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Existence and multiplicity of periodic solutions for nonautonomous second order Hamiltonian systems

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Abstract
Using the least action principle, the existence of periodic solutions for some nonautonomous second order Hamiltonian systems is obtained. Using minimax methods, the multiplicity of periodic solutions is obtained. Our results extend some previous results.

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1 Introduction and main results

Consider the nonautonomous second order Hamiltonian systems

$$\begin{cases} \ddot{u}(t) = \nabla F(t,u(t)), & \text{a.e. } t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where the constant $T > 0$, the function $F(t,x) = F_1(t,x) + F_2(t,x)$, and functions $F_1, F_2 \in C^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$ with conditions that $F_1(t+T,x) = F_1(t,x)$ and $F_2(t+T,x) = F_2(t,x)$ hold for all t and x.

Let $H_T^1 = \{u : [0,T] \to \mathbf{R}^n \mid u$ be absolutely continuous, $u(0) = u(T)$ and $\dot{u} \in L^2([0,T], \mathbf{R}^n)\}$ be a Hilbert space with the norm defined by

$$\|u\|_{H_T^1} = \left[\int_0^T \left(|u(t)|^2 + |\dot{u}(t)|^2\right) \mathrm{d}t\right]^{\frac{1}{2}}.$$

We denote the inner products in H_T^1 and $\mathbf{R}^n$ by $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ respectively.

Define the functional

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 \,\mathrm{d}t + \int_0^T \left(F\big(t,u(t)\big) - F(t,\mathbf{0})\right) \mathrm{d}t, \quad \forall u \in H_T^1.$$

It is well known that the solution of problem (1.1) corresponds to the critical point of φ.

As is well known, in much literature one has studied the existence of periodic solutions for problem (1.1) with many solvable conditions via a variational principle, such as the coercive condition in [1] and [2], the bi-even subquadratic potential condition in [3], the γ-quasisubadditive potential condition in [4], the bounded sublinear gradient condition in [5], the bounded linear gradient condition in [6] and [7], and other conditions in [8–13], *etc.* To the best of the authors' knowledge, [14] first studied problem (1.1) with $F(t,x)$ having the decomposition form $F(t,x) = F_1(t,x) + F_2(t,x)$, where F_1 is a subconvex function and F_2 satisfies

$$\left|\nabla F_2(t,x)\right| \leq f(t)|x|^\alpha + g(t), \tag{1.2}$$

where $\alpha \in [0,1), f,g \in L^1([0,T],\mathbf{R}_+)$ and

$$\frac{1}{|x|^{2\alpha}}\left[\frac{1}{\mu}\int_0^T F_1(t,\lambda x)\,\mathrm{d}t + \int_0^T F_2(t,x)\,\mathrm{d}t\right] \to +\infty, \quad |x| \to +\infty.$$

Afterward, [7] generalized the corresponding result with $\alpha = 1$ (see [7], Theorem 1).

In this paper, we obtain two new existence results (Theorem 1.1 and 1.2) improving the corresponding results in [5, 7, 14] without coercive condition.

A function $G : \mathbf{R}^n \to \mathbf{R}$ is said to be (λ,μ)-*subconvex* for some $\lambda, \mu > 0$, if

$$G\bigl(\lambda(x+y)\bigr) \leq \mu\bigl(G(x) + G(y)\bigr), \quad \forall x, y \in \mathbf{R}^n.$$

For our convenience, define two sets $\mathcal{G}$ and $\mathcal{H}$ as follows:

$$\mathcal{G} = \bigl\{G : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}, G(\cdot,x) \in L^1\bigl(\mathbf{R},\mathbf{R}^n\bigr) \mid G(t+T,x) = G(t,x), \\ G \text{ satisfies } (\mathrm{G}_1) \text{ and } (\mathrm{G}_2)\bigr\},$$

where

- (G_1) $G(t,x)$ is measurable in variable t for every $x \in \mathbf{R}^n$, continuous in variable x for a.e. $t \in [0,T]$ and satisfies $|G(t,x)| \leq a(|x|)b(t)$ for all $x \in \mathbf{R}^n$, a.e. $t \in [0,T]$, for some $a \in C(\mathbf{R}_+,\mathbf{R}_+)$ and some $b \in L^1([0,T],\mathbf{R}_+)$,
- (G_2) $G(t,x)$ is (λ,μ)-subconvex about x for some $\lambda, \mu > 0$.

$\mathcal{H} = \{h \in C(\mathbf{R}_+,\mathbf{R}_+) \mid h$ satisfies $(\mathrm{h}_1) - (\mathrm{h}_4)\}$, where

- (h_1) there exists a constant $K_0 > 0$ such that $h(s) \leq h(t) + K_0$, $\forall 0 \leq s \leq t$,
- (h_2) there exists a constant $C_0 > 0$ such that $h(s+t) \leq C_0(h(s) + h(t))$, $\forall s,t \in [0,+\infty)$,
- (h_3) there exist constants $K_1, K_2 > 0$ and $\alpha \in [0,1)$ such that

$$0 \leq h(t) \leq K_1 t^\alpha + K_2, \quad \forall t \in [0,+\infty),$$

- (h_4) $h(t) \to +\infty$ as $t \to +\infty$.

Theorem 1.1 *Assume the function $F = F_1 + F_2$ and there exist T-periodic functions $p,q,f,g \in L^1([0,T],\mathbf{R}_+)$, $G \in \mathcal{G}$ and a constant $\beta \in [0,2)$ such that*

- (H1) $G(t,x) \leq \min\{F_1(t,x), p(t)|x|^\beta + q(t)\}$ *for $|x| > M$ and a.e. $t \in [0,T]$,*
- (H2) *there exists a function $h \in \mathcal{H}$ such that $|\nabla F_2(t,x)| \leq f(t)h(|x|) + g(t)$ holds for all $x \in \mathbf{R}^n$ and a.e. $t \in [0,T]$,*

(H3) $\liminf_{|x|\to+\infty} \frac{1}{h^2(|x|)}[\frac{1}{\mu}\int_0^T G(t,\lambda x)\,\mathrm{d}t + \int_0^T F_2(t,x)\,\mathrm{d}t] > \frac{C_0^2 T}{3}(\int_0^T f(t)\,\mathrm{d}t)^2$, *where* h, C_0 *are as above.*

Then problem (1.1) *possesses at least one solution which minimizes* φ *in* H_T^1.

Obviously, if $G(t,x) = F_1(t,x)$ and $h(s) = s^\alpha$, then Theorem 1 in [14] is the direct corollary of Theorem 1.1. If $G(t,x) = F_1(t,x) \equiv 0$, then Theorem 1.1 in [5] is a special case of Theorem 1.1. Functions satisfying Theorem 1.1 do really exist, such as Example 3.1 and Example 3.3 in Section 3, which cannot be covered by Theorem 1 in [14], Theorem 1.1 in [6] and Theorem 1.1 in [5].

Theorem 1.2 *Assume function* $F = F_1 + F_2$ *and there exist* T*-periodic functions* $s, \nu \in L^1([0,T], \mathbf{R}_+)$ *such that*

(H1*) $G(t,x) \le \min\{F_1(t,x), p(t)|x|^2 + q(t)\}$ *for* $|x| > M$ *and a.e.* $t \in [0,T]$, *where* $G \in \mathcal{G}$,
(H4) *there exists a function* $h \in C(\mathbf{R}_+, \mathbf{R}_+)$ *satisfying* (h_1), (h_2), (h_4) *in* $\mathcal{H}$, *and*

- (h_3^*) *there exist constants* $K_1, K_2 > 0$ *such that* $0 \le h(t) \le K_1 t^2 + K_2$, $\forall t \in [0,+\infty)$ *such that* $F_2(t,x) \ge -s(t)h(|x|) - \nu(t)$ *holds for all* $x \in \mathbf{R}^n$ *and a.e.* $t \in [0,T]$, *where* $\int_0^T p(t)\,\mathrm{d}t + C_0 K_1 \int_0^T s(t)\,\mathrm{d}t < \frac{6}{T}$,

(H5) $\liminf_{|x|\to+\infty} \frac{1}{\mu h(|x|)} \int_0^T G(t,\lambda x)\,\mathrm{d}t > C_0 \int_0^T s(t)\,\mathrm{d}t$, *where* h *and* C_0 *are in* (H4).

Then problem (1.1) *possesses at least one solution which minimizes* φ *in* H_T^1.

Functions satisfying Theorem 1.2 do really exist, such as Example 3.2 in Section 3, which cannot be covered by Theorem 1.5 in [5] and Theorem 1 in [7].

Different from [7] and [14], rather than imposing the subconvex condition and the restriction of $|x|^m$ (generally, $0 \le m \le 2$) on $F_1(t,x)$, we use a subconvex condition on the function G in condition (H1) or condition (H1*), which shows that the function $F_1(t,x)$ in assumptions (H1) and (H1*) can be out of the control of both subconvex condition and $|x|^m$ ($\forall m \in \mathbf{R}$), such as Example 3.1 in Section 3.

There also exist some multiplicity results for problem (1.1), if $F(t,x)$ satisfies the following condition (H6), such as [5, 11, 15, 16] and [17].

(H6) *There exist constants* $k \in \mathbf{N}^*$, $\omega = \frac{2\pi}{T}$, $r > 0$, *such that*

$$\begin{aligned} -\frac{1}{2}(k+1)^2\omega^2|x|^2 &\le F(t,x) - F(t,\mathbf{0}) \\ &\le -\frac{1}{2}k^2\omega^2|x|^2 \quad \text{for all } |x| \le r \text{ and a.e. } t \in [0,T]. \end{aligned}$$

Among them, [15] obtained a multiplicity result if $F(t,x)$ satisfies (1.2). Afterward, [5] generalized the corresponding results in [15] and [17].

Different from the multiplicity results in [5, 11, 15, 16] and [17], we list our multiplicity results corresponding to Theorem 1.1 and Theorem 1.2 respectively.

Theorem 1.3 *Assume that* $F = F_1 + F_2$, *conditions* (H1), (H2), (H3), *and* (H6) *hold, then problem* (1.1) *has at least two nonzero solutions in* H_T^1.

If $G(t,x) = F_1(t,x) \equiv 0$, then Theorem 1.4 in [5] is a special case of our Theorem 1.3.

Theorem 1.4 *Assume that* $F = F_1 + F_2$, *conditions* (H1*), (H4), (H5), *and* (H6) *hold, then problem* (1.1) *has at least two nonzero solutions in* H_T^1.

There are functions F satisfying Theorem 1.3 (Theorem 1.4) but that cannot be covered by Theorem 1.4 in [5] (Theorem 1.8 in [5]), such as Example 3.3 in Section 3.

2 Proof of theorems

For every $u \in H_T^1$, set $\tilde{u}(t) = u(t) - \bar{u}$ with $\bar{u} = \frac{1}{T}\int_0^T u(t)\,\mathrm{d}t$. Page 9 of [1] tells us that

$$\|\tilde{u}\|_\infty^2 \le \frac{T}{12}\int_0^T \bigl|\dot{u}(t)\bigr|^2\,\mathrm{d}t \quad \text{(Sobolev's inequality)}.$$

Reference [14] tells us that $\|u\| = (|\bar{u}|^2 + \|\dot{u}\|_{L^2}^2)^{\frac{1}{2}}$ is equivalent to the norm $\|u\|_{H_T^1}$.

The least action principle (see [1]) *If φ is weakly lower semicontinuous on a reflexive Banach space X and has a bounded minimizing sequence, then φ has a minimum on X.*

Remark 2.1 If the functional φ is coercive, that is, $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$, then we have a bounded minimizing sequence $\{\varphi(u_m)\}$ such that $\varphi(u_m) \to \inf\varphi < +\infty$. In fact, for any minimizing sequence $\{\varphi(u_m)\}$, if $\{u_m\}$ is unbounded, then φ to be coercive implies that $\varphi(u_m) \to +\infty$, which is a contradiction.

Remark 2.2 Under the assumption that $F_1, F_2 \in C^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$, similar to the proof of Proposition 4 of [3], we claim that the weak solution satisfying (1.1) is the desired classical solution. In fact, if there exists a function $\hat{u} \in C^2(\mathbf{R}, \mathbf{R}^n)$ satisfying $\int_0^T (\dot{\hat{u}} + \nabla F(t,u)) \cdot \dot{h}\,\mathrm{d}t = 0$ for $\forall h \in H_T^1$, then we have $\int_0^T |\dot{u}(t) - \dot{\hat{u}}(t)|^2\,\mathrm{d}t = 0$, which implies that $|u(t) - \hat{u}(t)| \le \int_0^t |\dot{u}(s) - \dot{\hat{u}}(s)|\,\mathrm{d}s \le \sqrt{T}\|\dot{u}(s) - \dot{\hat{u}}(s)\|_{L^2} = 0$, $\forall t \in [0,T]$. Hence, $u \in C^2(\mathbf{R}, \mathbf{R}^n)$.

Lemma 2.1 (see Theorem 4 of [18]) *Let X be a Banach space with a direct sum decomposition $X = X_1 \oplus X_2$ with $k = \dim X_2 < \infty$ and let φ be a C^1 functional on X with $\varphi(\mathbf{0}) = 0$, which satisfies* (PS) *condition. Assume that for some $R > 0$,*

$$\begin{cases} \varphi(u) \ge 0, & u \in X_1 \text{ with } \|u\| \le R, \\ \varphi(u) \le 0, & u \in X_2 \text{ with } \|u\| \le R. \end{cases} \tag{2.1}$$

Assume also that φ is bounded below and $\inf_X \varphi < 0$. Then φ has at least two nonzero critical points.

Proof of Theorem 1.1 By condition (G_1), we see that $|G(t,x)| \le a_0 b(t)$ holds for $|x| \le M$, where $a_0 = \max_{0\le|x|\le M} a(|x|)$. Then by (H1), one has

$$F_1(t,x) \ge G(t,x) - a_0 b(t) - M_1, \quad \text{a.e. } t \in [0,T] \text{ and all } x \in \mathbf{R}^n, \tag{2.2}$$

where $M_1 = \max_{t\in[0,T]} \max_{|x|\le M} |F_1(t,x)|$.

From (2.2), (H1), and Sobolev's inequality, one has

$$\begin{aligned} \int_0^T F_1\bigl(t,u(t)\bigr)\,\mathrm{d}t &\ge \int_0^T G\bigl(t,u(t)\bigr)\,\mathrm{d}t - M_2 \\ &\ge \frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})\,\mathrm{d}t - \int_0^T G\bigl(t,-\tilde{u}(t)\bigr)\,\mathrm{d}t - M_2 \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})\,\mathrm{d}t-\int_{\{t\in[0,T]\mid|\tilde{u}(t)|>M\}} G\bigl(t,-\tilde{u}(t)\bigr)\,\mathrm{d}t\\
&\quad-\int_{\{t\in[0,T]\mid|\tilde{u}(t)|\leq M\}} G\bigl(t,-\tilde{u}(t)\bigr)\,\mathrm{d}t-M_2\\
&\geq \frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})\,\mathrm{d}t-\|\tilde{u}\|_\infty^\beta\int_0^T p(t)\,\mathrm{d}t-\int_0^T q(t)\,\mathrm{d}t-M_3\\
&\geq \frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})\,\mathrm{d}t-M_4\bigl(\|\dot{u}\|_{L^2}^\beta+1\bigr),\quad \forall u\in H_T^1,
\end{aligned}\tag{2.3}$$

where constants $M_2=M_1T+a_0\int_0^T b(t)\,\mathrm{d}t>0$, $M_3=M_2+a_0\int_0^T b(t)\,\mathrm{d}t>0$, and $M_4=\max\{(\frac{T}{12})^{\frac{\beta}{2}}\int_0^T p(t)\,\mathrm{d}t,\int_0^T q(t)\,\mathrm{d}t+M_3\}>0$.

Using (H2), (h_1)-(h_3), and Sobolev's inequality, we have

$$\begin{aligned}
&\left|\int_0^T\bigl(F_2\bigl(t,u(t)\bigr)-F_2(t,\bar{u})\bigr)\,\mathrm{d}t\right|\\
&\quad=\left|\int_0^T\int_0^1\bigl(\nabla F_2\bigl(t,\bar{u}+s\tilde{u}(t)\bigr),\tilde{u}(t)\bigr)\,\mathrm{d}s\,\mathrm{d}t\right|\\
&\quad\leq\int_0^T\int_0^1\bigl[f(t)h\bigl(|\bar{u}+s\tilde{u}(t)|\bigr)+g(t)\bigr]\bigl|\tilde{u}(t)\bigr|\,\mathrm{d}s\,\mathrm{d}t\\
&\quad\leq\int_0^T\int_0^1 C_0f(t)\bigl[h\bigl(|\bar{u}|\bigr)+h\bigl(s\bigl|\tilde{u}(t)\bigr|\bigr)\bigr]\bigl|\tilde{u}(t)\bigr|\,\mathrm{d}s\,\mathrm{d}t\\
&\qquad+K_0\int_0^T f(t)\bigl|\tilde{u}(t)\bigr|\,\mathrm{d}t+\int_0^T g(t)\bigl|\tilde{u}(t)\bigr|\,\mathrm{d}t\\
&\quad\leq\int_0^T C_0f(t)\bigl[h\bigl(|\bar{u}|\bigr)+h\bigl(|\tilde{u}(t)|\bigr)+K_0\bigr]\bigl|\tilde{u}(t)\bigr|\,\mathrm{d}t+\|\tilde{u}\|_\infty\left(K_0\int_0^T f(t)\,\mathrm{d}t+\int_0^T g(t)\,\mathrm{d}t\right)\\
&\quad\leq C_0\|\tilde{u}\|_\infty\int_0^T\bigl[h\bigl(|\bar{u}|\bigr)+h\bigl(\|\tilde{u}\|_\infty\bigr)+2K_0\bigr]f(t)\,\mathrm{d}t+M_5\|\dot{u}\|_{L^2}\\
&\quad\leq C_0\left[\frac{3}{C_0T}\|\tilde{u}\|_\infty^2+\frac{C_0T}{3}h^2\bigl(|\bar{u}|\bigr)\left(\int_0^T f(t)\,\mathrm{d}t\right)^2\right]\\
&\qquad+C_0h\bigl(\|\tilde{u}\|_\infty\bigr)\|\tilde{u}\|_\infty\int_0^T f(t)\,\mathrm{d}t+M_6\|\dot{u}\|_{L^2}\\
&\quad\leq\frac{1}{4}\|\dot{u}\|_{L^2}^2+\frac{C_0^2T}{3}h^2\bigl(|\bar{u}|\bigr)\|f\|_{L^1}^2\\
&\qquad+M_7\left[K_1\left(\frac{T}{12}\int_0^T\bigl|\dot{u}(t)\bigr|^2\,\mathrm{d}t\right)^{\frac{\alpha}{2}}+K_2\right]\|\dot{u}\|_{L^2}+M_6\|\dot{u}\|_{L^2}\\
&\quad=\frac{1}{4}\|\dot{u}\|_{L^2}^2+\frac{C_0^2T}{3}h^2\bigl(|\bar{u}|\bigr)\|f\|_{L^1}^2+M_8\|\dot{u}\|_{L^2}^{\alpha+1}+M_9\|\dot{u}\|_{L^2},\quad\forall u\in H_T^1,
\end{aligned}\tag{2.4}$$

where constants $M_i>0$, $i=5,6,7,8,9$.

Using (2.3) and (2.4), one has

$$\begin{aligned}
\varphi(u)&=\frac{1}{2}\|\dot{u}\|_{L^2}^2+\int_0^T F_1\bigl(t,u(t)\bigr)\,\mathrm{d}t+\int_0^T\bigl(F_2\bigl(t,u(t)\bigr)-F_2(t,\bar{u})\bigr)\,\mathrm{d}t\\
&\quad+\int_0^T F_2(t,\bar{u})\,\mathrm{d}t-\int_0^T F(t,\mathbf{0})\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}&\ge \frac{1}{4}\|\dot{u}\|_{L^2}^2 - \frac{C_0^2T}{3}h^2\big(|\bar{u}|\big)\|f\|_{L^1}^2 - M_4\big(\|\dot{u}\|_{L^2}^\beta + 1\big) - M_8\|\dot{u}\|_{L^2}^{\alpha+1} - M_9\|\dot{u}\|_{L^2}\\&\quad + \frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})\,\mathrm{d}t + \int_0^T F_2(t,\bar{u})\,\mathrm{d}t + C,\end{aligned} \tag{2.5}$$

where $C = -\int_0^T F(t,\mathbf{0})\,\mathrm{d}t$.

If $\|u\| \to +\infty$, but $0 \le |\bar{u}| < +\infty$, then (2.5), $\alpha \in [0,1)$, $\beta \in [0,2)$, and $\|u\| = (|\bar{u}|^2 + \|\dot{u}\|_{L^2}^2)^{\frac{1}{2}}$ imply that $\varphi(u) \to +\infty$. If $\|u\| \to +\infty$ with $|\bar{u}| \to +\infty$, using (2.5), we have

$$\begin{aligned}\varphi(u) &\ge \frac{1}{4}\|\dot{u}\|_{L^2}^2 - M_8\|\dot{u}\|_{L^2}^{\alpha+1} - M_9\|\dot{u}\|_{L^2} - M_4\big(\|\dot{u}\|_{L^2}^\beta + 1\big)\\&\quad + h^2\big(|\bar{u}|\big)\left[\frac{1}{h^2(|\bar{u}|)}\left(\frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})\,\mathrm{d}t + \int_0^T F_2(t,\bar{u})\,\mathrm{d}t\right) - \frac{C_0^2T}{3}\|f\|_{L^1}^2\right]\\&\quad + C.\end{aligned} \tag{2.6}$$

By (2.6), (h_4), (H3), $\alpha \in [0,1)$, $\beta \in [0,2)$, and $\|u\| = (|\bar{u}|^2 + \|\dot{u}\|_{L^2}^2)^{\frac{1}{2}}$, one has $\varphi(u) \to +\infty$. So we have $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$. According to the least action principle and Remark 2.1 and Remark 2.2, we complete our proof. □

Proof of Theorem 1.2 (H1*) still implies that (2.2) holds. Similarly to (2.3), from (2.2) and Sobolev's inequality, we have

$$\begin{aligned}\int_0^T F_1\big(t,u(t)\big)\,\mathrm{d}t &\ge \frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})\,\mathrm{d}t - \int_{\{t\in[0,T]\,|\,|\tilde{u}(t)|>M\}} G\big(t,-\tilde{u}(t)\big)\,\mathrm{d}t\\&\quad - \int_{\{t\in[0,T]\,|\,|\tilde{u}(t)|\le M\}} G\big(t,-\tilde{u}(t)\big)\,\mathrm{d}t - M_2\\&\ge \frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})\,\mathrm{d}t - \|\tilde{u}\|_\infty^2\int_0^T p(t)\,\mathrm{d}t - \int_0^T q(t)\,\mathrm{d}t - M_3\\&\ge \frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})\,\mathrm{d}t - \frac{T}{12}\|\dot{u}\|_{L^2}^2\int_0^T p(t)\,\mathrm{d}t\\&\quad - M_{10}, \quad \forall u \in H_T^1,\end{aligned} \tag{2.7}$$

where constant $M_{10} > 0$.

Using (H4) and Sobolev's inequality, we have

$$\begin{aligned}\int_0^T F_2\big(t,u(t)\big)\,\mathrm{d}t &\ge \int_0^T \big[-s(t)h\big(|u(t)|\big) - \nu(t)\big]\,\mathrm{d}t\\&\ge -\big[C_0\big(h\big(|\bar{u}|\big) + h\big(\|\tilde{u}\|_\infty\big)\big) + K_0\big]\int_0^T s(t)\,\mathrm{d}t - \int_0^T \nu(t)\,\mathrm{d}t\\&\ge -C_0h\big(|\bar{u}|\big)\int_0^T s(t)\,\mathrm{d}t - \frac{C_0K_1T}{12}\|\dot{u}\|_{L^2}^2\int_0^T s(t)\,\mathrm{d}t\\&\quad - M_{11}, \quad \forall u \in H_T^1,\end{aligned} \tag{2.8}$$

where constant $M_{11} > 0$.

From (2.7) and (2.8), we know that

$$\varphi(u) \geq \left(\frac{1}{2} - \frac{T}{12}\int_0^T p(t)\,\mathrm{d}t - \frac{C_0K_1T}{12}\int_0^T s(t)\,\mathrm{d}t\right)\|\dot{u}\|_{L^2}^2 + \frac{1}{\mu}\int_0^T G(t,\lambda\bar{u})\,\mathrm{d}t$$
$$- C_0h\bigl(|\bar{u}|\bigr)\int_0^T s(t)\,\mathrm{d}t - M_{12}, \quad \forall u \in H_T^1, \tag{2.9}$$

where constant $M_{12} > 0$.

If $\|u\| \to +\infty$, but $0 \leq |\bar{u}| < +\infty$, then (2.9), $\int_0^T p(t)\,\mathrm{d}t + C_0K_1\int_0^T s(t)\,\mathrm{d}t < \frac{6}{T}$ in (H4) and $\|u\| = (|\bar{u}|^2 + \|\dot{u}\|_{L^2}^2)^{\frac{1}{2}}$ imply that $\varphi(u) \to +\infty$. If $\|u\| \to +\infty$ with $|\bar{u}| \to +\infty$, then (2.9) implies that

$$\varphi(u) \geq \left(\frac{1}{2} - \frac{T}{12}\int_0^T p(t)\,\mathrm{d}t - \frac{C_0K_1T}{12}\int_0^T s(t)\,\mathrm{d}t\right)\|\dot{u}\|_{L^2}^2$$
$$+ h\bigl(|\bar{u}|\bigr)\left[\frac{1}{\mu}\frac{\int_0^T G(t,\lambda\bar{u})\,\mathrm{d}t}{h(|\bar{u}|)} - C_0\int_0^T s(t)\,\mathrm{d}t\right] - M_{12}. \tag{2.10}$$

By (2.10), (H4), and (H5), and $\|u\| = (|\bar{u}|^2 + \|\dot{u}\|_{L^2}^2)^{\frac{1}{2}}$, we get $\varphi(u) \to +\infty$. So one has $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$. According to the least action principle and Remark 2.1 and Remark 2.2, we complete our proof. □

Proof of Theorem 1.3 (*Theorem* 1.4) The idea comes from [11].

Step 1. We claim that functional φ satisfies the (PS) condition.

In fact, suppose a sequence $\{u_m\}$ is a (PS) sequence, that is, $\{\varphi(u_m)\}$ is bounded and $\varphi'(u_m) \to \mathbf{0}$ as $m \to +\infty$, then the proof of Theorem 1.1 (or Theorem 1.2) tells us that φ is coercive, which implies that $\{u_m\}$ is bounded. Hence, there exists a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$, such that $u_m \rightharpoonup u$ in H_T^1. By Sobolev's embedding theorem, we have $u_m \to u$ in $C([0,T],\mathbf{R}^n)$. A simple calculation tells us that

$$\begin{aligned}
&\int_0^T \bigl|\dot{u}_m(t) - \dot{u}(t)\bigr|^2\,\mathrm{d}t \\
&\quad = \bigl\langle \varphi'(u_m) - \varphi'(u), u_m - u\bigr\rangle - \int_0^T \bigl(\nabla F\bigl(t,u_m(t)\bigr) - \nabla F\bigl(t,u(t)\bigr), u_m(t) - u(t)\bigr)\,\mathrm{d}t \\
&\quad \to 0 \quad \text{as } m \to +\infty,
\end{aligned}$$

which implies that $\|\dot{u}_m - \dot{u}\|_{L^2} \to 0$. So we have $u_m \to u$ in H_T^1. Hence φ satisfies the (PS) condition.

Step 2. We check that (2.1) holds. Let $X = H_T^1$, $X = X_1 \oplus X_2$, $X_2 = X_1^\perp$ with

$$X_2 = \left\{\sum_{j=0}^k (a_j\cos j\omega t + b_j\sin j\omega t)\,\middle|\, a_j, b_j \in \mathbf{R}^n, j = 0,\ldots,k\right\}.$$

A simple calculation tells us

$$\int_0^T \bigl|u(t)\bigr|^2\,\mathrm{d}t = T\sum_{j=0}^k \frac{a_j^2 + b_j^2}{2} \quad \text{and} \quad \int_0^T \bigl|\dot{u}(t)\bigr|^2\,\mathrm{d}t = T\omega^2\sum_{j=0}^k j^2\frac{a_j^2 + b_j^2}{2}, \quad \forall u \in X_2.$$

By the Sobolev embedding theorem, there exists a constant $M_{13} > 0$ such that $\|u\|_\infty \leq M_{13}\|u\|_{H_T^1}$.

It follows from the condition (H6) that

$$\begin{aligned}\varphi(u) &\leq \frac{1}{2}\int_0^T \left|\dot{u}(t)\right|^2 \,\mathrm{d}t - \frac{1}{2}k^2\omega^2 \int_0^T \left|u(t)\right|^2 \,\mathrm{d}t \\ &\leq 0, \quad \forall u \in X_2 \text{ with } \|u\|_{H_T^1} \leq \frac{r}{M_{13}}.\end{aligned} \tag{2.11}$$

In the same way, we obtain

$$\begin{aligned}\varphi(u) &\geq \frac{1}{2}\int_0^T \left|\dot{u}(t)\right|^2 \,\mathrm{d}t - \frac{1}{2}(k+1)^2\omega^2 \int_0^T \left|u(t)\right|^2 \,\mathrm{d}t \\ &\geq 0, \quad \forall u \in X_1 \text{ with } \|u\|_{H_T^1} \leq \frac{r}{M_{13}}.\end{aligned}$$

Case 1. If $\inf_X \varphi < 0$, by Lemma 2.1, then we see that φ has at least two nonzero critical points.

Case 2. If $\inf_X \varphi \geq 0$, by (2.11), then we get $\varphi(u) = 0$ for all $u \in X_2$ with $\|u\|_{H_T^1} \leq \frac{r}{M_{13}}$. Therefore, φ has infinite many critical points.

We complete our proof. □

3 Examples

Example 3.1 Choose $T = 2$, functions $G \in C^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$, $F_i \in C^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$ $(i = 1, 2)$, and $r \in C^1(\mathbf{R}, \mathbf{R}^n)$ as follows. Let $G(t,x) = |x|^{\frac{3}{2}}$, $F_1(t,x) = \mathrm{e}^{|x|^2}$ and $F_2(t,x) = \ln^2(1+|x|^2) + (r(t),x)$, where $r(t) = (\sin(\pi t), 0, \ldots, 0)$.

We can check that $G(t,x)$ is $(1,\sqrt{8})$-subconvex. In fact, $|x+y|^{\frac{3}{2}} \leq ||x| + |y||^{\frac{3}{2}} \leq 2^{\frac{3}{2}}(|x|^{\frac{3}{2}} + |y|^{\frac{3}{2}})$. $G(t,x) \leq \min\{\mathrm{e}^{|x|^2}, |x|^{\frac{3}{2}}\}$ holds for all $t \in \mathbf{R}$, and $x \in \mathbf{R}^n$. If we choose $a(t) = t^{\frac{3}{2}}$ and $b(t) \equiv 1$, $p(t) \equiv 1$, $q(t) \equiv 0$, $\beta = \frac{3}{2}$, then assumption (H1) holds.

Obviously, $|\nabla F_2(t,x)| \leq 2\ln(1 + |x|^2) + 1$ holds for all $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$. If we choose $h(s) = \ln(1+s^2)$, $f(t) \equiv 2$, and $g(t) \equiv 2$, then assumption (H2) holds.

We also obtain $\lim_{|x|\to+\infty} \frac{\frac{1}{\sqrt{8}}\int_0^2 |x|^{\frac{3}{2}}\,\mathrm{d}t + \int_0^2 \ln^2(1+|x|^2)\,\mathrm{d}t + \int_0^2 (r(t),x)\,\mathrm{d}t}{\ln^2(1+|x|^2)} \to +\infty$, so assumption (H3) holds for $\lambda = 1$ and $\mu = \sqrt{8}$.

Therefore, the function $F(t,x) = \mathrm{e}^{|x|^2} + \ln^2(1+|x|^2) + (r(t),x)$ satisfies Theorem 1.1. Meanwhile, the function $F(t,x)$ is out of quadratic growth as $|x| \to +\infty$, which extends the sublinear conditions for $F(t,x)$ in Theorem 1 of [14], Theorem 1.1 of [5] and Theorem 1.1 of [6].

Example 3.2 Choose $T = \frac{4}{3}$, functions $G \in C^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$, $F_i \in C^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$ $(i = 1, 2)$ and $r \in C^1(\mathbf{R}, \mathbf{R}^n)$ as follows. Let $G(t,x) = F_1(t,x) = 2\sin^2(\frac{3\pi}{2}t)|x|^2$ and

$$F_2(t,x) = \frac{\cos(\frac{3\pi}{2}t)}{80\ln(100+|x|)}|x|^2 + \bigl(r(t),x\bigr),$$

where $r(t) = (\frac{1}{80}\sin(\frac{3}{2}\pi t), 0, \ldots, 0)$.

We can check that $G(t,x)$ is $(1,4)$-subconvex. In fact, $2\sin^2(\frac{3\pi}{2}t)|x+y|^2 \leq 2\sin^2(\frac{3\pi}{2}t)(|x| + |y|)^2 \leq 8\sin^2(\frac{3\pi}{2}t)(|x|^2 + |y|^2)$. When $|x|$ is large enough, we see that $G(t,x) \leq \min\{F_1(t,x),$

$2|x|^2\}$ holds for all $t \in \mathbf{R}$. If we choose $a(t) = 2t^2$ and $b(t) \equiv 1$, $p(t) \equiv 2$, $q(t) \equiv 0$, then assumption (H1*) holds.

For $F_2(t,x)$, there exists a constant $L > 0$ such that $F_2(t,x) \geq -\frac{|x|^2}{40} - L$ holds for all $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$. If we choose $h(s) = s^2$, $K_1 = 1$, $K_2 = 0$, $s(t) \equiv \frac{1}{40}$, $\nu(t) \equiv L$, and $C_0 = 4$, then assumptions (H4) and (H5) hold.

Therefore, the function $F(t,x) = 2\sin^2(\frac{3\pi}{2}t)|x|^2 + \frac{\cos(\frac{3\pi}{2}t)}{80\ln(100+|x|)}|x|^2 + (r(t),x)$ satisfies Theorem 1.2. However, $\frac{\int_0^{\frac{4}{3}} F(t,x)\,\mathrm{d}t}{|x|^2} < +\infty$, so $F(t,x)$ cannot be covered by Theorem 1.5 in [5] and Theorem 1.1 in [7] if we choose $h(t) = t$. In addition, if the function $F(t,x)$ has the above decomposition, then we see that $F_2(t,x)$ is out of control of assumption (H2). In this case, $F(t,x)$ cannot be covered by Theorem 1.1.

Example 3.3 Choose $T = 1$, $\omega = \frac{2\pi}{T}$, functions $F_i \in C^1(\mathbf{R} \times \mathbf{R}, \mathbf{R})$ $(i = 1, 2)$ and $G \in C^1(\mathbf{R} \times \mathbf{R}, \mathbf{R})$ as follows:

$$F_1(t,x) = \begin{cases} \frac{2}{3}\mathrm{e}^{x^2} - \frac{5\mathrm{e}}{18}, & |x| > 1, \\ \frac{2\mathrm{e}}{3}[-\frac{1}{4}\omega^2 x^2 + (\frac{1}{2}\omega^2 + \frac{3}{4})x^4 - (\frac{1}{4}\omega^2 + \frac{1}{6})x^6], & |x| \leq 1, \end{cases}$$

$$F_2(t,x) = \begin{cases} \frac{2}{3}\ln^2(1+x^2) - \frac{2}{3}\ln^2 2 + \frac{7\ln 2}{18}, & |x| > 1, \\ \frac{2\ln 2}{3}[-\frac{1}{4}\omega^2 x^2 + (\frac{1}{2}\omega^2 + \frac{3}{4})x^4 - (\frac{1}{4}\omega^2 + \frac{1}{6})x^6], & |x| \leq 1, \end{cases}$$

$F(t,x) = F_1(t,x) + F_2(t,x)$ and $G(t,x) = \frac{2}{3}|x|^{\frac{3}{2}}$.

We can check that $G(t,x)$ is $(1,\sqrt{8})$-subconvex. In fact, $\frac{2}{3}|x+y|^{\frac{3}{2}} \leq \frac{2\sqrt{8}}{3}(|x|^{\frac{3}{2}} + |y|^{\frac{3}{2}})$. We choose $L > 0$ large enough, $a(t) = t^{\frac{3}{2}}$ and $b(t) = 1$, then we see that $|G(t,x)| \leq a(|x|)b(t)$ holds for all $x \in \mathbf{R}$ and $t \in \mathbf{R}$. Setting $p(t) = \frac{2}{3}$, $q(t) \equiv 0$, $G(t,x) \leq \min\{\frac{2}{3}|x|^{\frac{3}{2}}, \frac{2}{3}\mathrm{e}^{|x|^2}\}$ holds for $|x|$ large enough and all $t \in \mathbf{R}$. So (H1) and (H1*) hold.

For any $x \in \mathbf{R}$, we see that $|\frac{\partial F_2(t,x)}{\partial x}| \leq 2\ln(1+x^2) + L$ holds for $t \in \mathbf{R}$ and

$$\lim_{|x|\to+\infty} \frac{\frac{\int_0^1 \frac{2}{3}|x|^{\frac{3}{2}}\,\mathrm{d}t}{\sqrt{8}} + \int_0^1 F_2(t,x)\,\mathrm{d}t}{\ln^2(1+x^2)} \to +\infty,$$

hence, assumptions (H2) and (H3) hold for $h(t) = \ln(1+t^2)$, $f(t) \equiv 2$, $g(t) \equiv L$, $\lambda = 1$, and $\mu = \sqrt{8}$.

In addition, $F_2(t,x) \geq -|x| - L_1$ holds for a positive constant L_1, all $x \in \mathbf{R}$, and $t \in \mathbf{R}$. Set $s(t) \equiv 1$, $\nu(t) \equiv L_1$, $h(t) = t$, $K_1 = 1$, $C_0 = 1$, and $\lambda = 1$, $\mu = \sqrt{8}$, then we have $\int_0^1 p(t)\,\mathrm{d}t + C_0 K_1 \int_0^1 s(t)\,\mathrm{d}t < 6$ and $\lim_{|x|\to+\infty} \frac{\int_0^1 \frac{2}{3}|x|^{\frac{3}{2}}\,\mathrm{d}t}{\sqrt{8}|x|} \to +\infty$, hence, assumptions (H4) and (H5) hold.

Choosing x with $|x|$ small enough, we can only consider the main part $-\frac{1}{6}(\mathrm{e} + \ln 2)\omega^2 x^2$ of $F(t,x)$, so we have $-\frac{1}{6}(\mathrm{e} + \ln 2) \in (-2, -\frac{1}{2})$, which implies that assumption (H6) holds for $k = 1$. Therefore, $F(t,x)$ satisfies both Theorem 1.3 and Theorem 1.4.

Obviously, $F(t,x)$ is out of quadratic growth as $|x| \to +\infty$. But Theorems 1.4 and 1.8 in [5] imply that ∇F satisfies the sublinearity or linearity condition. So Example 3.3 cannot be covered by Theorems 1.4 and 1.8 of [5].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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