a solution of the initial boundary value problem corresponding to zero external data $\mathcal{D}$. Then for all $t \in [0, \frac{T}{2})$ we have*

$$\int_B \rho\dot{u}_i(t)\dot{u}_i(t)\,\mathrm{d}v$$
$$= \int_0^t \int_B \big\{\dot{u}_{i,j}(t-s)A_{ijmn}[u_{n,m} + \varepsilon_{nmk}\varphi_k](t+s)$$
$$- \dot{u}_{i,j}(t+s)A_{ijmn}[u_{n,m} + \varepsilon_{nmk}\varphi_k](t-s)\big\}\,\mathrm{d}v\,\mathrm{d}s$$
$$+ \int_0^t \int_B \big[\dot{u}_{i,j}(t-s)B_{ijmn}\varphi_{n,m}(t+s) - \dot{u}_{i,j}(t+s)B_{ijmn}\varphi_{n,m}(t-s)\big]\,\mathrm{d}v\,\mathrm{d}s$$
$$+ \int_0^t \int_B \big[\dot{u}_{i,j}(t+s)D_{ij}\theta(t-s) - \dot{u}_{i,j}(t-s)D_{ij}\theta(t+s)\big]\,\mathrm{d}v\,\mathrm{d}s. \tag{38}$$

*Proof* We choose $\dot{u}_i^0(x) = 0, x \in \bar{B}$, $\tilde{u}_i(x,t) = 0$ on $\bar{\Sigma}_1 \times [0,\infty)$, $\tilde{t}_i(x,t) = 0$ on $\Sigma_2 \times [0,\infty)$ and $F_i(x,t) = 0$ on $\bar{B} \times [0,\infty)$ in (28) to obtain equation (38). □

## 4 Some useful remarks

We assume that *meas* $\Sigma_4 = 0$. Let $\{u_i, \varphi_j, \theta\}(x,t)$ be a solution of the initial boundary value problem corresponding to zero external data $\mathcal{D} = 0$. We might want to prove that

$\{u_i, \varphi_j, \theta\}(x,t) = 0$ in $\bar{B} \times [0,\infty)$. Let

$$\phi(t) = \int_0^t \int_B \frac{1}{\theta_0} K_{ij}\beta_i(s)\beta_j(s)\,\mathrm{d}v\,\mathrm{d}s \tag{39}$$

for all $t \in [0,\infty)$. If $\phi(t) = 0$ and $\theta_0 \neq 0, K_{ij} \neq 0$, then either $\beta_i(x,s) = 0$ or $\beta_j(x,s) = 0$ in $B \times [0,\infty)$. If $\beta_i(x,s) = 0$ then $\theta_{,i}(x,s) = 0$ in $B \times [0,\infty)$. By [10] and [1] and since $meas\Sigma_3 \neq 0$ and $\theta(x,t) = 0$ on $\bar{\Sigma}_3 \times [0,\infty)$ we have

$$\int_B \theta_{,i}(t)\theta_{,i}(t)\,\mathrm{d}v \geq \lambda \int_B \theta^2(t)\,\mathrm{d}v \tag{40}$$

where $\lambda > 0$, $\lambda =$ const. is the smallest eigenvalue of the fixed membrane problem. Therefore $\theta(x,t) = 0$ in $\bar{B} \times [0,\infty)$, which yields

$$\begin{aligned}
&\int_B \rho\dot{u}_i(t)\dot{u}_i(t)\,\mathrm{d}v \\
&\quad= -2\int_0^t \int_B A_{ijmn}\big[u_{n,m}(s) + \varepsilon_{nmk}\varphi_k(s)\big]\dot{u}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s - 2\int_0^t \int_B B_{ijmn}\varphi_{n,m}(s)\dot{u}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\quad= -2\int_0^t \int_B A_{ijmn}\varepsilon_{mn}(s)\dot{u}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s - 2\int_0^t \int_B B_{ijmn}\gamma_{mn}(s)\dot{u}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s,
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\int_B I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t)\,\mathrm{d}v \\
&\quad= -2\int_0^t \int_B \big\{B_{mnij}\big[u_{n,m}(s) + \varepsilon_{nmk}\varphi_k(s)\big] + C_{ijmn}\varphi_{n,m}(s)\big\}\dot{\varphi}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad+ 2\int_0^t \int_B \varepsilon_{ijk}\big\{A_{jkmn}\big[u_{n,m}(s) + \varepsilon_{nmk}\varphi_k(s)\big] + B_{jkmn}\varphi_{n,m}(s)\big\}\dot{\varphi}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\quad= -2\int_0^t \int_B \big\{B_{mnij}\varepsilon_{mn}(s) + C_{ijmn}\gamma_{mn}(s)\big\}\dot{\varphi}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\quad+ 2\int_0^t \int_B \varepsilon_{ijk}\big\{A_{jkmn}\varepsilon_{mn}(s) + B_{jkmn}\gamma_{mn}(s)\big\}\dot{\varphi}_{i,j}(s)\,\mathrm{d}v\,\mathrm{d}s.
\end{aligned} \tag{42}$$

By imposing certain conditions on the coefficients, we might prove that $\varphi(x,t) = 0$ and $u(x,t) = 0$ in $\bar{B} \times [0,\infty)$. Following [3], we know that

$$A_{klmn}\varepsilon_{kl}\varepsilon_{mn} + C_{klmn}\gamma_{kl}\gamma_{mn} + 2B_{klmn}\varepsilon_{kl}\gamma_{mn} \geq 0. \tag{43}$$

Now we derive some useful inequalities for the constitutive coefficients $E_{ij}$ and $D_{ij}$. Since $\theta(x,s) = 0$ on $\partial B \times [0,T)$ we have

$$\begin{aligned}
&-2\int_0^t \int_B \theta(s)E_{ij}\dot{\varphi}_{j,i}(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\quad= -2\int_0^t \int_B \big\{\big[E_{ij}\theta(s)\dot{\varphi}_j(s)\big]_{,i} - \big[E_{ij}\theta(s)\big]_{,i}\dot{\varphi}_j(s)\big\}\,\mathrm{d}v\,\mathrm{d}s \\
&\quad= -2\int_0^t \int_{\partial B} E_{ij}\theta(s)\dot{\varphi}_j(s)n_i\,\mathrm{d}a\,\mathrm{d}s + 2\int_0^t \int_B \big[E_{ij,i}\theta(s)\dot{\varphi}_j(s) + E_{ij}\theta_{,i}(s)\dot{\varphi}_j(s)\big]\,\mathrm{d}v\,\mathrm{d}s \\
&\quad= 2\int_0^t \int_B \left\{\left[\frac{\sqrt{\varepsilon_1}}{\sqrt{\rho_0}}E_{ij,i}\theta(s)\right]\left[\frac{\sqrt{\rho_0}}{\sqrt{\varepsilon_1}}\dot{\varphi}_j(s)\right] + \left[\frac{\sqrt{\varepsilon_2}}{\sqrt{\rho_0}}E_{ij}\theta_{,i}(s)\right]\left[\frac{\sqrt{\rho_0}}{\sqrt{\varepsilon_2}}\dot{\varphi}_j(s)\right]\right\}\mathrm{d}v\,\mathrm{d}s,
\end{aligned} \tag{44}$$

which by the Cauchy-Schwartz inequality and by the arithmetic-geometric mean inequality is bounded by

$$\begin{aligned}
&\leq 2\int_0^t \int_B \left\{ \left[ \frac{\sqrt{\varepsilon_1}}{\sqrt{\rho_0}} E_{ij,i}\theta(s) \right]^2 \right. \\
&\quad \left. + \left[ \frac{\sqrt{\varepsilon_2}}{\sqrt{\rho_0}} E_{ij}\theta_{,i}(s) \right]^2 \right\}^{\frac{1}{2}} \left\{ \left[ \frac{\sqrt{\rho_0}}{\sqrt{\varepsilon_1}} \dot{\varphi}_j(s) \right]^2 + \left[ \frac{\sqrt{\rho_0}}{\sqrt{\varepsilon_2}} \dot{\varphi}_j(s) \right]^2 \right\}^{\frac{1}{2}} \mathrm{d}v\,\mathrm{d}s \\
&\leq \int_0^t \int_B \left\{ \left[ \frac{\sqrt{\varepsilon_1}}{\sqrt{\rho_0}} E_{ij,i}\theta(s) \right]^2 \right. \\
&\quad \left. + \left[ \frac{\sqrt{\varepsilon_2}}{\sqrt{\rho_0}} E_{ij}\theta_{,i}(s) \right]^2 + \rho_0 \dot{\varphi}_j(s)\dot{\varphi}_j(s) \left( \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \right) \right\} \mathrm{d}v\,\mathrm{d}s. \qquad (45)
\end{aligned}$$

We assume that

$$m_E = \sup_{\bar{B}} (E_{ij}E_{ij})^{\frac{1}{2}} > 0, \qquad (46)$$

$$m_E^* = \sup_{\bar{B}} (E_{ij,i}E_{kj,k})^{\frac{1}{2}} > 0. \qquad (47)$$

We denote $Q_j^E = E_{ij}\theta_{,i}$ and

$$Q_j^E Q_j^E = E_{ij}\theta_{,i} Q_j^E \leq (E_{ij}E_{ij})^{\frac{1}{2}} \left( Q_j^E \theta_{,i} Q_j^E \theta_{,i} \right)^{\frac{1}{2}}, \qquad (48)$$

which yields

$$E_{ij}\theta_{,i}E_{kj}\theta_{,k} \leq m_E^2 \theta_{,i}\theta_{,i}. \qquad (49)$$

Therefore $\forall \varepsilon_1, \varepsilon_2 > 0$

$$\begin{aligned}
&-2\int_0^t \int_B \theta(s) E_{ij}\dot{\varphi}_{j,i}(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\quad \leq \int_0^t \int_B \left\{ \left( \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \right) \rho_0 \dot{\varphi}_j(s)\dot{\varphi}_j(s) + \frac{\varepsilon_1}{\rho_0} m_E^{*2} \theta^2(s) + \frac{\varepsilon_2}{\rho_0} m_E^2 \theta_{,i}(s)\theta_{,i}(s) \right\} \mathrm{d}v\,\mathrm{d}s, \qquad (50)
\end{aligned}$$

which by (40) is further bounded by

$$\leq \int_0^t \int_B \left\{ \left( \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \right) \rho_0 \dot{\varphi}_j(s)\dot{\varphi}_j(s) + \left( \frac{\varepsilon_1}{\rho_0} m_E^{*2} \lambda^{-1} + \frac{\varepsilon_2}{\rho_0} m_E^2 \right) \theta_{,i}(s)\theta_{,i}(s) \right\} \mathrm{d}v\,\mathrm{d}s. \qquad (51)$$

Since $\theta(x,t) = 0$ in $\bar{B} \times [0,\infty)$, by (35) and by (51) we obtain

$$\int_0^t \int_B \rho_0 \dot{\varphi}_j(s)\dot{\varphi}_j(s)\,\mathrm{d}v\,\mathrm{d}s \geq 0. \qquad (52)$$

Hence we do not need any condition on $\rho_0$ to prove the positivity of the integral above.

We perform similar computations for the constitutive coefficient $D_{ij}$. We have, since $\theta(x,s) = 0$ on $\partial B \times [0,T)$,

$$
\begin{aligned}
&-2\int_0^t \int_B \theta(s) D_{ij}\dot{u}_{j,i}(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\quad = -2\int_0^t \int_B \big\{\big[D_{ij}\theta(s)\dot{u}_j(s)\big]_{,i} - \big[D_{ij}\theta(s)\big]_{,i}\dot{u}_j(s)\big\}\,\mathrm{d}v\,\mathrm{d}s \\
&\quad = -2\int_0^t \int_{\partial B} D_{ij}\theta(s)\dot{u}_j(s)n_i\,\mathrm{d}a\,\mathrm{d}s + 2\int_0^t \int_B \big[D_{ij,i}\theta(s)\dot{u}_j(s) + D_{ij}\theta_{,i}(s)\dot{u}_j(s)\big]\,\mathrm{d}v\,\mathrm{d}s \\
&\quad = 2\int_0^t \int_B \left\{\left[\frac{\sqrt{\varepsilon_1}}{\sqrt{\rho_0}}D_{ij,i}\theta(s)\right]\left[\frac{\sqrt{\rho_0}}{\sqrt{\varepsilon_1}}\dot{u}_j(s)\right] + \left[\frac{\sqrt{\varepsilon_2}}{\sqrt{\rho_0}}D_{ij}\theta_{,i}(s)\right]\left[\frac{\sqrt{\rho_0}}{\sqrt{\varepsilon_2}}\dot{u}_j(s)\right]\right\}\,\mathrm{d}v\,\mathrm{d}s, \qquad (53)
\end{aligned}
$$

which by the Cauchy-Schwartz inequality and by the arithmetic-geometric mean inequality is bounded by

$$
\begin{aligned}
&\leq 2\int_0^t \int_B \left\{\left[\frac{\sqrt{\varepsilon_1}}{\sqrt{\rho_0}}D_{ij,i}\theta(s)\right]^2\right. \\
&\quad \left. + \left[\frac{\sqrt{\varepsilon_2}}{\sqrt{\rho_0}}D_{ij}\theta_{,i}(s)\right]^2\right\}^{\frac{1}{2}} \left\{\left[\frac{\sqrt{\rho_0}}{\sqrt{\varepsilon_1}}\dot{u}_j(s)\right]^2 + \left[\frac{\sqrt{\rho_0}}{\sqrt{\varepsilon_2}}\dot{u}_j(s)\right]^2\right\}^{\frac{1}{2}}\,\mathrm{d}v\,\mathrm{d}s \\
&\leq \int_0^t \int_B \left\{\left[\frac{\sqrt{\varepsilon_1}}{\sqrt{\rho_0}}D_{ij,i}\theta(s)\right]^2 + \left[\frac{\sqrt{\varepsilon_2}}{\sqrt{\rho_0}}D_{ij}\theta_{,i}(s)\right]^2 + \rho_0\dot{u}_j(s)\dot{u}_j(s)\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\right\}\,\mathrm{d}v\,\mathrm{d}s. \qquad (54)
\end{aligned}
$$

We assume that

$$
m_D = \sup_{\bar{B}}(D_{ij}D_{ij})^{\frac{1}{2}} > 0, \qquad (55)
$$

$$
m_D^* = \sup_{\bar{B}}(D_{ij,i}D_{kj,k})^{\frac{1}{2}} > 0. \qquad (56)
$$

We denote $Q_j^D = D_{ij}\theta_{,i}$ and

$$
Q_j^D Q_j^D = D_{ij}\theta_{,i}Q_j^D \leq (D_{ij}D_{ij})^{\frac{1}{2}}\big(Q_j^D\theta_{,i}Q_j^D\theta_{,i}\big)^{\frac{1}{2}}, \qquad (57)
$$

which yields

$$
D_{ij}\theta_{,i}D_{kj}\theta_{,k} \leq m_D^2\theta_{,i}\theta_{,i}. \qquad (58)
$$

Therefore $\forall\varepsilon_1, \varepsilon_2 > 0$

$$
\begin{aligned}
&-2\int_0^t \int_B \theta(s)D_{ij}\dot{u}_{j,i}(s)\,\mathrm{d}v\,\mathrm{d}s \\
&\quad \leq \int_0^t \int_B \left\{\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\rho_0\dot{u}_j(s)\dot{u}_j(s) + \frac{\varepsilon_1}{\rho_0}m_D^{*2}\theta^2(s) + \frac{\varepsilon_2}{\rho_0}m_D^2\theta_{,i}(s)\theta_{,i}(s)\right\}\,\mathrm{d}v\,\mathrm{d}s, \qquad (59)
\end{aligned}
$$

which by (40) is further bounded by

$$
\leq \int_0^t \int_B \left\{\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\rho_0\dot{u}_j(s)\dot{u}_j(s) + \left(\frac{\varepsilon_1}{\rho_0}m_D^{*2}\lambda^{-1} + \frac{\varepsilon_2}{\rho_0}m_D^2\right)\theta_{,i}(s)\theta_{,i}(s)\right\}\,\mathrm{d}v\,\mathrm{d}s. \qquad (60)
$$

Since $\theta(x,t) = 0$ in $\bar{B} \times [0,\infty)$, by (35) and by (60) we obtain

$$\int_0^t \int_B \rho_0 \dot{u}_j(s)\dot{u}_j(s)\,\mathrm{d}v\,\mathrm{d}s \geq 0. \tag{61}$$

Hence we do not need any condition on $\rho_0$ to prove the positivity of the integral above.

We say that $K_{ij}$ is a positive definite tensor if there exists a positive constant $k_0 > 0$ such that $\forall \xi_i$

$$K_{ij}\xi_i\xi_j \geq k_0\xi_i\xi_i. \tag{62}$$

Thus $k_0$ can be identified with the minimum of the positive eigenvalues of $K_{ij}$ on $\bar{B}$. Since $K_{ij}\theta_{,i}\theta_{,j} \geq k_0\theta_{,i}\theta_{,i}$ we have $\forall \varepsilon_1, \varepsilon_2 > 0, t \in [0,T)$

$$\begin{aligned} &-2\int_0^t \int_B \theta(s)E_{ij}\dot{\varphi}_{j,i}(s)\,\mathrm{d}v\,\mathrm{d}s \\ &\quad\leq \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\int_0^t \int_B \rho_0\dot{\varphi}_j(s)\dot{\varphi}_j(s)\,\mathrm{d}v\,\mathrm{d}s \\ &\qquad + \frac{\theta_0}{\rho_0 k_0}\left(\varepsilon_1 m_E^{*2}\lambda^{-1} + \varepsilon_2 m_E^2\right)\int_0^t \int_B \frac{1}{\theta_0}K_{ij}\theta_{,i}(s)\theta_{,j}(s)\,\mathrm{d}v\,\mathrm{d}s \end{aligned} \tag{63}$$

and

$$\begin{aligned} &-2\int_0^t \int_B \theta(s)D_{ij}\dot{u}_{j,i}(s)\,\mathrm{d}v\,\mathrm{d}s \\ &\quad\leq \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\int_0^t \int_B \rho_0\dot{u}_j(s)\dot{u}_j(s)\,\mathrm{d}v\,\mathrm{d}s \\ &\qquad + \frac{\theta_0}{\rho_0 k_0}\left(\varepsilon_1 m_D^{*2}\lambda^{-1} + \varepsilon_2 m_D^2\right)\int_0^t \int_B \frac{1}{\theta_0}K_{ij}\theta_{,i}(s)\theta_{,j}(s)\,\mathrm{d}v\,\mathrm{d}s. \end{aligned} \tag{64}$$

We choose parameters $\varepsilon_1, \varepsilon_2$ so small that

$$\alpha = 2 - \frac{\theta_0}{\rho_0 k_0}\left(\varepsilon_1 m_D^{*2}\lambda^{-1} + \varepsilon_2 m_D^2\right) - \frac{\theta_0}{\rho_0 k_0}\left(\varepsilon_1 m_E^{*2}\lambda^{-1} + \varepsilon_2 m_E^2\right) > 0. \tag{65}$$

Hence from equation (35) we have

$$\begin{aligned} &\int_B \frac{1}{\theta_0}c\theta^2(t)\,\mathrm{d}v + \alpha\int_0^t \int_B \frac{1}{\theta_0}K_{ij}\beta_i(s)\beta_j(s)\,\mathrm{d}v\,\mathrm{d}s \\ &\quad\leq \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\int_0^t \int_B \rho_0\dot{u}_j(s)\dot{u}_j(s)\,\mathrm{d}v\,\mathrm{d}s \\ &\qquad + \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\int_0^t \int_B \rho_0\dot{\varphi}_j(s)\dot{\varphi}_j(s)\,\mathrm{d}v\,\mathrm{d}s - 2\int_0^t \int_B \theta(s)D_{ij}\varepsilon_{ijk}\varphi_k(s)\,\mathrm{d}v\,\mathrm{d}s. \end{aligned} \tag{66}$$

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

**Author details**
[1]Department of Mathematics, Transilvania University of Braşov, Braşov, Romania. [2]Department of Mathematics, Texas A&M University - Kingsville, Kingsville, USA.

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