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On a generalization of the Dirichlet problem for the Poisson equation

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Abstract

In this paper, we investigate a generalization of the Dirichlet problem for the Poisson equation in a rectangular domain. We assume that the kth-order normal derivatives of an unknown function are given on lower and upper bases of the rectangle and that homogeneous boundary conditions of the first kind are given on the lateral sides. Under these conditions, we prove the existence of a unique regular solution of this problem.

MSC: Primary 35J25

Keywords: Poisson equation; Dirichlet problem; regular solution; uniqueness of a solution; existence of a solution

1 Introduction. Formulation of the problem

The boundary value problems for elliptic equations have been studied extensively by many authors (see, *e.g.*, [1, 2] and the references therein). In [3], the following problem for the homogeneous heat conduction equation in the domain $(0 < x < \infty, t > 0)$ was considered:

$$\sum_{k=1}^{m} a_k \frac{\partial^k u(0,t)}{\partial x^k} = f(x,t), \qquad u(x,0) = 0,$$

and the uniqueness and existence of the solution of this problem were proved. In [4], for the Laplace equation in an n-dimensional bounded domain D a problem with the boundary condition of the form

$$\frac{d^m u}{d\nu^m} = f(x), \quad x \in \partial D,$$

was investigated, and its Fredholm property was proved. For the Laplace, Poisson, and Helmholtz equations, the boundary value problems in the unit ball with higher-order derivatives in the boundary conditions were studied by Karachik [5–8], Sokolovskiĭ [9], and others. In the papers [4–8], the boundary conditions were given on the whole boundary. Therefore, the uniqueness of problems was proved within homogeneous polynomials of certain degree. In a rectangular domain for the heat conduction equation, the initial-boundary value problem with higher-order derivative in the initial condition was studied in [10]. Boundary value problems in the rectangular domains were studied by Sabitov (see, *e.g.*, [11–15]).

In the present paper, we consider the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \tag{1.1}$$

in the domain $\Omega = \{(x,y) : 0 < x < p, 0 < y < q\}$.

Problem Find a function $u(x,y) \in C^2(\overline{\Omega})$, $\frac{\partial^k u}{\partial y^k} \in C(\overline{\Omega})$ satisfying equation (1.1) in Ω and the following conditions:

$$u(0,y) = 0, \quad 0 < y < q, \tag{1.2}$$

$$u(p,y) = 0, \quad 0 < y < q, \tag{1.3}$$

$$\frac{\partial^k u}{\partial y^k}(x,0) = \varphi(x), \quad 0 < x < p, \tag{1.4}$$

$$\frac{\partial^k u}{\partial y^k}(x,q) = \psi(x), \quad 0 < x < p, \tag{1.5}$$

where k is a fixed nonnegative integer. If $k = 0$, then it is necessary for the functions $\varphi(x)$ and $\psi(x)$ to satisfy the following conditions: $\varphi(0) = \varphi(p) = 0$, $\psi(0) = \psi(q) = 0$. In case $k = 0$ and $f(x,y) = 0$ in Ω, problem (1.1)-(1.5) was studied in [2]. In the papers [16] and [17], the authors used similar procedures.

In this paper, our goal is to show the existence of a unique regular solution for this problem.

2 Uniqueness of the solution of problem (1.1)-(1.5)

Here, we prove the uniqueness of the solution of problem (1.1)-(1.5).

Theorem 2.1 *The solution of problem* (1.1)-(1.5) *is unique if it exists.*

Proof Assume that

$$\varphi(x) = 0, \qquad \psi(x) = 0, \quad 0 \le x \le p, \qquad f(x,y) = 0, \quad (x,y) \in \overline{\Omega}.$$

We will prove that $u(x,y) = 0$ in $\overline{\Omega}$. In order to show this, we refer to [18] and consider the integral

$$\alpha_n(y) = \int_0^p u(x,y) X_n(x)\,dx, \tag{2.1}$$

where

$$X_n(x) = \sqrt{\frac{2}{p}} \sin \lambda_n x\,dx, \quad \lambda_n = \frac{n\pi}{p}, n = 1, 2, \ldots$$

is a complete orthonormal system in $L_2[0,p]$ (see, *e.g.*, [19]). Differentiating (2.1) twice with respect to y, we get

$$\alpha_n''(y) = \int_0^p \frac{\partial^2 u}{\partial y^2} X_n(x)\,dx.$$

From the homogeneous equation (1.1) we have

$$\alpha_n''(y) = -\int_0^p u_{xx}(x,y)X_n(x)\,dx.$$

Applying integration by parts and using conditions (1.2) and (1.3), we get

$$\alpha_n''(y) - \lambda_n^2\alpha_n(y) = 0. \tag{2.2}$$

The general solution of equation (2.2) has the form

$$\alpha_n(y) = a_n e^{-\lambda_n y} + b_n e^{\lambda_n y},$$

where a_n and b_n are unknown *constant* coefficients. In order to find a_n and b_n, we use conditions (1.4) and (1.5), which imply

$$\alpha_n^{(k)}(0) = 0, \qquad \alpha_n^{(k)}(q) = 0. \tag{2.3}$$

The derivative $\alpha_n^{(k)}(y)$ has form

$$\alpha_n^{(k)}(y) = \lambda_n^k\bigl[(-1)^k a_n e^{-\lambda_n y} + b_n e^{\lambda_n y}\bigr].$$

Using (2.3), we have

$$\begin{cases} (-1)^k a_n + b_n = 0, \\ (-1)^k a_n e^{-\lambda_n q} + b_n e^{\lambda_n q} = 0. \end{cases}$$

The determinant of this system equals $(-1)^k 2\,\mathrm{sh}(\lambda_n q) \neq 0$. Therefore, $a_n = b_n = 0$. Consequently, $\alpha_n(y) = 0$. Finally, from completeness of the functions $X_n(x)$ in $L_2(0,p)$ and from (2.1) we obtain $u(x,y) = 0$ in $\overline{\Omega}$.

Theorem 2.1 is proved. □

3 Existence of the solution of problem (1.1)-(1.5)

In this section, we first construct a formal solution of problem (1.1)-(1.5). Then, we prove some lemmas on convergence of the series in the formal solution and its derivatives. Finally, we formulate the theorem on solvability of problem (1.1)-(1.5). We seek a formal solution of this problem in the form of Fourier series

$$u(x,y) = \sum_{n=1}^{\infty} u_n(y)X_n(x) \tag{3.1}$$

expanded along system $X_n(x)$. It is clear that $u(x,y)$ satisfies conditions (1.2)-(1.3). Assume that

$$f(x,y) \in C^2(\Omega), \qquad f(0,y) = f(p,y) = 0, \qquad \varphi(x) \in C^2[0,p], \qquad \varphi(0) = \varphi(p) = 0,$$
$$\psi(x) \in C^2[0,p], \qquad \psi(0) = \psi(p) = 0.$$

We expand the given functions $f(x,y)$, $\varphi(x)$, and $\psi(x)$ in the Fourier series along the functions $X_n(x)$:

$$f(x,y) = \sum_{n=1}^{\infty} f_n(y)X_n(x), \tag{3.2}$$

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n X_n(x), \tag{3.3}$$

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n X_n(x), \tag{3.4}$$

where

$$f_n(y) = \int_0^p f(x,y)X_n(x)\,dx, \tag{3.5}$$

$$\varphi_n = \int_0^p \varphi(x)X_n(x)\,dx, \tag{3.6}$$

$$\psi_n = \int_0^p \psi(x)X_n(x)\,dx. \tag{3.7}$$

Using Fourier's method, we get a solution of problem (1.1)-(1.5) in the form

$$\begin{aligned} u(x,y) = \sum_{n=1}^{\infty} X_n(x)\Biggl\{ & \frac{e^{\lambda_n(q-y)} - (-1)^k e^{-\lambda_n(q-y)}}{(-1)^k 2 \operatorname{sh}\lambda_n q}\Biggl[\frac{\varphi_n}{\lambda_n^k} - \sum_{s=0}^{[\frac{k-2}{2}]} \frac{f_n^{(k-2-2s)}(0)}{\lambda_n^{k-2s}} \\ & + \frac{1}{2\lambda_n}\int_0^q e^{-\lambda_n \eta} f_n(\eta)\,d\eta\Biggr] + \frac{(-1)^k e^{\lambda_n y} y - e^{-\lambda_n y}}{(-1)^k 2 \operatorname{sh}\lambda_n q}\Biggl[\frac{\psi_n}{\lambda_n^k} \\ & - \sum_{s=0}^{[\frac{k-2}{2}]} \frac{f_n^{(k-2-2s)}(q)}{\lambda_n^{k-2s}} + \frac{(-1)^k}{2\lambda_n}\int_0^q e^{-\lambda_n(q-\eta)} f_n(\eta)\,d\eta\Biggr] \\ & - \frac{1}{2\lambda_n}\int_0^y e^{-\lambda_n(y-\eta)} f_n(\eta)\,d\eta - \frac{1}{2\lambda_n}\int_y^q e^{-\lambda_n(\eta-y)} f_n(\eta)\,d\eta\Biggr\}. \end{aligned} \tag{3.8}$$

Now, let us consider the derivatives

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty} X_n(x)\Biggl\{ & \frac{e^{\lambda_n(q-y)} - (-1)^k e^{-\lambda_n(q-y)}}{(-1)^k 2 \operatorname{sh}\lambda_n q}\Biggl[\frac{\varphi_n}{\lambda_n^{k-2}} \\ & - \sum_{s=0}^{[\frac{k-2}{2}]} \frac{f_n^{(k-2-2s)}(0)}{\lambda_n^{k-2-2s}} + \frac{1}{2}\sum_{s=0}^{1} \frac{1}{\lambda_n^s}\bigl(f_n^{(s)}(0) - f_n^{(s)}(q)e^{-\lambda_n q}\bigr) \\ & + \frac{1}{2\lambda_n}\int_0^q f_n''(\eta)e^{-\lambda_n \eta}\,d\eta\Biggr] + \frac{(-1)^k e^{\lambda_n y} - e^{-\lambda_n y}}{(-1)^k 2 \operatorname{sh}\lambda_n q}\Biggl[\frac{\psi_n}{\lambda_n^{k-2}} \\ & - \sum_{s=0}^{[\frac{k-2}{2}]} \frac{f_n^{(k-2-2s)}(q)}{\lambda_n^{k-2-2s}} + \frac{(-1)^k}{2}\sum_{s=0}^{1} \frac{(-1)^s}{\lambda_n^s}\bigl[f_n^{(s)}(q) - f_n^{(s)}(0)e^{-\lambda_n q}\bigr] \end{aligned}$$

$$
\begin{aligned}
&+\frac{(-1)^k}{2\lambda_n}\int_0^q f_n''(\eta)e^{-\lambda_n(q-y)}\,d\eta\Bigg]-\frac{1}{2}\sum_{s=0}^{1}\frac{(-1)^s}{\lambda_n^s}\big[f_n^{(s)}(y)=f_n^{(s)}(0)e^{-\lambda_n y}\big]\\
&-\frac{1}{2\lambda_n}\int_0^y f_n''(\eta)e^{-\lambda_n(y-\eta)}\,d\eta-\frac{1}{2}\sum_{s=0}^{1}\frac{1}{\lambda_n^s}\big[f_n^{(s)}(y)-f_n^{(s)}(q)e^{-\lambda_n(q-y)}\big]\\
&-\frac{1}{2\lambda_n}\int_y^q f_n''(\eta)e^{-\lambda_n(\eta-y)}\,d\eta\Bigg\},
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2}={}&\sum_{n=1}^{\infty}X_n(x)\Bigg[\frac{e^{\lambda_n(q-y)}-(-1)^k e^{-\lambda_n(q-y)}}{(-1)^k 2\operatorname{sh}\lambda_n q}\Bigg[\frac{\varphi_n}{\lambda_{n^{k-2}}}-\sum_{s=0}^{[\frac{k=2}{2}]}\frac{f_n^{(k=2-2s)}(0)}{\lambda_n^{k-2-2s}}\\
&+\frac{1}{2}\sum_{s=0}^{1}\frac{1}{\lambda_n^s}\big(f_n^{(s)}(0)-f_n^{(s)}(q)e^{-\lambda_n q}\big)+\frac{1}{2\lambda_n}\int_0^q f_n''(\eta)e^{-\lambda_n\eta}\,d\eta\Bigg]\\
&+\frac{(-1)^k e^{\lambda_n y}-e^{-\lambda_n y}}{(-1)^k 2\operatorname{sh}\lambda_n q}\Bigg[\frac{\psi_n}{\lambda_n^{k-2}}-\sum_{s=0}^{[\frac{k-2}{2}]}\frac{f_n^{(k-2-2s)}(q)}{\lambda_n^{k-2-2s}}+\frac{(-1)^k}{2}\sum_{s=0}^{1}\frac{(-1)^s}{\lambda_n^s}\big(f_n^{(s)}(q)\\
&-f_n^{(s)}(0)e^{-\lambda_n q}\big)+\frac{(-1)^k}{2\lambda_n}\int_0^q f_n''(\eta)e^{-\lambda_n(q-\eta)}\,d\eta\Bigg]+f_n(y)-\frac{1}{2}\sum_{s=0}^{1}\frac{(-1)^s}{\lambda_n^s}\big[f_n^{(s)}(y)\\
&-f_n^{(s)}(0)e^{-\lambda_n y}\big]-\frac{1}{2\lambda_n}\int_0^y f_n''(\eta)e^{-\lambda_n(y-\eta)}\,d\eta-\frac{1}{2}\sum_{s=0}^{1}\frac{1}{\lambda_n^s}\big[f_n^{(s)}(y)\\
&-f_n^{(s)}(q)e^{-\lambda_n(q-y)}\big]-\frac{1}{2\lambda_n}\int_y^q f_n''(\eta)e^{-\lambda_n(\eta-y)}\,d\eta\Bigg],
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
\frac{\partial^k u}{\partial y^k}={}&\sum_{n=1}^{\infty}X_n(x)\Bigg\{\frac{\operatorname{sh}\lambda_n(q-y)}{\operatorname{sh}\lambda_n q}\Bigg[\varphi_n-\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{2s}f_n^{(k-2-2s)}(0)+\frac{1}{2}\sum_{s=0}^{k-1}\lambda_n^{k-2-s}\big[f_n^{(s)}(0)\\
&-f_n^{(s)}(q)e^{-\lambda_n q}\big]+\frac{1}{2\lambda_n}\int_0^q f_n^{(k)}(\eta)e^{-\lambda_n\eta}\,d\eta\Bigg]+\frac{\operatorname{sh}\lambda_n y}{\operatorname{sh}\lambda_n q}\Bigg[\psi_n-\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{2s}f_n^{(k-2-2s)}(q)\\
&+\frac{1}{2}\sum_{s=0}^{k-1}\frac{(-1)^{k+s}}{\lambda_n^{s+2-k}}\big(f_n^{(s)}(q)-f_n^{(s)}(0)e^{-\lambda_n q}\big)+\frac{1}{2\lambda_n}\int_0^q f_n^{(k)}(\eta)e^{-\lambda_n(q-\eta)}\,d\eta\Bigg]\\
&+\sum_{s=0}^{[\frac{k-2}{2}]}\lambda_n^{2s}f_n^{(k-2-2s)}(y)-\frac{(-1)^k}{2}\sum_{s=0}^{k-1}(-1)^s\lambda_n^{k-2-2s}\big[f_n^{(s)}(y)-f_n^{(s)}(0)e^{-\lambda_n y}\big]\\
&-\frac{1}{2\lambda_n}\int_0^y f_n^{(k)}(\eta)e^{-\lambda_n(y-\eta)}\,d\eta-\frac{1}{2}\sum_{s=0}^{k-1}\lambda_n^{k-2-s}\big[f_n^{(s)}(y)-f_n^{(s)}(q)e^{-\lambda_n(q-y)}\big]\\
&-\frac{1}{2\lambda_n}\int_y^q f_n^{(k)}(\eta)e^{-\lambda_n(\eta-y)}\,d\eta\Bigg\}.
\end{aligned}
\tag{3.11}
$$

Denote by $C_{x,y}^{1,0}(\overline{\Omega})$ the class of the functions $u(x,y)$ such that $u(x,y), u_x(x,y)\in C(\overline{\Omega})$. We have the following lemmas.

Lemma 3.1 *If*

$$f(x,y) \in C^{1,0}_{x,y}(\overline{\Omega}), \qquad \frac{\partial f}{\partial x} \in \mathrm{Lip}_\alpha[0,p], \quad 0 < \alpha < 1,$$

uniformly with respect to y, then the series in (3.2) *absolutely and uniformly converges in* $\overline{\Omega}$.

Proof Applying integration by parts to (3.5), we obtain

$$f_n(y) = \frac{1}{\lambda_n} f_n^{(1)}(y), \tag{3.12}$$

where

$$f_n^{(1)}(y) = \int_0^p \frac{\partial f}{\partial x}(x,y) \sqrt{\frac{2}{p}} \cos \lambda_n x \, dx.$$

According to [20], $|f_n^{(1)}(y)| \le \frac{c}{n^\alpha}$, where $c > 0$ is a constant. Then, $|f_n(y)| < \frac{cp}{\pi} \frac{1}{n^{1+\alpha}}$, and the series $\sum_{n=1}^\infty \frac{1}{n^{1+\alpha}}$ is convergent. Consequently, the series in (3.2) is absolutely and uniformly convergent in $\overline{\Omega}$.

Lemma 3.1 is proved. □

Lemma 3.2 *If*

$$\varphi(x) \in W_2^1(0,p), \qquad \varphi(0) = \varphi(p) = 0$$

and

$$\psi \in W_2^1(0,p), \qquad \psi(0) = \psi(p) = 0,$$

then the series (3.3) *and* (3.4) *are absolutely and uniformly convergent in* $[0,p]$.

Proof Integrating the integral in (3.6) by parts, we have

$$\varphi_n = \frac{1}{\lambda_n} \varphi_n^{(1)}, \tag{3.13}$$

where

$$\varphi_n^{(1)} = \int_0^p \varphi'(x) \sqrt{\frac{2}{p}} \cos \lambda_n x \, dx.$$

Taking into account equality (3.13) and applying the Hölder inequality to the sum (see, *e.g.*, [21]) of the series

$$\frac{p}{\pi} \sum_{n=1}^\infty \frac{1}{n} |\varphi_n^{(1)}|,$$

we get

$$\frac{p}{\pi} \sum_{n=1}^\infty \frac{1}{n} |\varphi_n^{(1)}| \le \frac{p}{\pi} \left(\sum_{n=1}^\infty \frac{1}{n^2} \right)^{\frac{1}{2}} \left(\sum_{n=1}^\infty |\varphi_n^{(1)}|^2 \right)^{\frac{1}{2}}.$$

Using Bessel's inequality (see, *e.g.*, [21]), we find

$$\left(\sum_{n=1}^{\infty}\left|\varphi_n^{(1)}\right|^2\right)^{\frac{1}{2}} \le \left\|\varphi'\right\|_{L_2(0,p)}.$$

Furthermore,

$$\frac{p}{\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}} = \frac{p}{\pi}\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}} = \frac{p}{\sqrt{6}}.$$

Hence, we have

$$\frac{p}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left|\varphi_n^{(1)}\right| \le \frac{p}{\sqrt{6}}\left\|\varphi'(x)\right\|_{L_2(0,p)}.$$

Consequently, the series (3.3) is absolutely and uniformly convergent in $[0,p]$. The proof of absolute and uniform convergence of the series (3.4) is analogous.

Lemma 3.2 is proved. □

Lemma 3.3 *If*

$$\varphi(x) \in W_2^3(0,p), \qquad \varphi(0) = \varphi(p) = 0, \qquad \varphi''(0) = \varphi''(p) = 0$$

and

$$\psi(x) \in W_2^3(0,p), \qquad \psi(0) = \psi(p) = 0, \qquad \psi''(0) = \psi''(p) = 0,$$

then, for any $k \ge 0$, *the series*

$$\sum_{n=1}^{\infty}\frac{e^{\lambda_n(q-y)} - (-1)^k e^{-\lambda_n(q-y)}}{(-1)^k 2\operatorname{sh}\lambda_n q}\frac{\varphi_n}{\lambda_n^{k-2}} \tag{3.14}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^k e^{\lambda_n y} - e^{-\lambda_n y}}{(-1)^k 2\operatorname{sh}\lambda_n q}\frac{\psi_n}{\lambda_n^{k-2}} \tag{3.15}$$

are absolutely and uniformly convergent in $\overline{\Omega}$.

Proof We show the inequalities

$$0 \le \frac{e^{\lambda_n(q-y)} - (-1)^k e^{-\lambda_n(q-y)}}{2\operatorname{sh}\lambda_n q} \le C_0, \tag{3.16}$$

$$0 \le \left|\frac{(-1)^k e^{\lambda_n y} - e^{-\lambda_n y}}{(-1)^k 2\operatorname{sh}\lambda_n q}\right| \le C_0, \tag{3.17}$$

where $C_0 = \frac{2}{1-e^{-2\frac{\pi q}{p}}}$.

Indeed,

$$\frac{e^{\lambda_n(q-y)} - (-1)^k e^{-\lambda_n(q-y)}}{e^{\lambda_n q} - e^{-\lambda_n q}} = \frac{e^{\lambda_n(q-y)}}{e^{\lambda_n q}} \frac{1 - (-1)^k e^{-2\lambda_n(q-y)}}{1 - e^{-2\lambda_n q}}.$$

Since

$$\frac{e^{\lambda_n(q-y)}}{e^{\lambda_n q}} \le 1, \qquad 1 - (-1)^k e^{-2\lambda_n(q-y)} \le 2, \qquad 1 - e^{-2\lambda_n q} > 1 - e^{-2\frac{\pi q}{p}},$$

we have

$$\frac{e^{\lambda_n(q-y)} - (-1)^k e^{-\lambda_n(q-y)}}{2 \operatorname{sh} \lambda_n q} \le \frac{2}{1 - e^{-2\frac{\pi q}{p}}} = C_0.$$

Inequality (3.16) is proved.

Similarly, one can verify inequality (3.17). In the case $k = 2$, the series (3.14) and (3.15) are absolutely and uniformly convergent in $\overline{\Omega}$ according to Lemma 3.2. When $k > 2$, the series (3.14) and (3.15) evidently are absolutely and uniformly convergent in $\overline{\Omega}$. Let $k = 0$. We consider the absolute value of the series in (3.14)

$$\sum_{n=1}^{\infty} \frac{e^{\lambda_n(q-y)} - (-1)^k e^{-\lambda_n(q-y)}}{2 \operatorname{sh} \lambda_n q} \lambda_n^2 |\varphi_n|.$$

In order to prove the convergence of the last series, we apply integration by parts in (3.6). We have

$$\varphi_n = -\frac{1}{\lambda_n^3} \varphi_n^{(3)}, \tag{3.18}$$

where

$$\varphi_n^{(3)} = \int_0^p \varphi'''(x) \sqrt{\frac{2}{p}} \cos \lambda_n x \, dx.$$

Using (3.16) and (3.18) in the last series, we get

$$C_0 \sum_{n=1}^{\infty} \lambda_n^2 \frac{1}{\lambda_n^3} \left|\varphi_n^{(3)}\right| = C_0 \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left|\varphi_n^{(3)}\right|.$$

Applying Hölder's inequality to the sum to the last series, we obtain

$$\begin{aligned} C_0 \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left|\varphi_n^{(3)}\right| &\le C_0 \frac{p}{\pi} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \left|\varphi_n^{(3)}\right|^2 \right)^{\frac{1}{2}} \\ &= C_0 \frac{p}{\pi} \frac{\pi}{\sqrt{6}} \left(\sum_{n=1}^{\infty} \left|\varphi_n^{(3)}\right|^2 \right)^{\frac{1}{2}} \\ &= \frac{C_0 p}{\sqrt{6}} \left(\sum_{n=1}^{\infty} \left|\varphi_n^{(3)}\right|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Using Bessel's inequality, we have

$$\left(\sum_{n=1}^{\infty}\left|\varphi_n^{(3)}\right|^2\right)^{\frac{1}{2}} \leq \left\|\varphi'''\right\|_{L_2(0,p)}.$$

Further, we have

$$C_0\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|\varphi_n^{(3)}\right| \leq \frac{C_0 p}{\sqrt{6}}\left\|\varphi'''\right\|_{L_2(0,p)}.$$

Consequently, the series (3.14) converges. Analogously, the proof of convergence of the series (3.15) can be obtained, and, thus, we do not give it here.

Lemma 3.3 is proved. □

Lemma 3.4 *If* $\frac{\partial^{k-2}f(x,y)}{\partial y^{k-2}} \in C(\overline{\Omega})$, *then the series*

$$\sum_{n=1}^{\infty}X_n(x)\frac{e^{\lambda_n(q-y)}-(-1)^k e^{-\lambda_n(q-y)}}{(-1)^k 2\operatorname{sh}\lambda_n q}\sum_{s=0}^{[\frac{k-2}{2}]}\frac{f_n^{(k-2-2s)}(0)}{\lambda_n^{k-2s}} \tag{3.19}$$

and

$$\sum_{n=1}^{\infty}X_n(x)\frac{(-1)^k e^{\lambda_n y}-e^{-\lambda_n y}}{(-1)^k 2\operatorname{sh}\lambda_n q}\sum_{s=0}^{[\frac{k-2}{2}]}\frac{f_n^{(k-2-2s)}(q)}{\lambda_n^{k-2s}} \tag{3.20}$$

are absolutely and uniformly convergent in $\overline{\Omega}$.

Proof By the condition of the lemma we have

$$\left|f_n^{(k-2-2s)}(a)\right| \leq C_1, \tag{3.21}$$

where $a = 0$ or $a = q$, and $C_1 > 0$ is constant. Taking into account (3.16), (3.17), and the last inequality, we conclude that the series

$$C_0 C_1\sqrt{\frac{2}{p}}\sum_{n=1}^{\infty}\sum_{s=0}^{[\frac{k-2}{2}]}\frac{1}{\lambda_n^{k-2s}} \tag{3.22}$$

is a majorant for the series (3.19) and (3.20). If $s = 0$ and $k \geq 2$, then the series (3.22) converges. At $s = [\frac{k-2}{2}]$, we have

$$k-2s = k-2\left[\frac{k-2}{2}\right] = \begin{cases} 3 & \text{if } k \text{ is odd}, \\ 2 & \text{if } k \text{ is even}. \end{cases}$$

Consequently, at $s = [\frac{k-2}{2}]$, if k is an odd number, then the series (3.22) has the form $\sum_{n=1}^{\infty}\frac{1}{\lambda_n^3}$; if k is an even number, then the series (3.22) has the form $\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}$. In both cases, the series (3.22) is convergent. Therefore, the series (3.19) and (3.20) are absolutely and uniformly convergent in $\overline{\Omega}$.

Lemma 3.4 is proved. □

Lemma 3.5 *If*

$$\frac{\partial^{m-1}f(x,y)}{\partial y^{m-1}} \in C(\overline{\Omega}), \qquad \frac{\partial^{m}f(x,y)}{\partial y^{m}} \in L_2(0,q),$$

then we have the estimate

$$\frac{1}{2\lambda_n}\left|\int_0^q e^{-\lambda_n(q-\eta)}f_n^{(m)}(\eta)\,d\eta\right| \le \frac{C}{n^{\frac{3}{2}}}\left\|f_n^{(m)}\right\|_{L_2(0,q)}, \tag{3.23}$$

where $C = (\frac{p}{2\pi})^{\frac{3}{2}}$, *and* $m = 2$ *or* $m = k$.

Proof Applying the Hölder inequality (see, *e.g.*,[21]) to the integral on the left-hand side of inequality (3.23), we obtain

$$\begin{aligned}\frac{1}{2\lambda_n}\left|\int_0^q e^{-\lambda_n(q-\eta)}f_n^{(m)}(\eta)\,d\eta\right| &\le \frac{1}{2\lambda_n}\left(\int_0^q e^{-2\lambda_n(q-\eta)}\,d\eta\right)^{\frac{1}{2}}\left(\int_0^q \left(f_n^{(m)}(\eta)\right)^2 d\eta\right)^{\frac{1}{2}} \\ &= \left[\frac{1}{2\lambda_n}\left(1-e^{-2\lambda_n q}\right)\right]^{\frac{1}{2}}\frac{1}{2\lambda_n}\left\|f_n^{(m)}\right\|_{L_2(0,q)} \\ &\le \frac{1}{(2\lambda_n)^{\frac{3}{2}}}\left\|f_n^{(m)}\right\|_{L_2(0,q)} = \frac{1}{(\frac{2n\pi}{p})^{\frac{3}{2}}}\left\|f_n^{(m)}\right\|_{L_2(0,q)} \\ &= \frac{C}{n^{\frac{3}{2}}}\left\|f_n^{(m)}\right\|_{L_2(0,q)}.\end{aligned}$$

Lemma 3.5 is proved. □

Lemma 3.6 *If* $\frac{\partial^{k-2}f(x,y)}{\partial y^{k-2}} \in C(\overline{\Omega})$, *then the series*

$$\sum_{n=1}^{\infty} X_n(x)\frac{e^{\lambda_n(q-y)}-(-1)^k e^{-\lambda_n(q-y)}}{(-1)^k 2\operatorname{sh}\lambda_n q}\sum_{s=0}^{[\frac{k-2}{2}]}\frac{f_n^{(k-2-2s)}(0)}{\lambda_n^{k-2-2s}} \tag{3.24}$$

and

$$\sum_{n=1}^{\infty} X_n(x)\frac{(-1)^k e^{\lambda_n y}-e^{-\lambda_n y}}{(-1)^k 2\operatorname{sh}\lambda_n q}\sum_{s=0}^{[\frac{k-2}{2}]}\frac{f_n^{(k-2-2s)}(q)}{\lambda_n^{k-2-2s}} \tag{3.25}$$

absolutely and uniformly converge in $\overline{\Omega}$.

Proof Taking into account (3.16) and (3.17), we conclude that the series

$$C_0\sqrt{\frac{2}{p}}\sum_{n=1}^{\infty}\sum_{s=0}^{[\frac{k-2}{2}]}\frac{|f_n^{(k-2-2s)}(a)|}{\lambda_n^{k-2-2s}}, \tag{3.26}$$

where $a = 0$ or $a = q$, is majorant for the series (3.24) and (3.25). If $s = [\frac{k-2}{2}]$, then we have

$$k-2-2s = k-2-2\left[\frac{k-2}{2}\right] = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.}\end{cases}$$

Therefore, in the case where k is an odd number, the series (3.26) has the form

$$\sum_{n=1}^{\infty} \frac{|f_n'(a)|}{\lambda_n}, \tag{3.27}$$

and if k is an even number, then the series (3.26) has the form

$$\sum_{n=1}^{\infty} \bigl|f_n(a)\bigr|. \tag{3.28}$$

Applying the Hölder inequality to the series (3.27), we get

$$\sum_{n=1}^{\infty} \frac{|f_n'(a)|}{\lambda_n} \le \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \bigl|f_n'(a)\bigr|^2\right)^{\frac{1}{2}}.$$

Using Bessel's inequality, we find

$$\left(\sum_{n=1}^{\infty} \bigl|f_n'(a)\bigr|^2\right)^{\frac{1}{2}} \le \left\| \frac{\partial f(x,a)}{\partial y} \right\|_{L_2(0,q)}.$$

Since $(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2})^{\frac{1}{2}} = \frac{p}{\sqrt{6}}$, the series (3.27) converges. In order to proof the convergence of the series (3.28), we integrate by parts the integral

$$f_n(a) = \int_0^p f(x,a) X_n(x)\, dx,$$

and we obtain

$$f_n(a) = \frac{1}{\lambda_n} f_n^{(1)}(a), \tag{3.29}$$

where

$$f_n^{(1)}(a) = \int_0^p \frac{\partial f(x,a)}{\partial x} \sqrt{\frac{2}{p}} \cos \lambda_n x\, dx.$$

Substituting (3.29) into the series (3.28), we find

$$\sum_{n=1}^{\infty} \bigl|f_n(a)\bigr| = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \bigl|f_n^{(1)}(a)\bigr|.$$

Further, as in the case of (3.27), we have

$$\sum_{n=1}^{\infty} \bigl|f_n(a)\bigr| \le \frac{p}{\sqrt{6}} \left\| \frac{\partial f(x,a)}{\partial x} \right\|_{L_2(0,p)}.$$

If $s \le [\frac{k-2}{2}]$, then the series (3.26) converges for both odd and even k. Consequently, the series (3.24) and (3.25) are absolutely and uniformly convergent in $\overline{\Omega}$.

Lemma 3.6 is proved. □

Lemma 3.7 *If*

$$\frac{\partial^{k-1} f(x,y)}{\partial x^i \, \partial y^j} \in C(\overline{\Omega}), \quad i,j = 0,1,\ldots,k-1, i+j = k-1, \qquad \frac{\partial^{2l} f}{\partial x^{2l}}(0,y) = \frac{\partial^{2l} f}{\partial x^{2l}}(p,y) = 0,$$

where

$$l = 0,1,\ldots, \begin{cases} \frac{k-2}{2} & \textit{if } k \textit{ is even}, \\ \frac{k-3}{2} & \textit{if } k \textit{ is odd}, \end{cases} \tag{3.30}$$

then the series

$$\sum_{n=1}^{\infty} \sum_{s=0}^{[\frac{k-2}{2}]} \lambda_n^{2s} \left| f_n^{(k-2-2s)}(a) \right| = \sum_{n=1}^{\infty} \Bigg[\left| f_n^{(k-2)}(a) \right| + \lambda_n^2 \left| f_n^{(k-4)}(a) \right| + \cdots + \begin{cases} \lambda_n^{k-2} |f_n(a)| & \textit{if } k \textit{ is even}, \\ \lambda_n^{k-3} |f_n'(a)| & \textit{if } k \textit{ is odd} \end{cases} \Bigg] \tag{3.31}$$

converges, where $a = 0$ or $a = q$.

Proof Let k be an even number. If $s = [\frac{k-2}{2}]$, then $2s = 2[\frac{k-2}{2}] = k-2$. If the series

$$\sum_{n=1}^{\infty} \lambda_n^{k-2} \left| f_n(a) \right| \tag{3.32}$$

converges, then the series (3.31) is also convergent, where

$$f_n(a) = \int_0^p f(x,a) \sqrt{\frac{2}{p}} \sin \lambda_n x \, dx.$$

Integrating the last integral by parts $k-1$ times, we have

$$f_n(a) = \frac{1}{\lambda_n^{k-1}} f_n^{(k-1,0)}(a), \tag{3.33}$$

where

$$f_n^{(k-1,0)}(a) = \int_0^p \frac{\partial^{k-1} f(x,a)}{\partial x^{k-1}} \sqrt{\frac{2}{p}} \sin \left[(k-1)\frac{\pi}{2} + x \right] dx.$$

Taking into account (3.33), the series (3.32) gives

$$\sum_{n=1}^{\infty} \lambda_n^{k-2} \left| f_n(a) \right| \le \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left| f_n^{(k-1,0)}(a) \right|.$$

Applying the Hölder inequality to the right-hand side of the last inequality, we have

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left| f_n^{(k-1,0)}(a) \right| \le \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \left| f_n^{(k-1,0)}(a) \right|^2 \right)^{\frac{1}{2}} = \frac{p}{\sqrt{6}} \left(\sum_{n=1}^{\infty} \left| f_n^{(k-1,0)}(a) \right|^2 \right)^{\frac{1}{2}}.$$

Using Bessel's inequality, we find

$$\left(\sum_{n=1}^{\infty}\left|f_n^{(k-1,0)}(a)\right|^2\right)^{\frac{1}{2}} \le \left\|\frac{\partial^{k-1}f(x,a)}{\partial x^{k-1}}\right\|_{L_2(0,p)}.$$

Hence, we obtain

$$\sum_{n=1}^{\infty}\lambda_n^{k-2}\left|f_n(a)\right| \le \frac{p}{\sqrt{6}}\left\|\frac{\partial^{k-1}f(x,a)}{\partial x^{k-1}}\right\|_{L_2(0,p)}.$$

The convergence of the series (3.32) is proved.

Let k be an odd number and $s = [\frac{k-2}{2}]$. In this case, $2s = 2[\frac{k-2}{2}] = k-3$. If the series

$$\sum_{n=1}^{\infty}\lambda_n^{k-3}\left|f_n'(a)\right| \tag{3.34}$$

is convergent, then the series (3.31) is also convergent for odd k. The proof of this assertion is analogous to the proof of convergence of the series (3.32).

Lemma 3.7 is proved. □

Lemma 3.8 *Let the conditions of Lemma* 3.7 *be satisfied. Then the series*

$$\sum_{n=1}^{\infty}\sum_{s=0}^{k-1}\lambda_n^{k-2-s}\left|f_n^{(s)}(a)\right|, \quad a = 0, q,$$

is convergent, and the series

$$\sum_{n=1}^{\infty}\sum_{s=0}^{k-1}\lambda_n^{k-2-s}\left|f_n^{(s)}(y) - f_n^{(0)}e^{-\lambda_n y}\right|$$

and

$$\sum_{n=1}^{\infty}\sum_{s=0}^{k-1}\lambda_n^{k-2-s}\left|f_n^{(s)}(y) - f_n^{(s)}(q)e^{-\lambda_n(q-y)}\right|$$

are absolutely and uniformly convergent.

Proof of this lemma is analogous to that of Lemma 3.7. Using the results of the presented lemmas, we get the following theorem.

Theorem 3.1 *Let the following conditions be satisfied*:

(i) $\varphi(x) \in W_2^3(0,p), \qquad \varphi(0) = \varphi(p) = 0, \qquad \varphi''(0) = \varphi''(p) = 0;$

(ii) $\psi \in W_2^3(0,p), \qquad \psi(0) = \psi(p) = 0, \qquad \psi''(0) = \psi''(p) = 0;$

(iii) $\dfrac{\partial^{k-1}f(x,y)}{\partial x^i\,\partial y^j} \in C(\overline{\Omega}), \quad i+j = k-1, i,j = 0,1,\dots,k-1,$

$\dfrac{\partial^k f(x,y)}{\partial y^k} \in L_2(\Omega), \qquad \dfrac{\partial^{2l}f}{\partial x^{2l}}(0,y) = \dfrac{\partial^{2l}f}{\partial x^{2l}}(p,y) = 0,$

where

$$l = 0, 1, \ldots, \begin{cases} \frac{k-2}{2} & \text{if } k \text{ is even,} \\ \frac{k-3}{2} & \text{if } k \text{ is odd.} \end{cases}$$

Then the series (3.8)-(3.11) *absolutely and uniformly converge in* $\overline{\Omega}$, *and solution* (3.8) *satisfies equation* (1.1) *in* Ω *and conditions* (1.2)-(1.5), *where* $u(x,y) \in C^2(\overline{\Omega})$, $\frac{\partial^k u}{\partial y^k} \in C(\overline{\Omega})$.

Proof Adding (3.9) and (3.10), we find that solution (3.8) satisfies equation (1.1) in Ω. From the properties of the functions $X_n(x)$ it follows that solution (3.8) satisfies conditions (1.2) and (1.3). The absolute and uniform convergence of the series (3.8) in $\overline{\Omega}$ follows from Lemma 3.3 with $m = k$ and from Lemmas 3.4 and 3.5. Therefore, $u(x,y) \in C(\overline{\Omega})$. The absolute and uniform convergence of the series (3.9) and (3.10) in $\overline{\Omega}$ follows from Lemmas 3.1-3.5. Hence, we have $\frac{\partial^2 u}{\partial x^2} \in C(\overline{\Omega})$, $\frac{\partial^2 u}{\partial y^2} \in C(\overline{\Omega})$. The absolute and uniform convergence of the series (3.11) in $\overline{\Omega}$ follows from the Lemmas 3.2, 3.5, 3.7, and 3.8. Therefore, $\frac{\partial^k u}{\partial y^k} \in C(\overline{\Omega})$. Consequently, $u(x,y) \in C^2(\overline{\Omega})$, $\frac{\partial^k u}{\partial y^k} \in C(\overline{\Omega})$. Taking the limit in (3.11) as $y \to 0$ and $y \to q$, we conclude that solution (3.8) satisfies conditions (1.4) and (1.5).

This proves the theorem. □

Competing interests

The author declares that there are no competing interests regarding the publication of this paper.

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