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Sign-changing solution and ground state solution for a class of (p,q)-Laplacian equations with nonlocal terms on $\mathbb{R}^N$

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Abstract

In the paper, we investigate the least energy sign-changing solution and the ground state solution of a class of (p,q)-Laplacian equations with nonlocal terms on $\mathbb{R}^N$. Applying the constraint variational method, the quantitative deformation lemma, and topological degree theory, we see that the equation has one least energy sign-changing solution u. Moreover, we regard c, d as parameters and give a convergence property of such a solution $u_{c,d}$ as $(c,d) \to 0$. Finally, using the Lagrange multiplier method, we obtain a ground state solution of the equation and show that the energy of u is strictly larger than two times the ground state energy.

Keywords: (p,q)-Laplacian equation; sign-changing solution; ground state solution; nonlocal term

1 Introduction

In this paper, we discuss the existence of a least energy sign-changing solution and a ground state solution of the following equation:

$$-\left(a + c\int_{\mathbb{R}^N} |\nabla u|^p\right)\Delta_p u - \left(b + d\int_{\mathbb{R}^N} |\nabla u|^q\right)\Delta_q u + h(x)|u|^{p-2}u + g(x)|u|^{q-2}u$$
$$= f(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $2 \le q < p < q^*$, $N < 2p$, $\Delta_m = \operatorname{div}(|\nabla u|^{m-2}\nabla u)$ is the m-Laplacian operator, $m^* = \infty$ for $N \le m$, and $m^* = Nm/(N-m)$ for $N > m$. a, b are positive constants, $c, d \ge 0$. We assume that h, g are continuous, coercive and positive functions.

When $c = d = 0$, equation (1.1) is the following (p,q)-Laplacian equation:

$$-a\Delta_p u - b\Delta_q u + h(x)|u|^{p-2}u + g(x)|u|^{q-2}u = f(u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

A special situation for (1.2) is the case where $p = q > 1$, *i.e.*, a single p-Laplacian equation. When $p = q = 2$, (1.2) becomes the nonlinear Laplacian type equation

$$-\Delta u + au = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.3}$$

Equation (1.2) appears, for example, as the stationary version of a general reaction-diffusion equation

$$u_t = \operatorname{div}\big[D(u)\nabla u\big] + f(x,u),$$

where u describes a concentration, $D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$ is the diffusion coefficient, and $f(x,u)$ is the reaction term connected with source and loss mechanisms. This equation has extensive applications in physics and related sciences such as biophysics, plasma physics, and chemical reaction design. Typically, in chemical and biological applications, the reaction term $f(x,u)$ is a polynomial of u with variable coefficients (see [1–4]).

The differential operator $\Delta_p + \Delta_q$ is known as the (p,q)-Laplacian operator, if $p \neq q$. The single p-Laplacian operator has been studied for at least four decades (see [1, 5–19]), whereas a deeper research involving the (p,q)-Laplacian operator has only arisen in the last decade (see [2–4, 20–22]).

In [16], the authors investigated the existence of a positive solution of equation (1.3) where $a > 0$ is a constant. In [8], the authors proved the existence of sign-changing solutions of equation (1.3) where $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and $\operatorname{ess\,inf} a > 0$. We also refer the interested reader to more related results as regards equation (1.3) in [9, 10] and the references therein. In [7], the authors proved the existence of least energy positive, negative, and sign-changing solutions for the p-Laplacian equation with potentials vanishing at infinity. In [1], the author obtained multiplicity solutions of the p-Laplacian equation with a critical nonlinearity. Since the (p,q)-Laplacian operator is not homogeneous, some technical difficulties appear when using the common methods of the elliptic equations. The existence of a nontrivial solution to equation (1.2) was obtained in [4, 20, 21]. In [4], the authors dealt with the situation $2 \leq q \leq p < N$ with $h \in L_+^{N/p}(\mathbb{R}^N)$ and $g \in L_+^{N/q}(\mathbb{R}^N)$, whereas in [21] the authors considered the case $1 < q < p < N$, but there h, g are positive constants. In [4, 21], the nonlinearity $f(x,s)$ was suitably controlled by the variable s as $s \to 0$ and also as $|s| \to \infty$, uniformly with respect to the variable x. In [20], the authors discussed the case that $1 < q < p < q^*$, $p < N$ with h, g continuous, positive, and coercive functions on $\mathbb{R}^N$ and $f(x,s)$ a Carathéodory function satisfying some conditions.

To the best of our knowledge, there is little work researching the sign-changing solution and the ground state of the (p,q)-Laplacian equations (1.1). Recently, Shuai in [23] discussed the following Kirchhoff type problem:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2)\Delta u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Motivated by [23], we investigate the sign-changing solution and the ground state solution of (p,q)-Laplacian equation with nonlocal terms.

In general, the working space to study (p,q)-Laplacian problems in a bounded domain Ω is $W_0^{1,p}(\Omega)$, by taking advantage of the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ for all $s \in [1, p^*)$. When the domain is the whole $\mathbb{R}^N$, Sobolev's embedding loses compactness. In order to overcome these difficulties, various methods have been developed. The radically symmetric Sobolev spaces have been applied to (1.3) (see [24, 25]), and the concentration-compactness principle or the constrained minimization method has been used to find solutions in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ (see [1, 3, 4, 21, 26]).

In this paper, we intend to choose an appropriate approach by taking into account the Banach space,

$$W = \left\{ u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \cap \mathcal{D}^{1,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} h|u|^p, \int_{\mathbb{R}^N} g|u|^q < \infty \right\}.$$

We recall that the space $\mathcal{D}^{1,m}(\mathbb{R}^N)$ is a reflexive Banach space which is characterized by (see [11])

$$\mathcal{D}^{1,m}(\mathbb{R}^N) = \left\{ u \in L^{m^*}(\mathbb{R}^N) : \frac{\partial u}{\partial x_i} \in L^m(\mathbb{R}^N) \right\}$$

and its norm is equivalent to the norm $\|\nabla u\|_{L^m(\mathbb{R}^N)}$. We denote the norm of $L^m(\mathbb{R}^N)$ as $|\cdot|_m$ hereafter. Moreover, $W^{1,m}(\mathbb{R}^N) \subset \mathcal{D}^{1,m}(\mathbb{R}^N) \hookrightarrow L^{m^*}(\mathbb{R}^N)$.

We take h, g as continuous, coercive, and positive functions on $\mathbb{R}^N$ and define normed spaces $(W_{p,a,h}, \|\cdot\|_1)$ and $(W_{q,b,g}, \|\cdot\|_2)$, respectively, by

$$W_{p,a,h} = \left\{ u \in \mathcal{D}^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} h|u|^p < \infty \right\},$$
$$W_{q,b,g} = \left\{ u \in \mathcal{D}^{1,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} g|u|^q < \infty \right\},$$

with norms

$$\|u\|_1 = \left(\int_{\mathbb{R}^N} \left[a|\nabla u|^p + h|u|^p \right] \right)^{1/p},$$
$$\|u\|_2 = \left(\int_{\mathbb{R}^N} \left[b|\nabla u|^q + g|u|^q \right] \right)^{1/q}.$$

Then $W_{p,a,h}$ and $W_{q,b,g}$ are reflexive Banach spaces. The embedding $W_{p,a,h} \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for all $s \in [p, p^*]$ and compact for all $s \in [p, p^*)$. Similarly, the embedding $W_{q,b,g} \hookrightarrow L^s(\mathbb{R}^N)$ is continuous if $s \in [q, q^*]$ and compact if $s \in [q, q^*)$ (see [20]).

Now we can define our working space W:

$$W = W_{p,a,h} \cap W_{q,b,g}$$

endowed with the norm

$$\|u\| = \|u\|_1 + \|u\|_2.$$

Then it is easy to see that W is a reflexive Banach space and the embedding $W \hookrightarrow L^s(\mathbb{R}^N)$ is continuous if $s \in [q, p^*]$ and compact if $s \in [q, p^*)$.

For brevity, we omit the integral domain $\mathbb{R}^N$ when no confusion arises hereafter.

We assume that $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the following hypotheses:

(f_1) $\lim_{s\to 0} f(s)/|s|^{q-1} = 0$;
(f_2) for some constant $r \in (2p, p^*)$, $\lim_{|s|\to\infty} f(s)/|s|^{r-1} = 0$;
(f_3) $\lim_{|s|\to\infty} F(s)/|s|^{2p} = \infty$, where $F(s) = \int_0^t f(t)\,\mathrm{d}t$ for all $s \in \mathbb{R}$;
(f_4) $f(s)/|s|^{2p-1}$ is increasing on $(-\infty, 0)$ and $(0, \infty)$, respectively.

Define the energy functional $I : W \to \mathbb{R}$ of (1.1) by

$$I(u) = \frac{1}{p}\|u\|_1^p + \frac{1}{q}\|u\|_2^q + \frac{c}{2p}\left(\int |\nabla u|^p\right)^2 + \frac{d}{2q}\left(\int |\nabla u|^q\right)^2 - \int F(u), \quad u \in W. \quad (1.4)$$

Then the functional I is well defined on W and belongs to $C^1(W, \mathbb{R})$. Moreover, for any $u, \varphi \in W$, we have

$$\begin{aligned}\langle I'(u), \varphi\rangle = {} & \int \left[a|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + h|u|^{p-2}u\varphi\right] + \int \left[b|\nabla u|^{q-2}\nabla u \cdot \nabla\varphi + g|u|^{q-2}u\varphi\right] \\ & + c\int |\nabla u|^p \int |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \\ & + d\int |\nabla u|^q \int |\nabla u|^{q-2}\nabla u \cdot \nabla\varphi - \int f(u)\varphi. \end{aligned} \quad (1.5)$$

A critical point of I corresponds to a solution of (1.1). Furthermore, if $u \in W$ is a solution of (1.1) with $u^\pm \neq 0$, then u is a sign-changing solution of (1.1), where

$$u^+(x) = \max\{u(x), 0\}, \qquad u^-(x) = \min\{u(x), 0\}.$$

Obviously, the energy functional $I_0 : W \to \mathbb{R}$ of (1.2) is given by

$$I_0(u) = \frac{1}{p}\|u\|_1^p + \frac{1}{q}\|u\|_2^q - \int F(u), \quad u \in W.$$

For $u \in W$,

$$I_0(u) = I_0(u^+) + I_0(u^-), \qquad \langle I_0'(u), u^\pm\rangle = \langle I_0'(u^\pm), u^\pm\rangle. \quad (1.6)$$

When $c, d > 0$, the nonlocal terms $(\int |\nabla u|^p)\Delta_p u$, $(\int |\nabla u|^q)\Delta_q u$ are involved in equation (1.1), for the functional I given by (1.4) it is apparent that

$$I(u) = I(u^+) + I(u^-) + \frac{c}{p}\int |\nabla u^+|^p \int |\nabla u^-|^p + \frac{d}{q}\int |\nabla u^+|^q \int |\nabla u^-|^q, \quad (1.7)$$

$$\langle I'(u), u^\pm\rangle = \langle I'(u^\pm), u^\pm\rangle + c\int |\nabla u^+|^p \int |\nabla u^-|^p + d\int |\nabla u^+|^q \int |\nabla u^-|^q. \quad (1.8)$$

Clearly, the functional I does no longer satisfy (1.6), since it contains two nonlocal terms. Hence, there may be some differences in investigating the sign-changing solution of equation (1.1) between $c, d > 0$ and $c = d = 0$.

In order to obtain a sign-changing solution of equation (1.1), we try to seek a minimizer of the functional I over the following constraint:

$$\mathcal{M} = \{u \in W : u^\pm \neq 0, \langle I'(u), u^+\rangle = \langle I'(u), u^-\rangle = 0\} \quad (1.9)$$

and

$$m = \inf\{I(u) : u \in \mathcal{M}\}. \quad (1.10)$$

Then we show that the minimizer is indeed a sign-changing solution of (1.1). As we have mentioned before, the functional I no longer satisfies the properties (1.6), so it is more difficult to prove that $\mathcal{M} \neq \emptyset$. Actually, we will obtain $\mathcal{M} \neq \emptyset$ by using the Brouwer fixed point theorem, which is different from the approach in [23].

In order to get the ground state solution of equation (1.1), let

$$\mathcal{N} = \left\{u \in W \setminus \{0\} : \langle I'(u), u\rangle = 0\right\}, \tag{1.11}$$

and consider the ground state energy

$$\tilde{m} = \inf\left\{I(u) : u \in \mathcal{N}\right\}. \tag{1.12}$$

Now, we state our main results.

Theorem 1.1 *If the assumptions* (f_1)-(f_4) *hold, then equation* (1.1) *has one least energy sign-changing solution.*

Theorem 1.2 *Suppose the assumptions* (f_1)-(f_4) *hold. For any sequence* $\{(c_n, d_n)\}$ *with* $c_n, d_n \geq 0$, *as* $(c_n, d_n) \to 0$, *there exists a subsequence, still denoted by* $\{(c_n, d_n)\}$, *such that* $u_{c_n,d_n} \to u_0$ *in* W, *and* u_0 *is a least energy sign-changing solution of equation* (1.2).

Theorem 1.3 *Suppose the assumptions* (f_1)-(f_4) *hold.*

(i) *There exists a ground state solution* v *of equation* (1.1).

(ii) $m > 2\tilde{m}$. *In particular, the ground state solution must maintain the sign unchanged.*

Remark 1.4 The three results above are also valid for (p, q)-Laplacian problems in a bounded domain Ω. Consider the following two problems:

$$\begin{cases} -(a + c\int_\Omega |\nabla u|^p)\Delta_p u - (b + d\int_\Omega |\nabla u|^q)\Delta_q u \\ \quad + h(x)|u|^{p-2}u + g(x)|u|^{q-2}u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.13}$$

and

$$\begin{cases} -a\Delta_p u - b\Delta_q u + h(x)|u|^{p-2}u + g(x)|u|^{q-2}u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, h, g are continuous and non-negative functions, including the case $h \equiv g \equiv 0$. Because the embedding $W_0^{1,m}(\Omega) \hookrightarrow L^s(\Omega)$ is continuous if $s \in [1, m^*]$ and compact if $s \in [1, m^*)$, we find solutions in the space $W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega)$, and then can also obtain the same conclusions as Theorems 1.1-1.3 for (1.13).

Both the conclusions of (1.1) and of (1.13) are true when $p = q$, *i.e.*, these results are true for a single p-Laplacian equation with nonlocal term.

The paper is organized as follows. In Section 2, we prove several lemmas, which are important to prove our main results. In Section 3, we first show that the minimizer of the constrained problem (1.9) is a sign-changing solution. Then we prove the convergence property of solutions of (1.1). Finally, we prove the existence of the ground state solution and give the energy comparison.

Throughout this paper, C and C_k denote various positive constants, which may vary from line to line.

2 Preliminaries

We use constraint minimization on $\mathcal{M}$ to seek a critical point of I. We begin this section by doing some preparation work.

Lemma 2.1 *Assume that* (f_1)-(f_4) *hold. If* $u \in W$ *with* $u \neq 0$, *then*

(i) $\lim_{s\to 0} \int \frac{f(su)u}{|s|^{q-1}} = 0$;
(ii) $\lim_{|s|\to\infty} \int \frac{f(su)u}{|s|^{2p-2}s} = \infty$;
(iii) $\lim_{|s|\to\infty} \int \frac{F(su)}{|s|^{2p}} = \infty$;
(iv) *moreover, if* $u^{\pm} \neq 0$, *then* $\lim_{|(s,t)|\to\infty} \int \frac{F(su^+)+F(tu^-)}{|s|^{2p}+|t|^{2p}} = \infty$.

Proof (i) By the conditions (f_1) and (f_2), for any given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\bigl|f(s)\bigr| \le \varepsilon |s|^{q-1} + C_\varepsilon |s|^{r-1}, \quad s \in \mathbb{R}, \tag{2.1}$$

$$\bigl|F(s)\bigr| \le \frac{\varepsilon}{q}|s|^q + \frac{C_\varepsilon}{r}|s|^r, \quad s \in \mathbb{R}. \tag{2.2}$$

By the condition (f_1), we have, for each $\eta \in \mathbb{R}$,

$$\lim_{s\to 0} \frac{f(s\eta)}{|s|^{q-1}} = 0. \tag{2.3}$$

Thus, by (2.1), (2.3) and the Lebesgue dominated convergence theorem, the conclusion (i) holds.

(ii) By the conditions (f_4) and (f_3), we have

$$\lim_{|s|\to\infty} \frac{f(s)}{|s|^{2p-2}s} = \infty. \tag{2.4}$$

It follows from (2.4) that, for any given $M_1 > 0$, there exists $R_1 > 0$ such that

$$\frac{f(s)s}{|s|^{2p}} \ge M_1, \quad |s| > R_1. \tag{2.5}$$

By the condition (f_1), we have

$$\lim_{s\to 0} \frac{f(s)s - M_1|s|^{2p}}{|s|^q} = 0.$$

Then there exists $C_{M_1} > 0$ such that

$$\frac{f(s)s - M_1|s|^{2p}}{|s|^q} \ge -C_{M_1}, \quad |s| \in (0, R_1]. \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$f(s)s \geq M_1|s|^{2p} - C_{M_1}|s|^q, \quad s \in \mathbb{R}. \tag{2.7}$$

It follows from (2.7) that

$$\liminf_{|s|\to\infty} \int \frac{f(su)u}{|s|^{2p-2}s} \geq M_1 \int |u|^{2p} - \lim_{|s|\to\infty} \frac{C_{M_1}}{|s|^{2p-q}} \int |u|^q = M_1 \int |u|^{2p}.$$

Then, by the arbitrariness of M_1, the conclusion (ii) is true.

(iii) By the condition (f_3), for any given $M_2 > 0$, there exists $R_2 > 0$ such that

$$\frac{F(s)}{|s|^{2p}} \geq M_2, \quad |s| > R_2. \tag{2.8}$$

By the condition (f_1), we have

$$\lim_{s\to 0} \frac{F(s) - M_2|s|^{2p}}{|s|^q} = 0.$$

Then there exists $C_{M_2} > 0$ such that

$$\frac{F(s) - M_2|s|^{2p}}{|s|^q} \geq -C_{M_2}, \quad |s| \in (0, R_2]. \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$F(s) \geq M_2|s|^{2p} - C_{M_2}|s|^q, \quad s \in \mathbb{R}. \tag{2.10}$$

Then it follows from (2.10) that

$$\liminf_{|s|\to\infty} \int \frac{F(su)}{|s|^{2p}} \geq M_2 \int |u|^{2p} - \lim_{|s|\to\infty} \frac{C_{M_2}}{|s|^{2p-q}} \int |u|^q = M_2 \int |u|^{2p}.$$

Thus, by the arbitrariness of M_2, the conclusion (iii) is also true.

(iv) For convenience, we denote functions $\psi_1(s) = \int F(su^+)$ and $\psi_2(s) = \int F(su^-)$ for all $s \in \mathbb{R}$. Then, by (iii), we have

$$\psi_1(s) \to \infty, \qquad \psi_2(s) \to \infty, \quad |s| \to \infty. \tag{2.11}$$

Because of (2.11) and the continuity of ψ_1, ψ_2, there exists $C > 0$ such that

$$\psi_1(s) \geq -C, \qquad \psi_2(s) \geq -C, \quad s \in \mathbb{R}. \tag{2.12}$$

By (iii), for any given $M > 0$, there exists $R > 0$ such that

$$\frac{\psi_1(s) + C}{|s|^{2p}} \geq 2M, \qquad \frac{\psi_2(s) + C}{|s|^{2p}} \geq 2M, \quad |s| \geq R. \tag{2.13}$$

When $|(s,t)| = \sqrt{s^2+t^2} \geq \sqrt{2}R$, by the inequality

$$\sqrt{s^2+t^2} \leq \sqrt{2}\max\{|s|,|t|\},$$

$\max\{|s|,|t|\} \geq R$. We may suppose that $|s| \geq |t|$, so that $|s| \geq R$. Combining with (2.12) and (2.13), we have

$$\frac{\psi_1(s)+C+\psi_2(t)+C}{|s|^{2p}+|t|^{2p}} \geq \frac{\psi_1(s)+C}{2|s|^{2p}} \geq M.$$

Then we have

$$\lim_{|(s,t)|\to\infty} \frac{\psi_1(s)+C+\psi_2(t)+C}{|s|^{2p}+|t|^{2p}} = \infty. \tag{2.14}$$

Therefore, it follows from (2.14) that (iv) holds. □

Remark 2.2 By the condition (f_4), for each $\eta \in \mathbb{R}\setminus\{0\}$, we see that $f(s\eta)\eta/|s|^{2p-1}$ is increasing on $(-\infty,0)$ and $(0,\infty)$, respectively. Therefore, for each $u \in W$ with $u \neq 0$, we see that $\int \frac{f(su)u}{s^{2p-1}}$ is increasing on $(0,\infty)$.

Now we start to check that the set $\mathcal{M}$ is nonempty.

For each $u \in W$ with $u^{\pm} \neq 0$, for convenience, we denote the positive numbers $A_{1,u} = (\int |\nabla u^+|^p)^2$, $A_{2,u} = (\int |\nabla u^-|^p)^2$, $A_{3,u} = \int |\nabla u^+|^p \int |\nabla u^-|^p$; $B_{1,u} = (\int |\nabla u^+|^q)^2$, $B_{2,u} = (\int |\nabla u^-|^q)^2$, $B_{3,u} = \int |\nabla u^+|^q \int |\nabla u^-|^q$.

Lemma 2.3 *Assume that* (f_1)-(f_4) *hold. If $u \in W$ with $u^{\pm} \neq 0$, then there is a unique pair (s_u,t_u) of positive numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}$.*

Proof For any given $u \in W$ with $u^{\pm} \neq 0$, we define a function $\Psi_u : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ by $\Psi_u(s,t) = I(su^+ + tu^-)$, where $\mathbb{R}_+ = [0,\infty)$, that is,

$$\begin{aligned}\Psi_u(s,t) = {}& \frac{1}{p}s^p\|u^+\|_1^p + \frac{1}{q}s^q\|u^+\|_2^q + \frac{c}{2p}A_{1,u}s^{2p} + \frac{c}{p}A_{3,u}s^pt^p \\ & + \frac{d}{2q}B_{1,u}s^{2q} + \frac{d}{q}B_{3,u}s^qt^q - \int F(su^+) \\ & + \frac{1}{p}t^p\|u^-\|_1^p + \frac{1}{q}t^q\|u^-\|_2^q + \frac{c}{2p}A_{2,u}t^{2p} \\ & + \frac{d}{2q}B_{2,u}t^{2q} - \int F(tu^-).\end{aligned} \tag{2.15}$$

For $s,t>0$, since

$$\begin{aligned}\nabla\Psi_u(s,t) &= \left(\frac{\partial\Psi_u}{\partial s}(s,t), \frac{\partial\Psi_u}{\partial t}(s,t)\right) \\ &= \left(\langle I'(su^+ + tu^-), u^+\rangle, \langle I'(su^+ + tu^-), u^-\rangle\right) \\ &= \left(\frac{1}{s}\langle I'(su^+ + tu^-), su^+\rangle, \frac{1}{t}\langle I'(su^+ + tu^-), tu^-\rangle\right),\end{aligned}$$

we have $su^{+} + tu^{-} \in \mathcal{M}$ if and only if (s,t) is a critical point of Ψ_u. Next we will prove the existence of a critical point of Ψ_u.

For any given $t_0 \in \mathbb{R}_+$, we have, for $s > 0$,

$$\begin{aligned}
&\frac{\partial}{\partial s}\Psi_u(s,t_0) \\
&\quad = s^{p-1}\|u^+\|_1^p + s^{q-1}\|u^+\|_2^q + cA_{1,u}s^{2p-1} + cA_{3,u}s^{p-1}t_0^p \\
&\qquad + dB_{1,u}s^{2q-1} + dB_{3,u}s^{q-1}t_0^q - \int f(su^+)u^+ \\
&\quad = s^{q-1}\biggl[s^{p-q}\|u^+\|_1^p + \|u^+\|_2^q + cA_{1,u}s^{2p-q} + cA_{3,u}s^{p-q}t_0^p \\
&\qquad + dB_{1,u}s^q + dB_{3,u}t_0^q - \int \frac{f(su^+)u^+}{s^{q-1}}\biggr] && (2.16) \\
&\quad = s^{2p-1}\biggl[\frac{1}{s^p}\|u^+\|_1^p + \frac{1}{s^{2p-q}}\|u^+\|_2^q + cA_{1,u} + \frac{t_0^p}{s^p}cA_{3,u} \\
&\qquad + \frac{dB_{1,u}}{s^{2p-2q}} + \frac{t_0^q dB_{3,u}}{s^{2p-q}} - \int \frac{f(su^+)u^+}{s^{2p-1}}\biggr]. && (2.17)
\end{aligned}$$

Since $u^+ \neq 0$, it follows from (2.16) and Lemma 2.1(i) that $\frac{\partial}{\partial s}\Psi_u(s,t_0) > 0$ for $s > 0$ small. It follows from (2.17) and Lemma 2.1(ii) that $\frac{\partial}{\partial s}\Psi_u(s,t_0) < 0$ for $s > 0$ large. Thus there exists $s_0 > 0$ such that $\frac{\partial}{\partial s}\Psi_u(s_0,t_0) = 0$.

Suppose that there exist s_1, s_2 with $0 < s_1 < s_2$ such that $\frac{\partial}{\partial s}\Psi_u(s_1,t_0) = \frac{\partial}{\partial s}\Psi_u(s_2,t_0) = 0$. Then (2.17) implies that

$$\begin{aligned}
&\frac{1}{s_i^p}\|u^+\|_1^p + \frac{1}{s_i^{2p-q}}\|u^+\|_2^q + cA_{1,u} + \frac{t_0^p}{s_i^p}cA_{3,u} + \frac{dB_{1,u}}{s_i^{2p-2q}} + \frac{t_0^q dB_{3,u}}{s_i^{2p-q}} \\
&\quad = \int \frac{f(s_iu^+)u^+}{s_i^{2p-1}}, \quad i = 1,2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\biggl(\frac{1}{s_1^p} - \frac{1}{s_2^p}\biggr)\|u^+\|_1^p + \biggl(\frac{1}{s_1^{2p-q}} - \frac{1}{s_2^{2p-q}}\biggr)\|u^+\|_2^q + \biggl(\frac{1}{s_1^p} - \frac{1}{s_2^p}\biggr)t_0^p cA_{3,u} \\
&\qquad + \biggl(\frac{1}{s_1^{2p-2q}} - \frac{1}{s_2^{2p-2q}}\biggr)dB_{1,u} + \biggl(\frac{1}{s_1^{2p-q}} - \frac{1}{s_2^{2p-q}}\biggr)t_0^q dB_{3,u} \\
&\quad = \int \biggl[\frac{f(s_1u^+)u^+}{s_1^{2p-1}} - \frac{f(s_2u^+)u^+}{s_2^{2p-1}}\biggr]. && (2.18)
\end{aligned}$$

But according to Remark 2.2, the right side of (2.18) is negative and (2.18) is absurd. Therefore there exists a unique $s_0 = s_0(t_0) > 0$ such that $\frac{\partial}{\partial s}\Psi_u(s_0,t_0) = 0$.

Now we can define a map $\varphi_1 : \mathbb{R}_+ \to (0,\infty)$ by $\varphi_1(t) = s(t)$, where $s(t)$ satisfies the properties just mentioned previously, with t in the place of t_0. By definition, we have

$$\frac{\partial \Psi_u}{\partial s}\bigl(\varphi_1(t),t\bigr) = 0, \quad t \in \mathbb{R}_+,$$

that is, for $t \geq 0$,

$$\begin{aligned}
&\varphi_1^{p-1}(t)\|u^+\|_1^p + \varphi_1^{q-1}(t)\|u^+\|_2^q + cA_{1,u}\varphi_1^{2p-1}(t) + cA_{3,u}\varphi_1^{p-1}(t)t^p \\
&\quad + dB_{1,u}\varphi_1^{2q-1}(t) + dB_{3,u}\varphi_1^{q-1}(t)t^q \\
&\quad = \int f(\varphi_1(t)u^+)u^+. \qquad (2.19)
\end{aligned}$$

We will prove some properties of the function φ_1.

(a_1) φ_1 has a positive lower bound.

In fact, suppose there exists $\{t_n\} \subset \mathbb{R}_+$ such that $\varphi_1(t_n) \to 0$. Then, by (2.19) and Lemma 2.1(i), we have

$$\|u^+\|_2^q \leq \lim_{n\to\infty} \int \frac{f(\varphi_1(t_n)u^+)u^+}{\varphi_1^{q-1}(t_n)} = 0.$$

This is absurd. Thus there exists $C > 0$ such that $\varphi_1(s) \geq C$ for all $s \in \mathbb{R}_+$.

(a_2) φ_1 is continuous.

In fact, let $t_n \to t_0$ in $\mathbb{R}_+$. We firstly prove that $\{\varphi_1(t_n)\}$ is bounded. Suppose, by contradiction, that there is a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\varphi_1(t_{n_k}) \to \infty$. It follows from (2.19) that

$$\begin{aligned}
&\frac{1}{\varphi_1^p(t_{n_k})}\|u^+\|_1^p + \frac{1}{\varphi_1^{2p-q}(t_{n_k})}\|u^+\|_2^q + cA_{1,u} + \frac{t_{n_k}^p}{\varphi_1^p(t_{n_k})}cA_{3,u} + \frac{dB_{1,u}}{\varphi_1^{2p-2q}(t_{n_k})} + \frac{t_{n_k}^q dB_{3,u}}{\varphi_1^{2p-q}(t_{n_k})} \\
&\quad = \int \frac{f(\varphi_1(t_{n_k})u^+)}{\varphi_1^{2p-1}(t_{n_k})}u^+. \qquad (2.20)
\end{aligned}$$

Letting $k \to \infty$ in (2.20), according to Lemma 2.1(ii), we have a contradiction $cA_{1,u} = \infty$. Thus, $\{\varphi_1(t_n)\}$ is bounded. For any subsequence $\{\varphi_1(t_n')\}$ of $\{\varphi_1(t_n)\}$, since $\{\varphi_1(t_n')\}$ is bounded, there exists a subsequence $\{\varphi_1(t_n'')\}$ of $\{\varphi_1(t_n')\}$ such that $\varphi_1(t_n'') \to s_0$ and it follows from (a_1) that $s_0 > 0$. Passing to the limit as $n \to \infty$ in (2.19) with $t = t_n''$, we get

$$\begin{aligned}
&s_0^{p-1}\|u^+\|_1^p + s_0^{q-1}\|u^+\|_2^q + cA_{1,u}s_0^{2p-1} + cA_{3,u}s_0^{p-1}t_0^p + dB_{1,u}s_0^{2q-1} + dB_{3,u}s_0^{q-1}t_0^q \\
&\quad = \int f(s_0u^+)u^+. \qquad (2.21)
\end{aligned}$$

Thus (2.17) and (2.21) imply

$$\frac{\partial\Psi_u}{\partial s}(s_0, t_0) = 0.$$

Consequently, by the uniqueness, $s_0 = \varphi_1(t_0)$. Therefore φ_1 is continuous.

(a_3) $\varphi_1(t) \leq t$ for t large.

In fact, if there exists a sequence $\{t_n\}$ with $t_n \to \infty$ such that $\varphi_1(t_n) > t_n$ for all $n \in \mathbb{N}$, then $\varphi_1(t_n) \to \infty$ and it follows from (2.20) that $\infty \leq cA_{1,u} + cA_{3,u}$. This is a contradiction. Thus $\varphi_1(t) \leq t$ for t large.

Similarly, for each $s \in \mathbb{R}_+$, we consider the function $\Psi_u(s, \cdot)$ and consequently, we can define a map $\varphi_2 : \mathbb{R}_+ \to (0, \infty)$ which satisfies

$$\frac{\partial\Psi_u}{\partial t}(s, \varphi_2(s)) = 0, \quad s \in \mathbb{R}_+,$$

that is, for $s \geq 0$,

$$\begin{aligned}
&\varphi_2^{p-1}(s)\|u^-\|_1^p + \varphi_2^{q-1}(s)\|u^-\|_2^q + cA_{2,u}\varphi_2^{2p-1}(s) + cA_{3,u}\varphi_2^{p-1}(s)s^p \\
&\quad + dB_{2,u}\varphi_2^{2q-1}(s) + dB_{3,u}\varphi_2^{q-1}(s)s^q \\
&= \int f\bigl(\varphi_2(s)u^-\bigr)u^-, \qquad (2.22)
\end{aligned}$$

and it also satisfies (a_1), (a_2), and (a_3) above.

Now we prove the existence of a critical point of Ψ_u by the Brouwer fixed point theorem. By (a_3), there exists $C_1 > 0$ such that $\varphi_1(t) \leq t$ for all $t > C_1$ and $\varphi_2(s) \leq s$ for all $s > C_1$. Let

$$C_2 = \max\Bigl\{\max_{t\in[0,C_1]} \varphi_1(t), \max_{s\in[0,C_1]} \varphi_2(s)\Bigr\}.$$

Let $\xi = \max\{C_1, C_2\}$. We define $T : [0,\xi] \to \mathbb{R}_+$ as $T(s) = \varphi_1(\varphi_2(s))$. Now we show $T(s) \in [0,\xi]$ for all $s \in [0,\xi]$. In fact, let $0 \leq s \leq \xi = \max\{C_1, C_2\}$. If $t = \varphi_2(s) > C_1$, then

$$T(s) = \varphi_1(t) \leq t = \varphi_2(s) \leq \begin{cases} s, & s > C_1, \\ \max_{s\in[0,C_1]} \varphi_2(s), & s \leq C_1, \end{cases}$$

so

$$T(s) \leq \max\{C_1, C_2\}.$$

If $t = \varphi_2(s) \leq C_1$, then

$$T(s) = \varphi_1(t) \leq \max_{t\in[0,C_1]} \varphi_1(t) \leq C_2.$$

Note that T is continuous. Then, by the Brouwer fixed point theorem, there exists $s_u \in [0,\xi]$ such that $\varphi_1(\varphi_2(s_u)) = s_u$. Let $t_u = \varphi_2(s_u)$. Then we have

$$s_u = \varphi_1(t_u), \qquad t_u = \varphi_2(s_u). \qquad (2.23)$$

Since $\varphi_i > 0$, (2.23) implies $s_u, t_u > 0$. By the definition we have

$$\frac{\partial \Psi_u}{\partial s}(s_u, t_u) = \frac{\partial \Psi_u}{\partial t}(s_u, t_u) = 0.$$

Thus, (s_u, t_u) is a critical point of Ψ_u.

Now we prove the uniqueness of (s_u, t_u). In fact, considering $w \in \mathcal{M}$ we have

$$\begin{aligned}
\nabla\Psi_w(1,1) &= \left(\frac{\partial \Psi_w}{\partial s}(1,1), \frac{\partial \Psi_w}{\partial t}(1,1)\right) \\
&= \bigl(\langle I'(w^+ + w^-), w^+\rangle, \langle I'(w^+ + w^-), w^-\rangle\bigr) = (0,0),
\end{aligned}$$

which implies that $(1,1)$ is a critical point of Ψ_w. Now we prove that $(1,1)$ is the unique critical point of Ψ_w with positive coordinates. In fact, we may suppose that (s_0, t_0) is also a critical point of Ψ_w with $0 < t_0 \leq s_0$. Then it follows from (2.19) and (2.22) that

$$\begin{aligned} & s_0^p \|w^+\|_1^p + s_0^q \|w^+\|_2^q + cA_{1,w}s_0^{2p} + cA_{3,w}s_0^p t_0^p + dB_{1,w}s_0^{2q} + dB_{3,w}s_0^q t_0^q \\ &\quad = \int f(s_0 w^+) s_0 w^+, && (2.24) \\ & t_0^p \|w^-\|_1^p + t_0^q \|w^-\|_2^q + cA_{2,w}t_0^{2p} + cA_{3,w}s_0^p t_0^p + dB_{2,w}t_0^{2q} + dB_{3,w}s_0^q t_0^q \\ &\quad = \int f(t_0 w^-) t_0 w^-. && (2.25) \end{aligned}$$

From (2.24) and $t_0 \leq s_0$, we have

$$s_0^p \|w^+\|_1^p + s_0^q \|w^+\|_2^q + c(A_{1,w} + A_{3,w})s_0^{2p} + d(B_{1,w} + B_{3,w})s_0^{2q} \geq \int f(s_0 w^+) s_0 w^+. \tag{2.26}$$

On the other hand, since $w \in \mathcal{M}$, we have

$$\|w^+\|_1^p + \|w^+\|_2^q + cA_{1,w} + cA_{3,w} + dB_{1,w} + dB_{3,w} = \int f(w^+) w^+. \tag{2.27}$$

Hence, from (2.26) and (2.27), we get

$$\begin{aligned} & \left(1 - \frac{1}{s_0^p}\right) \|w^+\|_1^p + \left(1 - \frac{1}{s_0^{2p-q}}\right) \|w^+\|_2^q + \left(1 - \frac{1}{s_0^{2p-2q}}\right) d(B_{1,w} + B_{3,w}) \\ &\quad \leq \int \left[f(w^+) w^+ - \frac{f(s_0 w^+) w^+}{s_0^{2p-1}} \right]. \end{aligned}$$

From the above inequality and Remark 2.2 we conclude that $s_0 \leq 1$ and then $0 < t_0 \leq s_0 \leq 1$.

Now we prove that $t_0 \geq 1$. In fact, from (2.25) and $0 < t_0 \leq s_0$, we have

$$t_0^p \|w^-\|_1^p + t_0^q \|w^-\|_2^q + c(A_{2,w} + A_{3,w})t_0^{2p} + d(B_{2,w} + B_{3,w})t_0^{2q} \leq \int f(t_0 w^-) t_0 w^-. \tag{2.28}$$

On the other hand, since $w \in \mathcal{M}$, we get

$$\|w^-\|_1^p + \|w^-\|_2^q + cA_{2,w} + cA_{3,w} + dB_{2,w} + dB_{3,w} = \int f(w^-) w^-. \tag{2.29}$$

Now from (2.28) and (2.29), we obtain

$$\begin{aligned} & \left(1 - \frac{1}{t_0^p}\right) \|w^-\|_1^p + \left(1 - \frac{1}{t_0^{2p-q}}\right) \|w^-\|_2^q + \left(1 - \frac{1}{t_0^{2p-2q}}\right) d(B_{2,w} + B_{3,w}) \\ &\quad \geq \int \left[f(w^-) w^- - \frac{f(t_0 w^-) w^-}{t_0^{2p-1}} \right]. \end{aligned}$$

By Remark 2.2, we conclude that $t_0 \geq 1$. Consequently, $t_0 = s_0 = 1$, this shows that $(1, 1)$ is the unique critical point of Ψ_w with positive coordinates.

Now we assume that $u \in W$ with $u^{\pm} \neq 0$ and (s_1, t_1), (s_2, t_2) are both critical points with positive coordinates for the map Ψ_u. Then

$$w_1 = s_1 u^+ + t_1 u^- \in \mathcal{M}, \qquad w_2 = s_2 u^+ + t_2 u^- \in \mathcal{M}.$$

Therefore,

$$w_2 = \left(\frac{s_2}{s_1}\right)s_1u^+ + \left(\frac{t_2}{t_1}\right)t_1u^- = \left(\frac{s_2}{s_1}\right)w_1^+ + \left(\frac{t_2}{t_1}\right)w_1^- \in \mathcal{M}.$$

Since $w_1 \in \mathcal{M}$ and $(\frac{s_2}{s_1}, \frac{t_2}{t_1})$ is a critical point of the map Ψ_{w_1} with positive coordinates, by the uniqueness we have

$$\frac{s_2}{s_1} = \frac{t_2}{t_1} = 1,$$

which implies that $(s_1, t_1) = (s_2, t_2)$. □

Lemma 2.4 *For a fixed $u \in W$ with $u^{\pm} \neq 0$, the vector (s_u, t_u), which was obtained in Lemma* 2.3, *is the unique maximum point of the function $\Psi_u(s,t)$.*

Proof From the proof of Lemma 2.3, (s_u, t_u) is the unique critical point of Ψ_u in $(0,\infty) \times (0,\infty)$. By (2.15), we have

$$\begin{aligned}
\Psi_u(s,t) \;=\; & \left(s^{2p} + t^{2p}\right)\left[\frac{1}{p}\frac{s^p}{s^{2p}+t^{2p}}\|u^+\|_1^p + \frac{1}{q}\frac{s^q}{s^{2p}+t^{2p}}\|u^+\|_2^q + \frac{1}{p}\frac{t^p}{s^{2p}+t^{2p}}\|u^-\|_1^p \right.\\
&\left. + \frac{1}{q}\frac{t^q}{s^{2p}+t^{2p}}\|u^-\|_2^q\right]\\
& + \left(s^{2p} + t^{2p}\right)\left[\frac{d}{2q}B_{1,u}\frac{s^{2q}}{s^{2p}+t^{2p}} + \frac{d}{q}B_{3,u}\frac{s^q t^q}{s^{2p}+t^{2p}} + \frac{d}{2q}B_{2,u}\frac{t^{2q}}{s^{2p}+t^{2p}}\right]\\
& + \left(s^{2p} + t^{2p}\right)\left[\frac{c}{2p}A_{1,u}\frac{s^{2p}}{s^{2p}+t^{2p}} + \frac{c}{p}A_{3,u}\frac{s^p t^p}{s^{2p}+t^{2p}} + \frac{c}{2p}A_{2,u}\frac{t^{2p}}{s^{2p}+t^{2p}}\right]\\
& - \left(s^{2p} + t^{2p}\right)\int \frac{F(su^+) + F(tu^-)}{s^{2p}+t^{2p}}\\
:= & \left(s^{2p} + t^{2p}\right)\left[\Psi_1(s,t) + \Psi_2(s,t) + \Psi_3(s,t) - \int \frac{F(su^+) + F(tu^-)}{s^{2p}+t^{2p}}\right].
\end{aligned}$$

It is clear that $\Psi_1(s,t), \Psi_2(s,t) \to 0$ as $|(s,t)| \to \infty$ and $\Psi_3(s,t)$ is bounded. Then, by Lemma 2.1(iv), we deduce that $\Psi_u(s,t) \to -\infty$ as $|(s,t)| \to \infty$. So it is sufficient to check that a maximum point cannot be obtained on the boundary of $\mathbb{R}_+ \times \mathbb{R}_+$. Without loss of generality, we may assume that $(0, \bar{t})$ is a maximum point of Ψ_u. Similar to (2.16), we can get $\frac{\partial}{\partial s}\Psi_u(s,\bar{t}) > 0$ for s small. Then $\Psi_u(s,\bar{t})$ is an increasing function with respect to s if s is small enough, the pair $(0,\bar{t})$ is not a maximum point of Ψ_u in $\mathbb{R}_+ \times \mathbb{R}_+$. □

Lemma 2.5 *Let* (f_1)-(f_4) *hold. Suppose that $u \in W$ with $u^{\pm} \neq 0$ such that $\langle I'(u), u^+\rangle \leq 0$, $\langle I'(u), u^-\rangle \leq 0$. Then the unique pair (s_u, t_u) of positive numbers obtained in Lemma* 2.3 *satisfies $0 < s_u, t_u \leq 1$.*

Proof We may suppose that $s_u \geq t_u > 0$. Since $s_u u^+ + t_u u^- \in \mathcal{M}$,

$$\begin{aligned}
& s_u^p\|u^+\|_1^p + s_u^q\|u^+\|_2^q + cA_{1,u}s_u^{2p} + cA_{3,u}s_u^{2p} + dB_{1,u}s_u^{2q} + dB_{3,u}s_u^{2q}\\
&\quad \geq s_u^p\|u^+\|_1^p + s_u^q\|u^+\|_2^q + cA_{1,u}s_u^{2p} + cA_{3,u}s_u^p t_u^p + dB_{1,u}s_u^{2q} + dB_{3,u}s_u^q t_u^q\\
&\quad = \int f\left(s_u u^+\right)s_u u^+. \qquad (2.30)
\end{aligned}$$

The assumption $\langle I'(u), u^+\rangle \leq 0$ gives

$$\|u^+\|_1^p + \|u^+\|_2^q + cA_{1,u} + cA_{3,u} + dB_{1,u} + dB_{3,u} \leq \int f(u^+)u^+. \tag{2.31}$$

Combining (2.30) and (2.31), we get

$$\begin{aligned}&\left(\frac{1}{s_u^p}-1\right)\|u^+\|_1^p + \left(\frac{1}{s_u^{2p-q}}-1\right)\|u^+\|_2^q + \left(\frac{1}{s_u^{2p-2q}}-1\right)(dB_{1,u}+dB_{3,u})\\ &\quad\geq \int\left[\frac{f(s_u u^+)u^+}{s_u^{2p-1}} - f(u^+)u^+\right].\end{aligned}$$

If $s_u > 1$, then the left side of this inequality is negative. But by Remark 2.2, the right side is positive, thus we must have $s_u \leq 1$. Then the proof is completed. □

From Lemma 2.3 and $\mathcal{M} \subset \mathcal{N}$, we know that $\mathcal{N}$ is nonempty and m, $\tilde{m}$ is well defined. Now we prove the following lemma.

Lemma 2.6 *Assume that* (f_1)-(f_4) *hold. If* $v \in W$ *with* $v \not\equiv 0$, *then there is a unique* $s_v \in (0,\infty)$ *such that* $s_v v \in \mathcal{N}$. *Moreover, if* $\langle I'(v), v\rangle \leq 0$, *then* $s_v \in (0,1]$.

Proof For fixed $v \in W$ with $v \not\equiv 0$ and $s \in (0,\infty)$, $sv \in \mathcal{N}$ if and only if $\langle I'(sv), sv\rangle = 0$, where

$$\begin{aligned}\langle I'(sv), sv\rangle &= s^q\left[s^{p-q}\|v\|_1^p + \|v\|_2^q + cs^{2p-q}\left(\int |\nabla v|^p\right)^2\right.\\ &\quad\left. + ds^q\left(\int |\nabla v|^q\right)^2 - \int \frac{f(sv)v}{s^{q-1}}\right] \qquad (2.32)\\ &= s^{2p}\left[\frac{1}{s^p}\|v\|_1^p + \frac{1}{s^{2p-q}}\|v\|_2^q + c\left(\int |\nabla v|^p\right)^2\right.\\ &\quad\left. + \frac{d}{s^{2p-2q}}\left(\int |\nabla v|^q\right)^2 - \int \frac{f(sv)v}{s^{2p-1}}\right]. \qquad (2.33)\end{aligned}$$

Since $v \not\equiv 0$, it follows from (2.32) and Lemma 2.1 (i) that $\langle I'(sv), sv\rangle > 0$ for $s > 0$ small. On the other hand, it follows from (2.33) and Lemma 2.1(ii) that $\langle I'(sv), sv\rangle < 0$ for $s > 0$ large. Thus there exists $s_v > 0$ such that $\langle I'(s_v v), s_v v\rangle = 0$.

Now we prove the uniqueness of s_v. Suppose that there exist s_1, s_2 with $0 < s_1 < s_2$ such that $\langle I'(s_1 v), s_1 v\rangle = \langle I'(s_2 v), s_2 v\rangle = 0$. Then (2.33) implies that

$$\frac{1}{s_i^p}\|v\|_1^p + \frac{1}{s_i^{2p-q}}\|v\|_2^q + c\left(\int |\nabla v|^p\right)^2 + \frac{d}{s_i^{2p-2q}}\left(\int |\nabla v|^q\right)^2 = \int \frac{f(s_i v)v}{s_i^{2p-1}}, \quad i = 1,2.$$

Hence,

$$\begin{aligned}&\left(\frac{1}{s_1^p} - \frac{1}{s_2^p}\right)\|v\|_1^p + \left(\frac{1}{s_1^{2p-q}} - \frac{1}{s_2^{2p-q}}\right)\|v\|_2^q + \left(\frac{d}{s_1^{2p-2q}} - \frac{d}{s_2^{2p-2q}}\right)\left(\int |\nabla v|^q\right)^2\\ &\quad= \int\left[\frac{f(s_1 v)v}{s_1^{2p-1}} - \frac{f(s_2 v)v}{s_2^{2p-1}}\right].\end{aligned}$$

Similar to (2.18), it is absurd in view of Remark 2.2.

Now we claim that $s_v \in (0,1]$. It follows from (2.33) that

$$\frac{1}{s_v^p}\|v\|_1^p + \frac{1}{s_v^{2p-q}}\|v\|_2^q + c\left(\int |\nabla v|^p\right)^2 + \frac{d}{s_v^{2p-2q}}\left(\int |\nabla v|^q\right)^2 = \int \frac{f(s_v v)v}{s_v^{2p-1}}. \tag{2.34}$$

The assumption $\langle I'(v), v\rangle \le 0$ gives

$$\|v\|_1^p + \|v\|_2^q + c\left(\int |\nabla v|^p\right)^2 + d\left(\int |\nabla v|^q\right)^2 \le \int f(v)v. \tag{2.35}$$

Combining (2.34) and (2.35), we have

$$\left(1-\frac{1}{s_v^p}\right)\|v\|_1^p + \left(1-\frac{1}{s_v^{2p-q}}\right)\|v\|_2^q + \left(1-\frac{1}{{s_v}^{2p-2q}}\right)d\left(\int |\nabla v|^q\right)^2 \le \int \left[f(v)v - \frac{f(s_v v)v}{s_v^{2p-1}}\right].$$

According to Remark 2.2, it is absurd if $s_v > 1$. Thus $s_v \in (0,1]$. □

Lemma 2.7 *Assume that* (f$_1$)-(f$_4$) *hold. Then* $m \ge \tilde{m} > 0$, *and* m, $\tilde{m}$ *can both be obtained.*

Proof (i) For any given $\varepsilon > 0$, by (2.1), we have

$$\int f(u)u \le \varepsilon \int |u|^q + C_\varepsilon \int |u|^r, \quad u \in W. \tag{2.36}$$

Hence, for some $\varepsilon > 0$ small, by the continuous embedding of $W_{p,a,h} \hookrightarrow L^r(\mathbb{R}^N)$ and $W_{q,b,g} \hookrightarrow L^q(\mathbb{R}^N)$, we get

$$\int f(u)u \le \frac{1}{2}\|u\|_2^q + C\|u\|_1^r, \quad u \in W. \tag{2.37}$$

For every $v \in \mathcal{N}$, we have $\langle I'(v), v\rangle = 0$, that is,

$$\|v\|_1^p + \|v\|_2^q + c\left(\int |\nabla v|^p\right)^2 + d\left(\int |\nabla v|^q\right)^2 = \int f(v)v. \tag{2.38}$$

For every $u \in \mathcal{M}$, we have $\langle I'(u), u^\pm\rangle = 0$, that is,

$$\|u^\pm\|_1^p + \|u^\pm\|_2^q + c\int |\nabla u|^p \int |\nabla u^\pm|^p + d\int |\nabla u|^q \int |\nabla u^\pm|^q = \int f(u^\pm)u^\pm. \tag{2.39}$$

Hence, for some $\varepsilon > 0$ small, it follows from (2.38), (2.39), (2.37), and (2.36) that

$$\|w\|_1^p + \|w\|_2^q \le \frac{1}{2}\|w\|_2^q + C\|w\|_1^r, \quad w = v, u^\pm, \tag{2.40}$$

$$\|w\|_1^p + \|w\|_2^q \le \varepsilon \int |w|^q + C_\varepsilon \int |w|^r, \quad w = v, u^\pm. \tag{2.41}$$

So, by (2.40), there exists a constant $\alpha > 0$, which is not dependent on c, d, such that

$$\|w\|_1 \ge \alpha, \quad w = v, u^\pm, \tag{2.42}$$

and then $\|v\|_1 \ge \alpha$, $\|u^\pm\|_1 \ge \alpha$.

By the condition (f_4) and $f \in C^1(\mathbb{R},\mathbb{R})$, we have

$$f'(s)s^2 - (2p-1)f(s)s \geq 0, \quad s \in \mathbb{R}. \tag{2.43}$$

By (2.43), we have

$$f(s)s - 2pF(s) \geq 0, \quad s \in \mathbb{R}. \tag{2.44}$$

Then

$$\begin{aligned} I(u) &= I(u) - \frac{1}{2p}\langle I'(u), u\rangle \\ &= \frac{1}{2p}\|u\|_1^p + \left(\frac{1}{q} - \frac{1}{2p}\right)\|u\|_2^q + \left(\frac{1}{2q} - \frac{1}{2p}\right)d\left(\int |\nabla u|^q\right)^2 \\ &\quad + \frac{1}{2p}\int \left[f(u)u - 2pF(u)\right] \\ &> \frac{1}{2p}\|u\|_1^p + \left(\frac{1}{q} - \frac{1}{2p}\right)\|u\|_2^q \geq \frac{1}{2p}\alpha^p. \end{aligned} \tag{2.45}$$

This implies that $\tilde{m} \geq \alpha^p/(2p)$.

(ii) Let $\{v_n\} \subset \mathcal{N}$ such that $I(v_n) \to \tilde{m}$. Then it follows from (2.45) that $\{v_n\}$ is bounded in W and there exists $v \in W$ such that $v_n \rightharpoonup v$ in W.

Let $\{u_n\} \subset \mathcal{M} \subset \mathcal{N}$ such that $I(u_n) \to m$. Then it follows from (2.45) that $\{u_n\}$ is bounded in W and there exists $u \in W$ such that $u_n^{\pm} \rightharpoonup u^{\pm}$ in W (see Lemma A.1).

Since $v_n \in \mathcal{N}$, $u_n \in \mathcal{M}$, it follows from (2.41) that

$$\alpha^p \leq \|w_n\|_1^p \leq \varepsilon \int |w_n|^q + C_\varepsilon \int |w_n|^r, \quad w_n = v_n, u_n^{\pm}.$$

Using the boundedness of $\{w_n\}$, there is $C_1 > 0$ such that

$$\alpha^p \leq \varepsilon C_1 + C_\varepsilon \int |w_n|^r.$$

Choosing $\varepsilon = \alpha^p/(2C_1)$, we get

$$\int |w_n|^r \geq \frac{\alpha^p}{2C_2},$$

where $C_2 = C_\varepsilon$. By the compactness of the embedding $W \hookrightarrow L^r(\mathbb{R}^N)$, we get

$$\int |w|^r \geq \frac{\alpha^p}{2C_2}, \quad w = v, u^{\pm}. \tag{2.46}$$

Thus $w \neq 0$. Equations (2.1) and (2.2) combined with the Lebesgue dominated convergence theorem give

$$\lim_{n\to\infty}\int f(w_n)w_n = \int f(w)w, \qquad \lim_{n\to\infty}\int F(w_n) = \int F(w). \tag{2.47}$$

(iii) By the weak lower semi-continuity of the norm, we have

$$\begin{aligned}&\|v\|_1^p + \|v\|_2^q + c\left(\int |\nabla v|^p\right)^2 + d\left(\int |\nabla v|^q\right)^2 \\ &\quad\le \liminf_{n\to\infty}\left\{\|v_n\|_1^p + \|v_n\|_2^q + c\left(\int |\nabla v_n|^p\right)^2 + d\left(\int |\nabla v_n|^q\right)^2\right\}.\end{aligned}$$

Then from (2.47) we get

$$\|v\|_1^p + \|v\|_2^q + c\left(\int |\nabla v|^p\right)^2 + d\left(\int |\nabla v|^q\right)^2 \le \int f(v)v. \tag{2.48}$$

From (2.48) and Lemma 2.6, there exists $s_v \in (0,1]$ such that $\bar{v} = s_v v \in \mathcal{N}$.

It follows from (2.43) that $G(s) = f(s)s - 2pF(s)$ is a non-negative function, increasing on $[0,\infty)$, and decreasing on $(-\infty,0]$. Then we have

$$\begin{aligned}\tilde{m} &\le I(\bar{v}) - \frac{1}{2p}\langle I'(\bar{v}),\bar{v}\rangle \\ &= \frac{1}{2p}\|\bar{v}\|_1^p + \left(\frac{1}{q}-\frac{1}{2p}\right)\|\bar{v}\|_2^q + \left(\frac{1}{2q}-\frac{1}{2p}\right)d\left(\int |\nabla\bar{v}|^q\right)^2 \\ &\quad + \frac{1}{2p}\int\left[f(\bar{v})\bar{v} - 2pF(\bar{v})\right] \\ &= \frac{1}{2p}s_v^p\|v\|_1^p + \left(\frac{1}{q}-\frac{1}{2p}\right)s_v^q\|v\|_2^q + \left(\frac{1}{2q}-\frac{1}{2p}\right)ds_v^{2q}\left(\int |\nabla v|^q\right)^2 \\ &\quad + \frac{1}{2p}\int\left[f(s_v v)s_v v - 2pF(s_v v)\right] \\ &\le \frac{1}{2p}\|v\|_1^p + \left(\frac{1}{q}-\frac{1}{2p}\right)\|v\|_2^q + \left(\frac{1}{2q}-\frac{1}{2p}\right)d\left(\int |\nabla v|^q\right)^2 \\ &\quad + \frac{1}{2p}\int\left[f(v)v - 2pF(v)\right] \\ &\le \liminf_{n\to\infty}\left\{I(v_n) - \frac{1}{2p}\langle I'(v_n),v_n\rangle\right\} = \tilde{m}.\end{aligned}$$

We then deduce that $s_v = 1$. Thus, $\bar{v} = v$ and $I(v) = \tilde{m}$.

(iv) By the weak lower semi-continuity of the norm, we have

$$\begin{aligned}&\|u^\pm\|_1^p + \|u^\pm\|_2^q + c\int |\nabla u|^p\int\left|\nabla u^\pm\right|^p + d\int |\nabla u|^q\int\left|\nabla u^\pm\right|^q \\ &\quad\le \liminf_{n\to\infty}\left\{\|u_n^\pm\|_1^p + \|u_n^\pm\|_2^q + c\int |\nabla u_n|^p\int\left|\nabla u_n^\pm\right|^p + d\int |\nabla u_n|^q\int\left|\nabla u_n^\pm\right|^q\right\}.\end{aligned}$$

Then from (2.47) we get

$$\|u^\pm\|_1^p + \|u^\pm\|_2^q + c\int |\nabla u|^p\int\left|\nabla u^\pm\right|^p + d\int |\nabla u|^q\int\left|\nabla u^\pm\right|^q \le \int f(u^\pm)u^\pm. \tag{2.49}$$

From (2.49) and Lemma 2.5, there exists $(s_u,t_u) \in (0,1] \times (0,1]$ such that

$$\bar{u} = s_u u^+ + t_u u^- \in \mathcal{M}.$$

Note that $G(s) = f(s)s - 2pF(s)$ is a non-negative function, increasing on $[0,\infty)$ and decreasing on $(-\infty,0]$. Then we have

$$\begin{aligned}
m &\le I(\bar{u}) - \frac{1}{2p}\langle I'(\bar{u}),\bar{u}\rangle \\
&= \frac{1}{2p}\|\bar{u}\|_1^p + \left(\frac{1}{q}-\frac{1}{2p}\right)\|\bar{u}\|_2^q + \left(\frac{1}{2q}-\frac{1}{2p}\right)d\left(\int|\nabla\bar{u}|^q\right)^2 + \frac{1}{2p}\int\left[f(\bar{u})\bar{u}-2pF(\bar{u})\right] \\
&= \frac{1}{2p}s_u^p\|u^+\|_1^p + \left(\frac{1}{q}-\frac{1}{2p}\right)s_u^q\|u^+\|_2^q + \left(\frac{1}{2q}-\frac{1}{2p}\right)ds_u^{2q}\left(\int|\nabla u^+|^q\right)^2 \\
&\quad + \frac{1}{2p}\int\left[f(s_u u^+)s_u u^+ - 2pF(s_u u^+)\right] + \left(\frac{1}{q}-\frac{1}{p}\right)ds_u^q t_u^q\int|\nabla u^+|^q\int|\nabla u^-|^q \\
&\quad + \frac{1}{2p}t_u^p\|u^-\|_1^p + \left(\frac{1}{q}-\frac{1}{2p}\right)t_u^q\|u^-\|_2^q + \left(\frac{1}{2q}-\frac{1}{2p}\right)dt_u^{2q}\left(\int|\nabla u^-|^q\right)^2 \\
&\quad + \frac{1}{2p}\int\left[f(t_u u^-)t_u u^- - 2pF(t_u u^-)\right] \\
&\le \frac{1}{2p}\|u\|_1^p + \left(\frac{1}{q}-\frac{1}{2p}\right)\|u\|_2^q + \left(\frac{1}{2q}-\frac{1}{2p}\right)d\left(\int|\nabla u|^q\right)^2 + \frac{1}{2p}\int\left[f(u)u-2pF(u)\right] \\
&\le \liminf_{n\to\infty}\left\{I(u_n) - \frac{1}{2p}\langle I'(u_n),u_n\rangle\right\} = m.
\end{aligned}$$

We then deduce that $s_u = t_u = 1$. Thus, $\bar{u} = u$ and $I(u) = m$. □

3 Proof of the main results

The purpose of this section is to prove our main results. We start to prove that the minimizer u for the minimization problem (1.10) is indeed a sign-changing solution of (1.1), that is, $I'(u) = 0$.

Proof of Theorem 1.1 Using the quantitative deformation lemma and topological degree theory, we prove that $I'(u) = 0$.

It is clear that $\langle I'(u),u^+\rangle = \langle I'(u),u^-\rangle = 0$. It follows from Lemma 2.4 that, for $(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $(s,t) \neq (1,1)$,

$$I(su^+ + tu^-) < I(u^+ + u^-) = m. \tag{3.1}$$

It follows from (2.46) that $\int|u^\pm|^r \ge \alpha^p/(2C_2) := \tau^r$. Then $|u^\pm|_r \ge \tau$. We denote by γ_r the embedding constant of $W \hookrightarrow L^r(\mathbb{R}^N)$.

If $I'(u) \neq 0$, then there exist $r_0, \rho > 0$ such that

$$\|I'(v)\| \ge \rho, \qquad \|v-u\| \le r_0. \tag{3.2}$$

Let $\delta \in (0, \min\{\tau/(2\gamma_r), r_0/3\})$ and let $\sigma \in (0, \min\{1/2, \delta/(2\|u\|_1), \delta/(2\|u\|_2)\})$. Let $D = (1-\sigma, 1+\sigma) \times (1-\sigma, 1+\sigma)$ and $\varphi(s,t) = su^+ + tu^-$ for all $(s,t) \in \overline{D}$. It follows from (3.1) that

$$\bar{m} = \max_{(s,t)\in\partial D} I(\varphi(s,t)) < m. \tag{3.3}$$

Let $\varepsilon = \min\{(m-\bar{m})/2, \rho\delta/8\}$ and $S = B(u,\delta)$. Then it follows from (3.2) that

$$\|I'(v)\| \geq 8\varepsilon/\delta, \quad v \in I^{-1}([m-2\varepsilon, m+2\varepsilon]) \cap S_{2\delta}. \tag{3.4}$$

Applying (3.4) and Lemma 2.3 in [27], p.38, there exists a deformation $\eta \in C([0,1] \times W, W)$ such that

(b_1) $\eta(1,v) = v$ if $v \notin I^{-1}([m-2\varepsilon, m+2\varepsilon]) \cap S_{2\delta}$;
(b_2) $\eta(1, I^{m+\varepsilon} \cap S) \subset I^{m-\varepsilon}$;
(b_3) $\|\eta(1,v) - v\| \leq \delta$ for all $v \in W$.

By Lemmas 2.4 and 2.7, for $(s,t) \in \overline{D}$, we know $I(\varphi(s,t)) \leq m < m+\varepsilon$, that is, $\varphi(s,t) \in I^{m+\varepsilon}$. Since

$$\begin{aligned}
\|\varphi(s,t) - u\|_1^p &= \|su^+ + tu^- - u^+ - u^-\|_1^p \\
&= |s-1|^p \|u^+\|_1^p + |t-1|^p \|u^-\|_1^p \\
&\leq \sigma^p \|u\|_1^p \\
&< (\delta/2)^p,
\end{aligned}$$

and similarly $\|\varphi(s,t) - u\|_2^q < (\delta/2)^q$, we have $\|\varphi(s,t) - u\| = \|\varphi(s,t) - u\|_1 + \|\varphi(s,t) - u\|_2 < \delta$. Thus $\varphi(s,t) \in S$. By (b_2), we have $I(\eta(1,\varphi(s,t))) < m - \varepsilon$. Then it is clear that

$$\max_{(s,t)\in\overline{D}} I(\eta(1,\varphi(s,t))) \leq m - \varepsilon < m. \tag{3.5}$$

We will prove that $\eta(1,\varphi(D)) \cap \mathcal{M} \neq \emptyset$, which is a contradiction with (3.5). Therefore, $I'(u) = 0$, that is, u is a sign-changing solution for equation (1.1). In fact, on $\overline{D}$ we also define $\psi(s,t) = \eta(1,\varphi(s,t))$ and

$$\begin{aligned}
\Phi_0(s,t) &= (\langle I'(\varphi(s,t)), su^+\rangle, \langle I'(\varphi(s,t)), tu^-\rangle) \\
&= (\langle I'(su^+ + tu^-), su^+\rangle, \langle I'(su^+ + tu^-), tu^-\rangle), \\
\Phi_1(s,t) &= (\langle I'(\psi(s,t)), \psi^+(s,t)\rangle, \langle I'(\psi(s,t)), \psi^-(s,t)\rangle).
\end{aligned}$$

Let

$$\begin{aligned}
P(s,t) &= \langle I'(su^+ + tu^-), su^+\rangle \\
&= s^p \|u^+\|_1^p + s^q \|u^+\|_2^q + cA_{1,u}s^{2p} + cA_{3,u}s^p t^p \\
&\quad + dB_{1,u}s^{2q} + dB_{3,u}s^q t^q - \int f(su^+)su^+, \\
Q(s,t) &= \langle I'(su^+ + tu^-), tu^-\rangle \\
&= t^p \|u^-\|_1^p + t^q \|u^-\|_2^q + cA_{2,u}t^{2p} + cA_{3,u}s^p t^p \\
&\quad + dB_{2,u}t^{2q} + dB_{3,u}s^q t^q - \int f(tu^-)tu^-.
\end{aligned}$$

By direct calculation, we have

$$\begin{aligned}
\frac{\partial P(s,t)}{\partial s}\bigg|_{(1,1)} &= (p-1)\|u^+\|_1^p + (q-1)\|u^+\|_2^q + (2p-1)cA_{1,u} + (p-1)cA_{3,u} \\
&\quad + (2q-1)dB_{1,u} + (q-1)dB_{3,u} - \int f'(u^+)|u^+|^2, \\
\frac{\partial P(s,t)}{\partial t}\bigg|_{(1,1)} &= pcA_{3,u} + qdB_{3,u}, \qquad \frac{\partial Q(s,t)}{\partial s}\bigg|_{(1,1)} = pcA_{3,u} + qdB_{3,u}, \\
\frac{\partial Q(s,t)}{\partial t}\bigg|_{(1,1)} &= (p-1)\|u^-\|_1^p + (q-1)\|u^-\|_2^q + (2p-1)cA_{2,u} + (p-1)cA_{3,u} \\
&\quad + (2q-1)dB_{2,u} + (q-1)dB_{3,u} - \int f'(u^-)|u^-|^2.
\end{aligned}$$

By (2.43), we get

$$\frac{\partial P(s,t)}{\partial s}\bigg|_{(1,1)} < -(pcA_{3,u} + qdB_{3,u})$$

and

$$\frac{\partial Q(s,t)}{\partial t}\bigg|_{(1,1)} < -(pcA_{3,u} + qdB_{3,u}).$$

Set the matrix

$$M = \begin{bmatrix} \frac{\partial P(1,1)}{\partial s} & \frac{\partial P(1,1)}{\partial t} \\ \frac{\partial Q(1,1)}{\partial s} & \frac{\partial Q(1,1)}{\partial t} \end{bmatrix}.$$

Then we get

$$J_{\Phi_0}(1,1) = \det M > 0.$$

Since Φ_0 is C^1, $(1,1)$ is the unique isolated zero of Φ_0, by Lemmas 1.5 and 1.6 in [28], p.112, we have

$$\deg(\Phi_0, D, 0) = \operatorname{ind}(\Phi_0, (1,1)) = \operatorname{sgn} J_{\Phi_0}(1,1) = 1.$$

It follows from (3.3), $\bar{m} < m - 2\varepsilon$, and (b_1) above that $\varphi = \psi$ on ∂D. Hence $\deg(\Phi_1, D, 0) = \deg(\Phi_0, D, 0) = 1$. Thus there exists a pair $(s_0, t_0) \in D$ such that $\Phi_1(s_0, t_0) = 0$. Since $|u^\pm|_r \geq \tau$, $(s_0, t_0) \in D$, we have $|\varphi^+(s_0,t_0)|_r = s_0|u^+|_r \geq \tau/2$ and $|\varphi^-(s_0,t_0)|_r = t_0|u^-|_r \geq \tau/2$. By (b_3), we have

$$|\psi(s_0,t_0) - \varphi(s_0,t_0)|_r \leq \gamma_r \|\psi(s_0,t_0) - \varphi(s_0,t_0)\| \leq \gamma_r \delta.$$

This implies that

$$|\psi^\pm(s_0,t_0) - \varphi^\pm(s_0,t_0)|_r \leq |\psi(s_0,t_0) - \varphi(s_0,t_0)|_r \leq \gamma_r \delta.$$

Thus we have

$$\left|\psi^{\pm}(s_0,t_0)\right|_r \geq \left|\varphi^{\pm}(s_0,t_0)\right|_r - \gamma_r\delta \geq \frac{\tau}{2} - \gamma_r\delta > 0.$$

That is, $\psi^{\pm}(s_0,t_0) \neq 0$. Then $\eta(1,\varphi(s_0,t_0)) = \psi(s_0,t_0) \in \mathcal{M}$. □

Now, we are in a situation to prove Theorem 1.2. We first claim that equation (1.2) has one least energy sign-changing solution.

Remark 3.1 The proof of equation (1.1) is still suited to (1.2) where $c = d = 0$ although the proof for equation (1.2) may be easier than (1.1). So, if the assumptions (f_1)-(f_4) hold, there exists a least energy sign-changing solution of (1.2).

In the following, we regard $c, d \geq 0$ as parameters in equation (1.1). Let $u_{c,d} \in W$ be the least energy sign-changing solution of (1.1) obtained in Theorem 1.1. The relative functional and constraint are denoted by $I_{c,d}$ and $\mathcal{M}_{c,d}$, respectively. We will analyze the convergence property of $u_{c,d}$ as $(c,d) \to 0$.

Proof of Theorem 1.2 (1) We first of all claim that, for any sequence $\{(c_n,d_n)\}$ with $(c_n,d_n) \to 0$ as $n \to \infty$, $\{u_{c_n,d_n}\} := \{u_n\}$ is bounded in W. In fact, choosing a function $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi^{\pm} \neq 0$, $\int h|\varphi|^p, \int g|\varphi|^q < \infty$. For any $c, d \in [0,1]$, since $c, d \leq 1$, Lemma 2.1(ii) implies that there exists a pair (μ_1,μ_2) of positive numbers, which does not depend on c, d, such that

$$\begin{cases} \mu_1^p\|\varphi^+\|_1^p + \mu_1^q\|\varphi^+\|_2^q + cA_{1,\varphi}\mu_1^{2p} + cA_{3,\varphi}\mu_1^p\mu_2^p \\ \quad + dB_{1,\varphi}\mu_1^{2q} + dB_{3,\varphi}\mu_1^q\mu_2^q - \int f(\mu_1\varphi^+)\mu_1\varphi^+ < 0, \\ \mu_2^p\|\varphi^-\|_1^p + \mu_2^q\|\varphi^-\|_2^q + cA_{2,\varphi}\mu_2^{2p} + cA_{3,\varphi}\mu_1^p\mu_2^p \\ \quad + dB_{2,\varphi}\mu_2^{2q} + dB_{3,\varphi}\mu_1^q\mu_2^q - \int f(\mu_2\varphi^-)\mu_2\varphi^- < 0. \end{cases}$$

In view of Lemmas 2.3 and 2.5, for any $c, d \in [0,1]$, there is a unique pair $(s_\varphi(c,d), t_\varphi(c,d)) \in (0,1] \times (0,1]$ such that

$$\bar{\varphi} = s_\varphi(c,d)\mu_1\varphi^+ + t_\varphi(c,d)\mu_2\varphi^- \in \mathcal{M}_{c,d}.$$

Thus, for any $c, d \in [0,1]$, combining (2.1) and (2.2), we have

$$\begin{aligned} I_{c,d}(u_{c,d}) &\leq I_{c,d}(\bar{\varphi}) - \frac{1}{2p}\langle I'_{c,d}(\bar{\varphi}),\bar{\varphi}\rangle \\ &= \frac{1}{2p}\|\bar{\varphi}\|_1^p + \left(\frac{1}{q} - \frac{1}{2p}\right)\|\bar{\varphi}\|_2^q + \left(\frac{1}{2q} - \frac{1}{2p}\right)d\left(\int |\nabla\bar{\varphi}|^q\right)^2 \\ &\quad + \frac{1}{2p}\int\left[f(\bar{\varphi})\bar{\varphi} - 2pF(\bar{\varphi})\right] \\ &\leq \frac{1}{2p}\|\bar{\varphi}\|_1^p + \left(\frac{1}{q} - \frac{1}{2p}\right)\|\bar{\varphi}\|_2^q + \left(\frac{1}{2q} - \frac{1}{2p}\right)d\left(\int |\nabla\bar{\varphi}|^q\right)^2 \\ &\quad + \frac{1}{2p}\int\left[C_1|\bar{\varphi}|^q + C_2|\bar{\varphi}|^r\right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2p}\mu_1^p\|\varphi^+\|_1^p + \frac{1}{2p}\mu_2^p\|\varphi^-\|_1^p + \left(\frac{1}{q} - \frac{1}{2p}\right)\mu_1^q\|\varphi^+\|_2^q + \left(\frac{1}{q} - \frac{1}{2p}\right)\mu_2^q\|\varphi^-\|_2^q \\
&\quad + \left(\frac{1}{2q} - \frac{1}{2p}\right)\left[B_{1,\varphi}\mu_1^{2q} + 2B_{3,\varphi}\mu_1^q\mu_2^q + B_{2,\varphi}\mu_2^{2q}\right] \\
&\quad + \frac{C_1}{2p}\int\left[\mu_1^q|\varphi^+|^q + \mu_2^q|\varphi^-|^q\right] + \frac{C_2}{2p}\int\left[\mu_1^r|\varphi^+|^r + \mu_2^r|\varphi^-|^r\right] := C_0,
\end{aligned}$$

where C_0 does not depend on c, d. It follows from (2.45) that, for n large enough,

$$C_0 \geq I_n(u_n) = I_n(u_n) - \frac{1}{2p}\langle I_n'(u_n), u_n\rangle \geq \frac{1}{2p}\|u_n\|_1^p + \left(\frac{1}{q} - \frac{1}{2p}\right)\|u_n\|_2^q,$$

where I_n denotes I_{c_n,d_n}. Then $\{u_n\}$ is bounded in W.

(2) There exists a subsequence of $\{(c_n, d_n)\}$, still denoted by $\{(c_n, d_n)\}$, such that $u_n \rightharpoonup u_0$ in W. Now we will prove that u_0 is a sign-changing solution of (1.2). Indeed, by (2.1), the Hölder inequality and the compactness of the embedding $W \hookrightarrow L^s(\mathbb{R}^N)$ for $s = q, r$, we get

$$\begin{aligned}
&\int \left|f(u_n)(u_n - u_0)\right| \\
&\quad \leq \int\left[C_1|u_n|^{q-1}|u_n - u_0| + C_2|u_n|^{r-1}|u_n - u_0|\right] \\
&\quad \leq C_1|u_n|_q^{q-1}|u_n - u_0|_q + C_2|u_n|_r^{r-1}|u_n - u_0|_r \to 0
\end{aligned} \tag{3.6}$$

and

$$\int\left|f(u_0)(u_n - u_0)\right| \to 0. \tag{3.7}$$

We have

$$\begin{aligned}
&\langle I_n'(u_n) - I_0'(u_0), u_n - u_0\rangle \\
&\quad = \int a\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0\right)\cdot\nabla(u_n - u_0) \\
&\qquad + \int h\left(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0\right)(u_n - u_0) \\
&\qquad + \int b\left(|\nabla u_n|^{q-2}\nabla u_n - |\nabla u_0|^{q-2}\nabla u_0\right)\cdot\nabla(u_n - u_0) \\
&\qquad + \int g\left(|u_n|^{q-2}u_n - |u_0|^{q-2}u_0\right)(u_n - u_0) \\
&\qquad + c_n\int|\nabla u_n|^p\int|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(u_n - u_0) \\
&\qquad + d_n\int|\nabla u_n|^q\int|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla(u_n - u_0) \\
&\qquad - \int f(u_n)(u_n - u_0) + \int f(u_0)(u_n - u_0) \\
&\quad \geq C_1\int a\left|\nabla(u_n - u_0)\right|^p + C_2\int h|u_n - u_0|^p \\
&\qquad + C_3\int b\left|\nabla(u_n - u_0)\right|^q + C_4\int g|u_n - u_0|^q
\end{aligned}$$

$$
\begin{aligned}
&+ c_n \int |\nabla u_n|^p \int |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u_0) \\
&+ d_n \int |\nabla u_n|^q \int |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla (u_n - u_0) \\
&- \int f(u_n)(u_n - u_0) + \int f(u_0)(u_n - u_0) \\
\geq{}& C\bigl(\|u_n - u_0\|_1^p + \|u_n - u_0\|_2^q\bigr) + c_n \int |\nabla u_n|^p \int |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u_0) \\
&+ d_n \int |\nabla u_n|^q \int |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla (u_n - u_0) \\
&- \int f(u_n)(u_n - u_0) + \int f(u_0)(u_n - u_0),
\end{aligned}
$$

where we have used the inequality $(|\xi|^{s-2}\xi - |\eta|^{s-2}\eta, \xi - \eta) \geq C_s|\xi - \eta|^s$ for all $\xi, \eta \in \mathbb{R}^N$ and $s \geq 2$ (see [29]). By (3.6) and (3.7), it follows that, passing to the limit on $n \to \infty$, we deduce $\|u_n - u_0\|_1^p + \|u_n - u_0\|_2^q \to 0$ as $n \to \infty$. Then $u_n \to u_0$ in W. It follows from (2.42) that $\|u_n^{\pm}\|_1 \geq \alpha$, where α is not dependent on c_n, d_n. By Lemma A.2, we know that $\|u_0^{\pm}\|_1 \geq \alpha$. So u_0 changes sign and u_0 is a solution of (1.2).

(3) Now we prove that u_0 is also a least energy sign-changing solution of (1.2). In fact, suppose that v_0 is a least energy sign-changing solution of (1.2). Then by Lemma 2.3, for each $c_n, d_n \geq 0$, there is a unique pair (s_n, t_n) of positive numbers such that

$$s_n v_0^+ + t_n v_0^- \in \mathcal{M}_n,$$

where $\mathcal{M}_n$ denotes $\mathcal{M}_{c_n,d_n}$. Then we have

$$
\begin{aligned}
&s_n^p \|v_0^+\|_1^p + s_n^q \|v_0^+\|_2^q + c_n s_n^{2p} \biggl(\int |\nabla v_0^+|^p\biggr)^2 + d_n s_n^{2q} \biggl(\int |\nabla v_0^+|^q\biggr)^2 \\
&\quad + c_n s_n^p t_n^p \int |\nabla v_0^+|^p \int |\nabla v_0^-|^p + d_n s_n^q t_n^q \int |\nabla v_0^+|^q \int |\nabla v_0^-|^q = \int f(s_n v_0^+) s_n v_0^+
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
&t_n^p \|v_0^-\|_1^p + t_n^q \|v_0^-\|_2^q + c_n t_n^{2p} \biggl(\int |\nabla v_0^-|^p\biggr)^2 + d_n t_n^{2q} \biggl(\int |\nabla v_0^-|^q\biggr)^2 \\
&\quad + c_n s_n^p t_n^p \int |\nabla v_0^+|^p \int |\nabla v_0^-|^p + d_n s_n^q t_n^q \int |\nabla v_0^+|^q \int |\nabla v_0^-|^q = \int f(t_n v_0^-) t_n v_0^-.
\end{aligned}
\tag{3.9}
$$

We first claim that (s_n, t_n) is bounded. Otherwise, we may assume there is a subsequence such that $s_n \geq t_n$, $s_n \to \infty$. Thus, by (3.8) and Lemma 2.1(ii), we get $0 = \infty$. This is a contradiction. Therefore, there are $s_0, t_0 \geq 0$ and a subsequence (s_n, t_n) such that $(s_n, t_n) \to (s_0, t_0)$. If $s_0 = 0$, by (3.8) and Lemma 2.1(i), we have

$$\|v_0^+\|_2^q = \lim_{n\to\infty} \int \frac{f(s_n v_0^+) v_0^+}{s_n^{q-1}} = 0.$$

It is absurd in view of $v_0^+ \neq 0$. Thus $s_0 \neq 0$. We also get $t_0 \neq 0$ by (3.9) in the same way.

Recall that $v_0^{\pm}$ satisfies

$$\|v_0^{\pm}\|_1^p + \|v_0^{\pm}\|_2^q = \int f(v_0^{\pm})v_0^{\pm}. \tag{3.10}$$

Passing to the limit on $n \to \infty$ in (3.8), we have

$$\frac{1}{s_0^p}\|v_0^+\|_1^p + \frac{1}{s_0^{2p-q}}\|v_0^+\|_2^q = \int \frac{f(s_0 v_0^+)v_0^+}{s_0^{2p-1}}. \tag{3.11}$$

Combining (3.10) and (3.11) we get

$$\left(\frac{1}{s_0^p} - 1\right)\|v_0^+\|_1^p + \left(\frac{1}{s_0^{2p-q}} - 1\right)\|v_0^+\|_2^q = \int \left[\frac{f(s_0 v_0^+)v_0^+}{s_0^{2p-1}} - f(v_0^+)v_0^+\right]. \tag{3.12}$$

As follows from (3.12) and Remark 2.2, we just get $s_0 = 1$. And $t_0 = 1$ is similar available. That yields

$$(s_n, t_n) \to (1,1), \quad n \to \infty. \tag{3.13}$$

Now, we can prove u_0 is a least energy sign-changing solution of (1.2). In fact, from (3.13), we have

$$I_0(v_0) \le I_0(u_0) = \lim_{n\to\infty} I_n(u_n) \le \lim_{n\to\infty} I_n(s_n v_0^+ + t_n v_0^-) = I_0(v_0^+ + v_0^-) = I_0(v_0).$$

Thus, $I_0(u_0) = I_0(v_0)$ and then u_0 is a least energy sign-changing solution of (1.2). This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3 (i) Let $\mathcal{N}$ and $\tilde{m}$ be given by (1.11) and (1.12), respectively. We prove that the minimizer v for the minimization problem (1.12) is indeed a ground state solution of (1.1), that is, $I'(v) = 0$.

We define a functional $H(u) = \langle I'(u), u\rangle$ for all $u \in W$, that is,

$$H(u) = \|u\|_1^p + \|u\|_2^q + c\left(\int |\nabla u|^p\right)^2 + d\left(\int |\nabla u|^q\right)^2 - \int f(u)u, \quad u \in W. \tag{3.14}$$

Since $v \in \mathcal{N}$, $I(v) = \tilde{m}$, there is a Lagrange multiplier $l \in \mathbb{R}$ such that

$$I'(v) - lH'(v) = 0. \tag{3.15}$$

Hence, $l\langle H'(v), v\rangle = \langle I'(v), v\rangle = 0$. But it follows from (3.14), (2.43), and $H(v) = 0$ that

$$\begin{aligned}\langle H'(v), v\rangle &= p\|v\|_1^p + q\|v\|_2^q + 2pc\left(\int |\nabla v|^p\right)^2 + 2qd\left(\int |\nabla v|^q\right)^2 - \int \left[f'(v)v^2 + f(v)v\right] \\ &< -p\|v\|_1^p - (2p-q)\|v\|_2^q - (2p-2q)d\left(\int |\nabla v|^q\right)^2 < 0.\end{aligned}$$

Thus we have $l = 0$. Then (3.15) implies that $I'(v) = 0$. Therefore, v is a ground state solution of (1.1).

(ii) From Theorem 1.1, we know that equation (1.1) has a least energy sign-changing solution u. Let $u = u^+ + u^-$ with $u^\pm \neq 0$. Then, combining $\langle I'(u^\pm), u^\pm\rangle < \langle I'(u), u^\pm\rangle = 0$ and Lemma 2.6, there is a unique $t_{u^+}, s_{u^-} \in (0,1)$ such that $t_{u^+}u^+, s_{u^-}u^- \in \mathcal{N}$. Then it follows from the definition of $\tilde{m}$ that

$$\begin{aligned}
\tilde{m} &\le I(t_{u^+}u^+) \\
&= I(t_{u^+}u^+) - \frac{1}{2p}\langle I'(t_{u^+}u^+), t_{u^+}u^+\rangle \\
&= \frac{1}{2p}t_{u^+}^p \|u^+\|_1^p + \left(\frac{1}{q} - \frac{1}{2p}\right)t_{u^+}^q \|u^+\|_2^q + \left(\frac{1}{2q} - \frac{1}{2p}\right)dt_{u^+}^{2q}\left(\int |\nabla u^+|^q\right)^2 \\
&\quad + \frac{1}{2p}\int \left[f(t_{u^+}u^+)t_{u^+}u^+ - 2pF(t_{u^+}u^+)\right] \\
&< \frac{1}{2p}\|u^+\|_1^p + \left(\frac{1}{q} - \frac{1}{2p}\right)\|u^+\|_2^q + \left(\frac{1}{2q} - \frac{1}{2p}\right)d\left(\int |\nabla u^+|^q\right)^2 \\
&\quad + \frac{1}{2p}\int \left[f(u^+)u^+ - 2pF(u^+)\right].
\end{aligned} \tag{3.16}$$

Similarly, we have

$$\begin{aligned}
\tilde{m} &< \frac{1}{2p}\|u^-\|_1^p + \left(\frac{1}{q} - \frac{1}{2p}\right)\|u^-\|_2^q + \left(\frac{1}{2q} - \frac{1}{2p}\right)d\left(\int |\nabla u^-|^q\right)^2 \\
&\quad + \frac{1}{2p}\int \left[f(u^-)u^- - 2pF(u^-)\right].
\end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17), we have

$$\begin{aligned}
2\tilde{m} &< \frac{1}{2p}\left(\|u^+\|_1^p + \|u^-\|_1^p\right) + \left(\frac{1}{q} - \frac{1}{2p}\right)\left(\|u^+\|_2^q + \|u^-\|_2^q\right) \\
&\quad + \left(\frac{1}{2q} - \frac{1}{2p}\right)d\left[\left(\int |\nabla u^+|^q\right)^2 + \left(\int |\nabla u^-|^q\right)^2\right] \\
&\quad + \frac{1}{2p}\int \left[f(u^+)u^+ - 2pF(u^+) + f(u^-)u^- - 2pF(u^-)\right] \\
&< \frac{1}{2p}\|u\|_1^p + \left(\frac{1}{q} - \frac{1}{2p}\right)\|u\|_2^q + \left(\frac{1}{2q} - \frac{1}{2p}\right)d\left(\int |\nabla u|^q\right)^2 \\
&\quad + \frac{1}{2p}\int \left[f(u)u - 2pF(u)\right] \\
&= I(u)u - \frac{1}{2p}\langle I'(u), u\rangle \\
&= I(u) = m.
\end{aligned}$$

That is, $m > 2\tilde{m}$. This implies that $\tilde{m}$ cannot be obtained by a sign-changing solution. This completes the proof. □

Appendix

Lemma A.1 *If $u_n \rightharpoonup u$ in W, then $u_n^\pm \rightharpoonup u^\pm$ in W.*

Proof We only prove $u_n^+ \rightharpoonup u^+$ in W, $u_n^- \rightharpoonup u^-$ in W is similar.

Since $u_n \rightharpoonup u$ in W, $\{u_n\}$ is bounded in W. Moreover, $\{u_n^+\}$ is bounded in W. Then there exist a subsequence $\{u_{n_k}^+\}$ and $v \in W$ such that

$$u_{n_k}^+ \rightharpoonup v \quad \text{in } W.$$

Then

$$\begin{aligned} &u_{n_k}^+ \to v \quad \text{in } L^s(\mathbb{R}^N), s \in [q, p^*), \\ &u_{n_{k_j}}^+(x) \to v(x) \quad \text{a.e. on } \mathbb{R}^N. \end{aligned}$$

Since $\{u_n\}$ is bounded in W, we have

$$\begin{aligned} &u_{n_{k_j}} \rightharpoonup u \quad \text{in } W, \\ &u_{n_{k_j}} \to u \quad \text{in } L^s(\mathbb{R}^N), s \in [q, p^*), \\ &u_{n_{k_{j_l}}}(x) \to u(x) \quad \text{a.e. on } \mathbb{R}^N, \\ &u_{n_{k_{j_l}}}^+(x) \to u^+(x) \quad \text{a.e. on } \mathbb{R}^N. \end{aligned}$$

Thus, $v = u^+$. Then the proof is completed. □

Lemma A.2 *If $u_n \to u$ in W, then $u_n^\pm \to u^\pm$ in W.*

Proof Since $u_n \to u$ in W, by the definition of W and Lemma A.1, we have

$$\begin{aligned} &u_n \to u \quad \text{in } W_{p,a,h}, \qquad u_n \to u \quad \text{in } W_{q,b,g}, \\ &u_n \rightharpoonup u \quad \text{in } W_{p,a,h}, \qquad u_n \rightharpoonup u \quad \text{in } W_{q,b,g}, \\ &u_n^\pm \rightharpoonup u^\pm \quad \text{in } W_{p,a,h}, \qquad u_n^\pm \rightharpoonup u^\pm \quad \text{in } W_{q,b,g}. \end{aligned}$$

By the weak lower semi-continuity of norm, we get

$$\|u^\pm\|_1^p \le \liminf_{n\to\infty} \|u_n^\pm\|_1^p.$$

Then

$$\begin{aligned} \|u^+\|_1^p + \|u^-\|_1^p &= \|u\|_1^p \\ &= \lim_{n\to\infty} \|u_n\|_1^p \\ &\ge \liminf_{n\to\infty} \|u_n^+\|_1^p + \liminf_{n\to\infty} \|u_n^-\|_1^p \\ &\ge \|u^+\|_1^p + \|u^-\|_1^p. \end{aligned}$$

Hence,

$$\|u^\pm\|_1^p = \liminf_{n\to\infty} \|u_n^\pm\|_1^p.$$

Then there exists a subsequence $\{u_{n_k}\}$ such that

$$\lim_{k\to\infty} \|u_{n_k}^{\pm}\|_1^p = \|u^{\pm}\|_1^p.$$

Similarly, we can get

$$\lim_{j\to\infty} \|u_{n_{k_j}}^{\pm}\|_2^q = \|u^{\pm}\|_2^q.$$

Then

$$\lim_{j\to\infty} \|u_{n_{k_j}}^{\pm}\| = \|u^{\pm}\|.$$

Combined with the fact that W is a reflexive Banach space, we can get $u_n^{\pm} \to u^{\pm}$ in W. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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