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Blow-up phenomena for a nonlinear parabolic problem with p-Laplacian operator under nonlinear boundary condition

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Abstract
In this paper, we study the blow-up phenomena for a positive solution of a nonlinear parabolic problem with p-Laplacian operator under a nonlinear boundary condition. The sufficient conditions which ensure that the blow-up does occur at finite time are presented by constructing some appropriate auxiliary functions and using first-order differential inequality technique. Moreover, a lower bound and an upper bound for the blow-up time are derived when blow-up happens.

MSC: 35B40; 35K35

Keywords: nonlinear parabolic equations; blow-up; p-Laplacian operator; Robin boundary condition

1 Introduction

The mathematical investigation of the blow-up phenomena of a solution to nonlinear parabolic equations and systems has received a great deal of attention during the last few decades [1–6]. The authors in [7, 8] considered an initial-boundary value problem for parabolic equations of the form

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u^p - |\nabla u|^q & \text{in } \mathcal{O} \times (0,\infty), \\ u = 0 & \text{on } \partial\mathcal{O} \times (0,\infty), \\ u(x,0) = h(x) \geq 0 & \text{in } \mathcal{O}. \end{cases} \tag{1}$$

Here $\mathcal{O}$ is a bounded domain in $\mathbb{R}^3$, $\triangle$ is the Laplace operator, ∇ is the gradient operator, $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$. They proved that problem (1) blows up at finite time T^* if $1 < p \leq 5$ and $1 < q < \frac{2p}{p+1}$. Soon et al. in [1] gave a lower bound for the blow-up time T^* under the above condition. Shortly afterwards, the relative result in [1] was extended to the case with nonlinear boundary condition by Liu [9]. Further, Enache in [10] considered a more complicated case, in which he investigated the following class of quasilinear initial-boundary value problems:

$$\begin{cases} u_t = \operatorname{div}(b(u)\nabla u) + f(u) & \text{in } \mathcal{O} \times (0,\infty), \\ \frac{\partial u}{\partial n} + \kappa u = 0 & \text{on } \partial\mathcal{O} \times (0,\infty), \\ u(x,0) = h(x) \geq 0 & \text{in } \mathcal{O}. \end{cases} \tag{2}$$

Here n is the unit outer normal vector of $\partial\mathcal{O}$, and $\frac{\partial u}{\partial n}$ is outward normal derivative of u on the boundary $\partial\mathcal{O}$ which is assumed to be sufficiently smooth. Under the suitable assumptions on the functions b, f, and h, the author established a sufficient condition to guarantee the occurrence of the blow-up. Moreover, a lower bound for the blow-up time was obtained.

However, there are few papers on blow-up phenomena of the problem with a p-Laplacian operator except [11], in which Zhou considered the following:

$$\begin{cases} u_t = \operatorname{div}(u|\nabla u|^{p-2}\nabla u) + (\gamma+1)|\nabla u|^p & \text{in } \mathcal{O}\times(0,\infty), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\mathcal{O}\times(0,\infty), \\ u(x,0) = h(x) \geq 0 & \text{in } \mathcal{O}. \end{cases} \tag{3}$$

He proved that problem (3) blows up at finite time T^* when $0 < \gamma < 1$. But he did not give any bounds to the scale T^*.

In this text, we consider the more complicated case than the ones in (1)-(3),

$$\big(a(u)\big)_t = \operatorname{div}\big(b(u)|\nabla u|^{p-2}\nabla u\big) + \gamma b'(u)|\nabla u|^p + f(u) \tag{4}$$

with the following nonlinear boundary condition:

$$\frac{\partial u}{\partial n} + g(u) = 0 \tag{5}$$

and the initial condition

$$u(x,0) = h(x) \geq 0. \tag{6}$$

In the process of deriving the lower bound, we make the following assumptions:

(A1) The parameters of problem (4) satisfy $0 \leq \gamma \leq 2$, $p > 2$.

(A2) The function $g(s)$ satisfies

$$g(s) = \sum_{i=1}^{n} \kappa_i s^{\sigma_i},$$

where κ_is and σ_is are nonnegative constants.

Since the initial data $h(x)$ in (6) is nonnegative, it is easy to see that the solution u to problem (4)-(6) is nonnegative in $\mathcal{O}\times(0,\infty)$ by the parabolic maximum principles [12, 13]. In Section 2, we plan to present the sufficient conditions which guarantee the occurrence of the blow-up. In Section 3, we will find a lower bound for the blow-up time when blow-up occurs.

2 The blow-up solution

In this section we mainly seek the sufficient conditions for the blow-up. To this end, we define some auxiliary functions of the form

$$G(s) = 2\int_0^s y b(y)^{(p-1)p-1} a'(y)\,\mathrm{d}y,$$

$$\begin{aligned}
&A(t) = \int_O G\big(u(x,t)\big)\,\mathrm{d}x,\\
&H_i(s) = \int_0^s y^{p\sigma_i-\sigma_i} b(y)^{p(p-1)}\,\mathrm{d}y, \quad i = 1,2,\ldots,n,\\
&\sigma = \max\{\sigma_i, i = 1,2,\ldots,n\}, \qquad F(s) = \int_0^s f(s)b(s)^{(p-1)p-1}\,\mathrm{d}s,\\
&B(t) = \int_{\mathcal{O}} F(u)\,\mathrm{d}x - \frac{1}{p}\int_{\mathcal{O}} b(u)^{(p-1)p}\big[(\nabla u)^2\big]^{\frac{p}{2}}\,\mathrm{d}x - \sum_{i=1}^n \kappa_i^{p-1}\int_{\partial\mathcal{O}} H_i(u)\,\mathrm{d}x,
\end{aligned} \tag{7}$$

where $u(x,t)$ is the solution of problem (3).

The main result of this section is formulated in the following theorem.

Theorem 2.1 *Let $u(x,t)$ be the solution of problem* (4)-(6). *Assume that*

$$sf(s)b(s)^{(p-1)p-1} \geq p(1+\alpha)F(s), \quad s > 0, \tag{8}$$

$$\lim_{y\to\infty} y^{\sigma p-\sigma+1} b(y)^{p(p-1)} = 0 \quad \text{and} \quad B(0) \geq 0, \tag{9}$$

where α is a positive constant. Then $u(x,t)$ blows up as some finite time T^* such that

$$T^* \leq M^{-1}A(0)^{1-\frac{1}{2}(p\sigma-\sigma+1)(1+\alpha)},$$

where M is a positive constant to be determined later.

Proof We first compute

$$\begin{aligned}
A'(t) &= \int_{\mathcal{O}} G'\big(u(x,t)\big)u_t\,\mathrm{d}x\\
&= 2\int_{\mathcal{O}} ub(u)^{(p-1)p-1}\big[\operatorname{div}\big(b(u)|\nabla u|^{p-2}\nabla u\big) + \gamma b'(u)|\nabla u|^p + f(u)\big]\,\mathrm{d}x\\
&= 2\int_{\mathcal{O}} uf(u)b(u)^{(p-1)p-1}\,\mathrm{d}x\\
&\quad + \big[\gamma - 2\big((p-1)p-1\big)\big]\int_{\mathcal{O}} ub(u)^{(p-1)p-1}b'(u)\big[(\nabla u)^2\big]^{\frac{p}{2}}\,\mathrm{d}x\\
&\quad - 2\int_{\mathcal{O}} b(u)^{(p-1)p}\big[(\nabla u)^2\big]^{\frac{p}{2}}\,\mathrm{d}x - 2\sum_{i=1}^n \kappa_i^{p-1}\int_{\partial\mathcal{O}} b(u)^{(p-1)p}u^{p\sigma_i-\sigma_i+1}\,\mathrm{d}x.
\end{aligned}$$

Noting that $b' \leq 0$ and $\gamma \leq 2$, we drop the nonnegative terms to obtain

$$\begin{aligned}
A'(t) \geq{}& 2\int_{\mathcal{O}} uf(u)b(u)^{(p-1)p-1}\,\mathrm{d}x - 2\int_{\mathcal{O}} b(u)^{(p-1)p}\big[(\nabla u)^2\big]^{\frac{p}{2}}\,\mathrm{d}x\\
&- 2\sum_{i=1}^n \kappa_i^{p-1}\int_{\partial\mathcal{O}} b(u)^{(p-1)p}u^{p\sigma_i-\sigma_i+1}\,\mathrm{d}x.
\end{aligned} \tag{10}$$

Next, we prove

$$(p\sigma_i - \sigma_i + 1)H(u) \geq u^{p\sigma_i-\sigma_i+1}b(u)^{p(p-1)}. \tag{11}$$

Use the method of integration by parts and consider condition (9). Then we obtain

$$\begin{aligned} H_i(u) &= \int_0^u y^{p\sigma_i-\sigma_i} b(y)^{p(p-1)}\,\mathrm{d}y \\ &= y^{p\sigma_i-\sigma_i+1} b(y)^{p(p-1)} \int_0^u - (p\sigma_i-\sigma_i)\int_0^u y^{p\sigma_i-\sigma_i} b(y)^{p(p-1)}\,\mathrm{d}y \\ &\quad - p(p-1)\int_0^u y^p b(y)^{p(p-1)-1} b'(y)\,\mathrm{d}y \\ &\geq u^{p\sigma_i-\sigma_i+1} b(u)^{p(p-1)} - (p\sigma_i-\sigma_i)\int_0^u y^{p\sigma_i-\sigma_i} b(y)^{p(p-1)}\,\mathrm{d}y \\ &= u^{p\sigma_i-\sigma_i+1} b(u)^{p(p-1)} - (p\sigma_i-\sigma_i) H_i(u). \end{aligned}$$

Thus, we prove (11). Further, inserting (8) and (11) into (10) gives

$$\begin{aligned} A'(t) &\geq 2(p\sigma-\sigma+1)(1+\alpha)\int_{\mathcal{O}} F(u)\,\mathrm{d}x \\ &\quad - 2(1+\alpha)\int_{\mathcal{O}} b(u)^{(p-1)p}\big[(\nabla u)^2\big]^{\frac{p}{2}}\,\mathrm{d}x \\ &\quad - 2(p\sigma-\sigma+1)(1+\alpha)\sum_{i=1}^n \kappa_i^{p-1}\int_{\partial\mathcal{O}} H_i(u)\,\mathrm{d}x \\ &\geq 2(p\sigma-\sigma+1)(1+\alpha)B(t). \end{aligned} \tag{12}$$

On the other hand, computing $B(t)$ in (12) gives

$$\begin{aligned} B'(t) &= \int_{\mathcal{O}} f(u) b(u)^{(p-1)p-1} u_t\,\mathrm{d}x \\ &\quad - (p-1)\int_{\mathcal{O}} b(u)^{(p-1)p-1} b'(u) u_t \big[(\nabla u)^2\big]^{\frac{p}{2}}\,\mathrm{d}x \\ &\quad - \int_{\mathcal{O}} b(u)^{(p-1)p}\big[(\nabla u)^2\big]^{\frac{p}{2}-1}\nabla u\nabla u_t\,\mathrm{d}x \\ &\quad - \sum_{i=1}^n \kappa_i^{p-1}\int_{\partial\mathcal{O}} H'{}_i(u) u_t\,\mathrm{d}x \\ &= \int_{\mathcal{O}} f(u) b(u)^{(p-1)p-1} u_t\,\mathrm{d}x \\ &\quad - (p-1)\int_{\mathcal{O}} b(u)^{(p-1)p-1} b'(u) u_t \big[(\nabla u)^2\big]^{\frac{p}{2}}\,\mathrm{d}x \\ &\quad - \int_{\mathcal{O}} b(u)^{(p-1)p}\big[(\nabla u)^2\big]^{\frac{p}{2}-1}\nabla u\nabla u_t\,\mathrm{d}x \\ &\quad - \sum_{i=1}^n \kappa_i^{p-1}\int_{\partial\mathcal{O}} u^{p\sigma_i-\sigma_i} b(u)^{p(p-1)} u_t\,\mathrm{d}x \\ &= \int_{\mathcal{O}} b(u)^{(p-1)p-1} u_t\Big\{f(u) + b'(u)\big((\nabla u)^2\big)^{\frac{p}{2}} \\ &\quad + b(u)\cdot\operatorname{div}\Big[\big((\nabla u)^2\big)^{\frac{p}{2}}\Big]\Big\}\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\geq \int_{\mathcal{O}} b(u)^{(p-1)p-1} u_t \bigl\{ f(u) + (\gamma+1) b'(u) \bigl((\nabla u)^2 \bigr)^{\frac{p}{2}} \\
&\quad + b(u) \cdot \operatorname{div} \bigl[\bigl((\nabla u)^2 \bigr)^{\frac{p}{2}} \bigr] \bigr\} \, \mathrm{d}x \\
&= \int_{\mathcal{O}} b(u)^{(p-1)p-1} u_t \bigl(a(u) \bigr)_t \, \mathrm{d}x \\
&= \int_{\mathcal{O}} b(u)^{(p-1)p-1} a'(u) (u_t)^2 \, \mathrm{d}x.
\end{aligned}
$$

Since $a' > 0$ and $B(0) \geq 0$, we see that $B(t)$ is a nondecreasing function satisfying

$$
B(t) \geq 0.
$$

Multiplying (12) by $B(t)$ and using the Hölder inequality, we obtain

$$
\begin{aligned}
0 &\leq (1+\alpha) A'(t) B(t) \\
&\leq \frac{1}{2(p\sigma - \sigma + 1)} \bigl(A'(t) \bigr)^2 \\
&= \frac{2}{(p\sigma - \sigma + 1)} \biggl(\int_{\mathcal{O}} u b(u)^{(p-1)p-1} a'(u) u_t \, \mathrm{d}x \biggr)^2 \\
&\leq \frac{2}{(p\sigma - \sigma + 1)} B'(t) \biggl(\int_{\mathcal{O}} u b(u)^{(p-1)p-1} a'(u) u^2 \, \mathrm{d}x \biggr).
\end{aligned}
\tag{13}
$$

We further prove that

$$
G(u) \geq u^2 b(u)^{(p-1)p-1} a'(u). \tag{14}
$$

Noting $b' \leq 0$, $a' > 0$, and $a'' \leq 0$, and using the method of integration by parts, we derive

$$
\begin{aligned}
G(u) &= s^2 b(s)^{(p-1)p-1} a'(s) \int_0^u - \int_0^u s b(s)^{(p-1)p-1} a'(s) \, \mathrm{d}s \\
&\quad - \bigl((p-1)p - 1 \bigr) \int_0^u s^2 b(s)^{(p-1)p-2} b'(s) a'(s) \, \mathrm{d}s \\
&\quad - \int_0^u s^2 b(s)^{(p-1)p-1} a''(s) \, \mathrm{d}s \\
&\geq u^2 b(u)^{(p-1)p-1} a'(u) - G(u).
\end{aligned}
$$

Thus, we prove (14) and substitute it into (13). Then we get

$$
\begin{aligned}
(1+\alpha) A'(t) B(t) &\leq \frac{2}{p\sigma - \sigma + 1} B'(t) \biggl(\int_{\mathcal{O}} G(u) \, \mathrm{d}x \biggr) \\
&= \frac{2}{p\sigma - \sigma + 1} B'(t) A(t),
\end{aligned}
$$

which leads to

$$
\frac{\mathrm{d}}{\mathrm{d}t} \bigl(A^{-\frac{1}{2}(p\sigma - \sigma + 1)(1+\alpha)} B \bigr) \geq 0. \tag{15}
$$

Integrating (15) from 0 to t gives

$$\frac{B(t)}{B(0)} \geq \left(\frac{A(t)}{A(0)}\right)^{\frac{1}{2}(p\sigma-\sigma+1)(1+\alpha)}.$$

This and (12) imply that

$$A'(t) \geq 2(p\sigma - \sigma + 1)(1+\alpha)B(0) \cdot A(0)^{-\frac{1}{2}(p\sigma-\sigma+1)(1+\alpha)}A(t)^{\frac{1}{2}(p\sigma-\sigma+1)(1+\alpha)}$$

or

$$\frac{A'(t)}{A(t)^{\frac{1}{2}(p\sigma-\sigma+1)(1+\alpha)}} \geq 2(p\sigma - \sigma + 1)(1+\alpha)B(0)A(0)^{-\frac{1}{2}(p\sigma-\sigma+1)(1+\alpha)}. \tag{16}$$

Use the fact that $p > 2$, $\sigma > 0$ and integrate (16) from 0 to t. Then we deduce that

$$A(t)^{1-\frac{1}{2}(p\sigma-\sigma+1)(1+\alpha)} \leq A(0)^{1-\frac{1}{2}(p\sigma-\sigma+1)(1+\alpha)} - Mt, \tag{17}$$

where

$$M = 2\left[\frac{1}{2}(p\sigma - \sigma + 1)(1+\alpha) - 1\right](p\sigma - \sigma + 1) \cdot (1+\alpha)B(0)A(0)^{-\frac{1}{2}(p\sigma-\sigma+1)(1+\alpha)}.$$

Inequality (17) cannot hold for $A(0)^{1-\frac{p}{2}(1+\alpha)} - Mt \leq 0$, that is, for

$$t \geq M^{-1}A(0)^{1-\frac{1}{2}(p\sigma-\sigma+1)(1+\alpha)}.$$

Hence, we conclude that the solution u of problem (4)-(6) blows up at some finite time T^* with upper bound $M^{-1}A(0)^{1-\frac{1}{2}(p\sigma-\sigma+1)(1+\alpha)}$. The proof is complete. □

3 Lower bound for blow-up time

In this section we seek the lower bound for the blow-up time T^*. To this end, we define an auxiliary function of the form

$$v(s) = \int_0^s \frac{a'(y)}{b(y)}\,\mathrm{d}y, \tag{18}$$

$$E(t) = \int_{\mathcal{O}} \left[v\big(u(x,t)\big)\right]^{\mu p+2}\,\mathrm{d}y \quad \text{with } \mu \geq 1. \tag{19}$$

Moreover, we have to point out that (18) indicates

$$\Delta v = \frac{a'(u)}{b(u)}\Delta u, \tag{20}$$

which is very important to prove the following theorem.

Theorem 3.1 *Suppose that $\mathcal{O} \subset \mathbb{R}^3$ is a bounded convex domain. Further, assume that the nonlinear functions a, b, and f satisfy*

$$0 < f(s) \leq \delta b(s)\left(\int_0^s \nu(y)\,\mathrm{d}y\right)^{p-1}, \quad s > 0, \tag{21}$$

where δ is a positive constant independent of a, b, and f. Then the blow-up time T^* is bounded below by

$$T^* \geq \int_{E(0)}^{+\infty} \frac{\mathrm{d}\xi}{A_0 + A_1\xi + A_2\xi^{\frac{3}{2}} + A_3\xi^3 + A_4\xi^{\frac{2(\mu p+2)-p}{2(p-2)(\mu p+2)}}},$$

where A_0, A_1, A_2, A_3, and A_4 are positive constants to be determined later.

Proof We first compute

$$\begin{aligned}
E'(t) &= (\mu p+2)\int_{\mathcal{O}} \nu^{\mu p+1}\frac{a'(u)}{b(u)}u_t\,\mathrm{d}x \\
&= (\mu p+2)\int_{\mathcal{O}} \nu^{\mu p+1}\frac{1}{b(u)}\big[\operatorname{div}\big(b(u)|\nabla u|^{p-2}\nabla u\big) \\
&\quad +\gamma b'(u)|\nabla u|^p + f(u)\big]\,\mathrm{d}x \\
&= -\kappa^{p-1}(\mu p+2)\int_{\partial\mathcal{O}} \nu^{\mu p+1}|u|^{(p-1)\sigma}\,\mathrm{d}x \\
&\quad - (\mu p+2)(\mu p+1)\int_{\mathcal{O}} \nu^{\mu p}\nabla\nu|\nabla u|^{p-2}\nabla u\,\mathrm{d}x \\
&\quad + (\mu p+2)(1+\gamma)\int_{\mathcal{O}} \nu^{\mu p+1}\frac{b'(u)}{b(u)}|\nabla u|^p\,\mathrm{d}x \\
&\quad + (\mu p+2)\int_{\mathcal{O}} \nu^{\mu p+1}\frac{f(u)}{b(u)}\,\mathrm{d}x \\
&\leq -\kappa^{p-1}(\mu p+2)\int_{\partial\mathcal{O}} \nu^{\mu p+1}|u|^{(p-1)\sigma}\,\mathrm{d}x \\
&\quad - (\mu p+2)(\mu p+1)\int_{\mathcal{O}} \nu^{\mu p}\nabla\nu|\nabla u|^{p-2}\nabla u\,\mathrm{d}x \\
&\quad + (\mu p+2)(1+\gamma)\int_{\mathcal{O}} \nu^{\mu p+1}\frac{b'(u)}{b(u)}|\nabla u|^p\,\mathrm{d}x \\
&\quad + \delta(\mu p+2)\int_{\mathcal{O}} \nu^{\mu p+p}\,\mathrm{d}x.
\end{aligned} \tag{22}$$

The last inequality holds due to condition (21). Further, in view of (20), (21), and $b' \leq 0$, we drop some non-positive terms in (22) to get

$$\begin{aligned}
E'(t) &\leq -(\mu p+2)(\mu p+1)\int_{\mathcal{O}} \left(\frac{b(u)}{a'(u)}\right)^{p-1}\nu^{\mu p}|\nabla\nu|^p\,\mathrm{d}x \\
&\quad + \delta(\mu p+2)\int_{\mathcal{O}} \nu^{\mu p+p}\,\mathrm{d}x.
\end{aligned} \tag{23}$$

Using the fact that $b(s) \geq b_m > 0$ and $0 < a'(s) \leq a'_M$, (23) becomes

$$E'(t) \leq -(\mu p+2)(\mu p+1)(\mu+1)^{-p}\left(\frac{b_m}{a'_M}\right)^{p-1}\int_{\mathcal{O}}\left|\nabla v^{\mu+1}\right|^p\,\mathrm{d}x + \delta(\mu p+2)\int_{\mathcal{O}} v^{\mu p+p}\,\mathrm{d}x. \tag{24}$$

Next, we seek to bound $\delta(\mu p+2)\int_{\mathcal{O}} v^{\mu p+p}\,\mathrm{d}x$ in terms of $E(t)$ and $\int_{\mathcal{O}}|\nabla v^{\mu+1}|^p\,\mathrm{d}x$. By means of the Hölder and Young inequalities, we have

$$\begin{aligned}
\int_{\mathcal{O}} v^{\mu p+p}\,\mathrm{d}x &\leq |\mathcal{O}|^{\frac{2}{\mu p+p+1}}\left(\int_{\mathcal{O}} v^{\mu p+p+1}\,\mathrm{d}x\right)^{\frac{\mu p+p}{\mu p+p+2}}\\
&\leq \frac{2}{\mu p+p+2}|\mathcal{O}| + \frac{\mu p+p}{\mu p+p+2}\int_{\mathcal{O}} v^{\mu p+p+2}\,\mathrm{d}x\\
&\leq \frac{2}{\mu p+p+1}|\mathcal{O}| + \frac{\mu p+p}{\mu p+p+2}\\
&\quad\cdot\left(\int_{\mathcal{O}} v^{\frac{3}{2}(\mu p+2)}\,\mathrm{d}x\right)^{\frac{2p}{\mu p+2}}\left(\int_{\mathcal{O}} v^{\mu p+2}\,\mathrm{d}x\right)^{\frac{\mu p+2-2p}{\mu p+2}}\\
&\leq \frac{2}{\mu p+p+2}|\mathcal{O}| + \frac{\mu p+p}{\mu p+p+2}\frac{2p}{\mu p+2}\int_{\mathcal{O}} v^{\frac{3}{2}(\mu p+2)}\,\mathrm{d}x\\
&\quad+ \frac{\mu p+p}{\mu p+p+2}\frac{\mu p+2-2p}{\mu p+2}\int_{\mathcal{O}} v^{\mu p+2}\,\mathrm{d}x.
\end{aligned} \tag{25}$$

Using the integral inequality derived in [1] (see (2.16)), namely

$$\int_{\mathcal{O}} u^{\frac{3}{2}(\mu p+2)}\,\mathrm{d}x \leq \frac{3^{\frac{3}{4}}}{2\rho_0{}^{\frac{3}{2}}}E(t)^{\frac{3}{2}} + \frac{\sqrt{2}}{3^{\frac{3}{4}}}\left(\frac{\rho_1}{\rho_0}+1\right)^{\frac{3}{2}}\left[\frac{E(t)^3}{4\chi^3} + \frac{3}{4}\chi\int_{\mathcal{O}}\left|\nabla u^{\frac{1}{2}(\mu p+2)}\right|^2\,\mathrm{d}x\right],$$

(25) becomes

$$\begin{aligned}
\int_{\mathcal{O}} v^{\mu p+p}\,\mathrm{d}x &\leq \frac{2}{\mu p+p+2}|\mathcal{O}| + \frac{\mu p+p}{\mu p+p+2}\frac{2p}{\mu p+2}\frac{3^{\frac{3}{4}}}{2\rho_0{}^{\frac{3}{2}}}E(t)^{\frac{3}{2}}\\
&\quad+ \frac{\mu p+p}{\mu p+p+2}\frac{2p}{\mu p+2}\frac{\sqrt{2}}{3^{\frac{3}{4}}}\left(\frac{\rho_1}{\rho_0}+1\right)^{\frac{3}{2}}\\
&\quad\cdot\left[\frac{E(t)^3}{4\chi^3} + \frac{3}{4}\chi\int_{\mathcal{O}}\left|\nabla u^{\frac{1}{2}(\mu p+2)}\right|^2\,\mathrm{d}x\right]\\
&\quad+ \frac{\mu p+p}{\mu p+p+2}\frac{\mu p+2-2p}{\mu p+2}\int_{\mathcal{O}} v^{\mu p+2}\,\mathrm{d}x.
\end{aligned} \tag{26}$$

For simplicity, let $w = v^{1+ns}$. Again by using the Hölder and Young inequalities, we obtain

$$\begin{aligned}
&\int_{\mathcal{O}}\left|\nabla v^{\frac{1}{2}(\mu p+2)}\right|^2\,\mathrm{d}x\\
&\quad\leq \frac{(\mu p+1)^2}{4(\mu+1)^2}\left(\int_{\mathcal{O}}|\nabla w|^p\,\mathrm{d}x\right)^{\frac{2}{p}}\left(\int_{\mathcal{O}} w^{\frac{p(\mu p+2)}{(p-2)(\mu+1)}-\frac{2p}{p-2}}\,\mathrm{d}x\right)^{\frac{p-2}{p}}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(\mu p+1)^2}{2p(\mu+1)^2}\int_{\mathcal{O}}|\nabla w|^p\,\mathrm{d}x \\
&\quad + \frac{p-2}{p}\frac{(\mu p+2)^2}{4(\mu+1)^2}\int_{\mathcal{O}} w^{\frac{p(\mu p+2)}{(p-2)(\mu+1)}-\frac{2p}{p-2}}\,\mathrm{d}x \\
&\leq \frac{(\mu p+1)^2}{2p(\mu+1)^2}\int_{\mathcal{O}}\left|\nabla v^{1+\mu}\right|^p\,\mathrm{d}x \\
&\quad + \frac{p-2}{p}|\mathcal{O}|^{1-\frac{2(\mu p+2)-p}{2(p-2)(\mu p+2)}}\frac{(\mu p+1)^2}{4(\mu+1)^2}E(t)^{\frac{2(\mu p+2)-p}{2(p-2)(\mu p+2)}},
\end{aligned}$$

combining which with (26) yields

$$\begin{aligned}
&\delta(\mu p+2)\int_{\mathcal{O}} u^{\mu p+p}\,\mathrm{d}x \\
&\leq A_0 + A_1E(t) + A_2E(t)^{\frac{3}{2}} + A_3E(t)^3 \\
&\quad + A_4E(t)^{\frac{2(\mu p+2)-p}{2(p-2)(\mu p+2)}} + \chi A_5\int_{\mathcal{O}}\left|\nabla v^{1+\mu}\right|^p\,\mathrm{d}x,
\end{aligned} \tag{27}$$

where χ is a positive constant to be determined later,

$$\begin{aligned}
A_0 &= \frac{2\delta(\mu p+2)}{\mu p+p+2}|\mathcal{O}|, \qquad A_1 = \delta(\mu p+2)\frac{\mu p+p}{\mu p+p+2}\frac{\mu p+2-2p}{\mu p+2}, \\
A_2 &= \frac{3^{\frac{3}{4}}}{2\rho_0{}^{\frac{3}{2}}}\delta(\mu p+2)\frac{\mu p+p}{\mu p+p+2}\frac{2p}{\mu p+2}, \\
A_3 &= \frac{\delta(\mu p+2)}{4\chi_2^3}\frac{\mu p+p}{\mu p+p+2}\frac{2p}{\mu p+2}\frac{\sqrt{2}}{3^{\frac{3}{4}}}\left(\frac{\rho_1}{\rho_0}+1\right)^{\frac{3}{2}}, \\
A_4 &= \frac{3}{4}\frac{\sqrt{2}}{3^{\frac{3}{4}}}\left(\frac{\rho_1}{\rho_0}+1\right)^{\frac{3}{2}}\delta(\mu p+2)\frac{\mu p+p}{\mu p+p+2} \\
&\quad \cdot \frac{2p}{\mu p+2}\frac{p-2}{p}|\mathcal{O}|^{1-\frac{2(\mu p+2)-p}{2(p-2)(\mu p+2)}}\frac{(\mu p+1)^2}{4(\mu+1)^2}\chi, \\
A_5 &= \frac{3}{4}\frac{\sqrt{2}}{3^{\frac{3}{4}}}\left(\frac{\rho_1}{\rho_0}+1\right)^{\frac{3}{2}}\delta(\mu p+2)\frac{\mu p+p}{\mu p+p+2}\frac{2p}{\mu p+2}\frac{(\mu p+1)^2}{2p(\mu+1)^2}.
\end{aligned}$$

Finally, inserting (27) into (24), we obtain

$$\begin{aligned}
E'(t) &\leq -(\mu p+2)(\mu p+1)(\mu+1)^{-p}\frac{b_m}{a'_M}\int_{\mathcal{O}}\left|\nabla v^{\mu+1}\right|^p\,\mathrm{d}y \\
&\quad + A_0 + A_1E(t) + A_2E(t)^{\frac{3}{2}} + A_3E(t)^3 \\
&\quad + A_4E(t)^{\frac{2(\mu p+2)-p}{2(p-2)(\mu p+2)}} + \chi A_5\int_{\mathcal{O}}\left|\nabla v^{1+\mu}\right|^p\,\mathrm{d}x.
\end{aligned} \tag{28}$$

To make use of (28), we choose

$$\chi = A_5^{-1}(\mu p+2)(\mu p+1)(\mu+1)^{-p}\left(\frac{b_m}{a'_M}\right)^{p-1}$$

to arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq A_0 + A_1E(t) + A_2E(t)^{\frac{3}{2}} + A_3E(t)^3 + A_4E(t)^{\frac{2(\mu p+2)-p}{2(p-2)(\mu p+2)}}. \tag{29}$$

An integration of the differential inequality (29) from 0 to t implies that

$$\int_{E(0)}^{E(t)} \frac{\mathrm{d}\xi}{A_0 + A_1\xi + A_2\xi^{\frac{3}{2}} + A_3\xi^3 + A_4\xi^{\frac{2(\mu p+2)-p}{2(p-2)(\mu p+2)}}} \leq t$$

from which we derive a lower bound for T^*, that is,

$$T^* \geq \int_{E(0)}^{+\infty} \frac{\mathrm{d}\xi}{A_0 + A_1\xi + A_2\xi^{\frac{3}{2}} + A_3\xi^3 + A_4\xi^{\frac{2(\mu p+2)-p}{2(p-2)(\mu p+2)}}}.$$

Thus, the proof is complete. □

Remark 3.2 Theorem 3.1 remains valid if we assume that g is a positive $L^p(\mathbb{R}_+)$ function replacing the one in Assumption (A2).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this paper and they read and approved the final manuscript.

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