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# Multiplicity of small negative-energy solutions for a class of semilinear elliptic systems

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**Abstract**

This paper is concerned with the following semilinear elliptic systems:

$$\begin{cases} -\Delta u + V(x)u = H(x)F_u(x,u,v), & x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = H(x)F_v(x,u,v), & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad v(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where $V(x)$, $H(x)$ are nonnegative continuous functions. Under some appropriate assumptions on $V(x)$, $H(x)$, and $F(x,u,v)$, we prove the existence of infinitely many small negative-energy solutions by using the fountain theorem established by Zou. Recent results from the literature are extended.

**MSC:** 35B38; 35J20

**Keywords:** semilinear elliptic systems; multiple solutions; variant fountain theorem; variational methods

## 1 Introduction

In this paper, we consider the existence and multiplicity of solutions to the following semilinear elliptic systems:

$$\begin{cases} -\Delta u + V(x)u = H(x)F_u(x,u,v), & x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = H(x)F_v(x,u,v), & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad v(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $V(x)$, $H(x)$ are nonnegative continuous functions, we assume that the functions $V(x)$, $H(x)$, and $F(x,u,v)$ satisfy the following hypotheses:

($H_1$) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) \geq a_0 > 0$, where $a_0 > 0$ is *a* constant. Moreover, for any $M > 0$, $\operatorname{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty$, where meas denotes the Lebesgue measure in $\mathbb{R}^N$.

($H_2$) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$, $|F_u(x,u,v)| \leq c(|(u,v)| + |(u,v)|^{p-1})$, and $|F_v(x,u,v)| \leq c(|(u,v)| + |(u,v)|^{q-1})$ for some $1 < p,q < 2$, where $c$ is a positive constant, and $|(u,v)| = (u^2+v^2)^{\frac{1}{2}}$.

($H_3$) $F(x,0,0) = 0$, $F(x,u,v) \geq 0$ for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2$, and for some $1 < \mu < 2$, there exists $c_1 > 0$ such that $F(x,u,v) \geq c_1|(u,v)|^{\mu}$.

(H$_4$) $H(x) \geq 0$ and $H(x) \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}) \cap L^{\frac{2}{2-q}}(\mathbb{R}^N, \mathbb{R}) \cap L^{\frac{2}{2-\mu}}(\mathbb{R}^N, \mathbb{R}) \cap L^{\infty}(\mathbb{R}^N, \mathbb{R})$.
(H$_5$) $F(x,u,v) = F(x,-u,-v)$ for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2$.

When $\Omega$ is a bounded domain of $\mathbb{R}^N$, the problem

$$\begin{cases} -\Delta u = \lambda(a(x)u + b(x)v) + F_u(x,u,v) & \text{in } \Omega, \\ -\Delta v = \lambda(b(x)u + c(x)v) + F_v(x,u,v) & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

which is related to reaction-diffusion systems that appear in chemical and biological phenomena, including the steady and unsteady state situation (see [1–4]), has been extensively investigated in recent years. For the results on existence, multiple solutions, and positive solutions to problem (1.2), we refer the readers to [1, 4–9] and the references therein. Qu and Tang [5] obtained the existence and multiplicity of weak solutions of problem (1.2) by using the Ekeland variational principle, the mountain pass theorem, and the saddle point theorem in critical point theory, and by applying the local linking theorem and the saddle point theorem some new existence theorems of weak solutions were obtained by Duan *et al.* [6]. In [7], by using Morse theory the multiplicity of solutions was obtained for cooperative elliptic systems at resonance. Costa and Magalhães [8, 9] researched subquadratic perturbation problems of semiliner elliptic systems by minimax methods.

Recently, the problems in the whole space $\mathbb{R}^N$ were considered in some works. For example, see [10–15] and the references therein. Cao and Tang [10] studied the following Schrödinger systems:

$$\begin{cases} -\Delta u + V(x)u = F_u(x,u,v), & x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = F_v(x,u,v), & x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

Under suitable assumptions on $F(x,u,v)$, they obtained the existence of infinitely many solutions characterized by the number of nodes of each component. When $N \geq 3$, $V(x) = 0$, $F_u(x,u,v) = p(x)f(v)$, and $F_v(x,u,v) = q(x)g(u)$, Zhang *et al.* [11] obtained the existence and nonexistence of entire solutions to (1.3). Wu [12] obtained five new critical point theorems on the product spaces and studied three existence theorems for the sequence of high-energy solutions to problem (1.3), whereas Zhou *et al.* [13] established the existence of high-energy solutions to (1.3) under some conditions that are weaker than those in [12], which unify and sharply improve the recent results in [14].

Inspired by all these facts, the aim of this paper is to study the multiplicity of small negative-energy solutions to problem (1.1) via variational methods, which have been widely used to study Schrödinger equations; see [16–23] and the references therein. To the best of our knowledge, there has been few works concerning this case up to now.

Now, we state our main results.

**Theorem 1.1** *Suppose that conditions* (H$_1$)-(H$_5$) *hold. Then problem* (1.1) *possesses infinitely many solutions* $\{(u_k, v_k)\}$ *satisfying*

$$\frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u_k|^2 + V(x)u_k^2\right) dx + \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v_k|^2 + V(x)v_k^2\right) dx \\ - \int_{\mathbb{R}^N} H(x)F(x,u_k,v_k)\, dx \to 0^- \quad \text{as } k \to \infty.$$

The remainder of this paper is as follows. In Section 2, we present some preliminary results. In Section 3, we give a proof of the main result.

## 2 Variational setting and preliminaries

In this section, we outline the variational framework for problem (1.1) and give some preliminary lemmas.

Let

$$H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

with the norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)\,dx\right)^{\frac{1}{2}}.$$

Let

$$X = \left\{u \in H^1(\mathbb{R}^N) \,\Big|\, \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx < +\infty\right\}$$

with the inner product and norm

$$\langle u, v\rangle_X = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\,dx, \quad \|u\|_X = \langle u, u\rangle_X^{\frac{1}{2}}.$$

As usual, for $1 \leq p < +\infty$, we let

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u(x)|^p\,dx\right)^{\frac{1}{p}}, \quad u \in L^p(\mathbb{R}^N),$$

and

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |u(x)|, \quad u \in L^\infty(\mathbb{R}^N).$$

Then $E = X \times X$ is a Hilbert space with the inner product

$$\langle (u,v),(\varphi,\psi)\rangle = \langle u,\varphi\rangle_X + \langle v,\psi\rangle_X, \quad (u,v),(\varphi,\psi) \in X \times X,$$

and the norm

$$\|(u,v)\|^2 = \langle (u,v),(u,v)\rangle = \|u\|_X^2 + \|v\|_X^2, \quad (u,v),(\varphi,\psi) \in X \times X.$$

Define the functional $I$ on $E$ by

$$I(u,v) = \frac{1}{2}\|(u,v)\|^2 - \int_{\mathbb{R}^N} H(x)F(x,u,v)\,dx. \tag{2.1}$$

Then a weak solution of system (1.1) is a critical point of $I$ if $I$ is continuously differentiable on $E$.

Moreover, we have the following compactness lemma.

**Lemma 2.1** *Under assumption* ($H_1$), *the embedding* $E \hookrightarrow L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$ *is continuous for* $2 \le r \le 2^*$, *and* $E \hookrightarrow L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$ *is compact for* $2 \le r < 2^*$.

*Proof* By Lemma 3.4 in [24] we know that, under assumption ($H_1$), the embedding $X \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for $r \in [2, 2^*]$ and that $X \hookrightarrow L^r(\mathbb{R}^N)$ is compact for $r \in [2, 2^*)$, that is, there exist constants $C_r > 0$ such that $\|u\|_r \le C_r \|u\|_X$, $\forall u \in X$, and for any bounded sequence $\{u_n\} \subset X$, there exists a subsequence of $\{u_n\}$ such that $u_n \to u_0$ in $L^r(\mathbb{R}^N)$, $r \in [2, 2^*)$. Therefore, for any $(u, v) \in E$, there exists $C > 0$ such that

$$\|(u,v)\|_r^r \le C\big(\|u\|_r^r + \|u\|_r^r\big) \le C\big(\|u\|_X^r + \|u\|_X^r\big) \le C\|(u,v)\|^r,$$

that is, $\|(u,v)\|_r \le C\|(u,v)\|$, so that $E \hookrightarrow L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$ is continuous for $2 \le r \le 2^*$. On the other hand, suppose that $\{(u_n, v_n)\} \subset E$ are bounded, that is, $\{u_n\}$ and $\{v_n\}$ are bounded in $X$, then there exist subsequences $\{u_n\}$ and $\{v_n\}$ such that

$$u_n \to u_0, \qquad v_n \to v_0 \quad \text{in } L^r(\mathbb{R}^N), r \in [2, 2^*).$$

Therefore,

$$0 \le \|u_n - u_0\|_r^r + \|v_n - v_0\|_r^r \le \|(u_n, v_n) - (u_0, v_0)\|_r^r \le C\big(\|u_n - u_0\|_r^r + \|v_n - v_0\|_r^r\big) \to 0$$

as $n \to \infty$, that is,

$$(u_n, v_n) \to (u_0, v_0) \quad \text{in } L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^N), r \in [2, 2^*),$$

so that $E \hookrightarrow L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$ is compact for $r \in [2, 2^*)$. The proof is complete. □

**Lemma 2.2** *If assumptions* ($H_1$)-($H_2$) *hold, then* $I \in C^1(E, R)$,

$$\begin{aligned}\langle I'(u,v), (\varphi, \psi)\rangle = {} & \int_{\mathbb{R}^N} \nabla u \nabla \varphi \, dx + \int_{\mathbb{R}^N} V(x) u\varphi \, dx - \int_{\mathbb{R}^N} H(x) F_u(x,u,v)\varphi \, dx \\ & + \int_{\mathbb{R}^N} \nabla v \nabla \psi \, dx + \int_{\mathbb{R}^N} V(x) v\psi \, dx - \int_{\mathbb{R}^N} H(x) F_v(x,u,v)\psi \, dx, \end{aligned} \tag{2.2}$$

*and* $\Psi' : E \to E^*$ *is compact, where* $\Psi(u,v) = \int_{\mathbb{R}^N} H(x) F(x,u,v)\, dx$.

*Proof* The proof is similar to that of Lemma 3.1 in [12]; we omit it. □

To complete the proof of our theorem, the following theorem will be needed in our argument. Let $E$ be a Banach space with norm $\|\cdot\|$ and $E = \overline{\bigoplus_{j \in N} X_j}$ with $\dim X_j < \infty$ for any $j \in N$. Set $Y_k = \bigoplus_{j=0}^k X_j$, $Z_k = \overline{\bigoplus_{j=k+1}^\infty X_j}$. Consider the $C^1$ functional

$$\varphi_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1, 2],$$

where $A, B : E \to \mathbb{R}$ are two functionals.

**Theorem 2.1** ([25], Theorem 2.1) *Suppose that the functional* $\varphi_\lambda(u)$ *satisfies*:

($C_1$) *$\varphi_\lambda(u)$ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Furthermore, $\varphi_\lambda(-u) = \varphi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$.*
($C_2$) *$B(u) \geq 0$; $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite-dimensional subspace of $E$.*
($C_3$) *There exists $\rho_k > r_k > 0$ such that*

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \varphi_\lambda(u) \geq 0 > b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \varphi_\lambda(u), \quad \lambda \in [1,2],$$

$$d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \varphi_\lambda(u) \to 0 \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

*Then there exist $\lambda_n \to 1$ and $u(\lambda_n) \in Y_n$ such that*

$$\varphi'_{\lambda_n}|_{Y_n}\big(u(\lambda_n)\big) = 0, \qquad \varphi_{\lambda_n}\big(u(\lambda_n)\big) \to c_k \in \big[d_k(2), b_k(1)\big] \quad \text{as } n \to \infty.$$

*In particular, if $\{u(\lambda_n)\}$ has a convergent subsequence for every $k$, then $\varphi_1$ has many nontrivial critical points $\{u_k\} \subset E \setminus \{0\}$ satisfying $\varphi_1(u_k) \to 0^-$ as $n \to \infty$.*

## 3 Proof of the main result

In order to apply Theorem 2.1 to prove our main result, we define $A$, $B$, and $\varphi_\lambda$ on our working space $E$ by

$$A(u,v) = \frac{1}{2}\big\|(u,v)\big\|^2, \qquad B(u,v) = \int_{\mathbb{R}^N} H(x)F(x,u,v)\,dx, \tag{3.1}$$

and

$$\varphi_\lambda(u,v) = A(u,v) - \lambda B(u,v) = \frac{1}{2}\big\|(u,v)\big\|^2 - \lambda \int_{\mathbb{R}^N} H(x)F(x,u,v)\,dx \tag{3.2}$$

for all $(u,v) \in E$ and $\lambda \in [1,2]$. Obviously, $\varphi_\lambda(u,v) \in C^1(E,R)$ for all $\lambda \in [1,2]$. We choose a completely orthonormal basis $\{e_j : j \in N\}$ of $X$ and let $X_j = \operatorname{span}\{e_j\}$ for all $j \in N$. Then $Y_k = \operatorname{span}\{e_1, \dots, e_k\}$, $Z_k = Y_k^\perp$, and $E = (Y_k \times Y_k) \oplus (Z_k \times Z_k)$. Note that $\varphi_1 = I$, where $I$ is defined in (2.1).

**Lemma 3.1** *Suppose that conditions* ($H_1$), ($H_3$), *and* ($H_4$) *hold. Then $B(u,v) \geq 0$. Furthermore, $B(u,v) \to \infty$ as $\|(u,v)\| \to \infty$ on any finite-dimensional subspace of $E$.*

*Proof* Evidently, by ($H_3$) and ($H_4$), $B(u,v) = \int_{\mathbb{R}^N} H(x)F(x,u,v)\,dx \geq 0$, Now we claim that for any finite-dimensional subspace $\widetilde{E} \subset E$, there exists $\varepsilon > 0$ such that

$$\operatorname{meas}\big\{x \in \mathbb{R}^N : H(x)\big|(u,v)\big|^\mu \geq \varepsilon \big\|(u,v)\big\|^\mu\big\} \geq \varepsilon, \quad \forall (u,v) \in \widetilde{E}. \tag{3.3}$$

Arguing by contradiction, we assume that there exists a sequence $\{(u_n, v_n)\}_{n \in \mathbb{N}} \subset \widetilde{E} \setminus \{(0,0)\}$ such that

$$\operatorname{meas}\left\{x \in \mathbb{R}^N : H(x)\big|(u_n, v_n)\big|^\mu \geq \frac{1}{n}\big\|(u_n, v_n)\big\|^\mu\right\} < \frac{1}{n}. \tag{3.4}$$

Set $(s_n, w_n) = \frac{(u_n, v_n)}{\|(u_n, v_n)\|} \subset \widetilde{E} \setminus \{(0,0)\}$, then $\|(s_n, w_n)\| = 1$ for all $n \in \mathbb{N}$, and

$$\operatorname{meas}\left\{x \in \mathbb{R}^N : H(x)\big|(s_n, w_n)\big|^\mu \geq \frac{1}{n}\right\} < \frac{1}{n}. \tag{3.5}$$

Since $\dim \widetilde{E} < \infty$, it follows from the compactness of the unit sphere of $\widetilde{E}$ that there exists a subsequence, say $\{(s_n, w_n)\}$, such that $(s_n, w_n) \to (s_0, w_0)$ in $\widetilde{E}$. It is easy to verify that $\|(s_0, w_0)\| = 1$. In view of the equivalence of the norms on the finite-dimensional space $\widetilde{E}$, we have $(s_n, w_n) \to (s_0, w_0)$ in $L^2(\mathbb{R}^N)$, that is,

$$\int_{\mathbb{R}^N} \left|(s_n, w_n) - (s_0, w_0)\right|^2 dx \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

By (3.6) and the Hölder inequality we have

$$\begin{aligned} &\int_{\mathbb{R}^N} H(x)\left|(s_n, w_n) - (s_0, w_0)\right|^\mu dx \\ &\quad \leq \left\|H(x)\right\|_{\frac{2}{2-\mu}} \left(\int_{\mathbb{R}^N} \left|(s_n, w_n) - (s_0, w_0)\right|^2 dx\right)^{\frac{\mu}{2}} \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.7}$$

Therefore, there exist $\xi_1, \xi_2 > 0$ such that

$$\operatorname{meas}\left\{x \in \mathbb{R}^N : H(x)\left|(s_0, w_0)\right|^\mu \geq \xi_1\right\} \geq \xi_2. \tag{3.8}$$

Otherwise, we get

$$\operatorname{meas}\left\{x \in \mathbb{R}^N : H(x)\left|(s_0, w_0)\right|^\mu \geq \frac{1}{n}\right\} = 0, \tag{3.9}$$

which implies that

$$0 \leq \int_{\mathbb{R}^N} H(x)\left|(s_0, w_0)\right|^{\mu+2} dx \leq \frac{\|(s_0, w_0)\|_2^2}{n} \leq \frac{C^2\|(s_0, w_0)\|^2}{n} = \frac{C^2}{n} \to 0 \quad \text{as } n \to \infty.$$

Hence, $(s_0, w_0) = 0$, which contradicts with $\|(s_0, w_0)\| = 1$. Therefore, (3.8) holds.

Now let

$$\begin{aligned} \Omega_0 &= \operatorname{meas}\left\{x \in \mathbb{R}^N : H(x)\left|(s_0, w_0)\right|^\mu \geq \xi_1\right\}, \\ \Omega_n &= \operatorname{meas}\left\{x \in \mathbb{R}^N : H(x)\left|(s_0, w_0)\right|^\mu < \frac{1}{n}\right\}, \\ \Omega_n^c &= \mathbb{R}^N \setminus \Omega_n = \operatorname{meas}\left\{x \in \mathbb{R}^N : H(x)\left|(s_0, w_0)\right|^\mu \geq \frac{1}{n}\right\}. \end{aligned}$$

Then by (3.5) and (3.8) we get

$$\begin{aligned} \operatorname{meas}(\Omega_n \cap \Omega_0) &= \operatorname{meas}\left(\Omega_0 \setminus \left(\Omega_n^c \cap \Omega_0\right)\right) \\ &\geq \operatorname{meas}(\Omega_0) - \operatorname{meas}\left(\Omega_n^c \cap \Omega_0\right) \\ &\geq \xi_2 - \frac{1}{n} \end{aligned}$$

for all positive integer $n$. Let $n$ be large enough such that $\xi_2 - \frac{1}{n} \geq \frac{\xi_2}{2}$ and $\frac{\xi_1}{2^\mu} - \frac{1}{n} \geq \frac{\xi_1}{2^{\mu+1}}$. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} H(x)\big|(s_n, w_n) - (s_0, w_0)\big|^\mu \, dx \\
&\quad \geq \int_{\Omega_n \cap \Omega_0} H(x)\big|(s_n, w_n) - (s_0, w_0)\big|^\mu \, dx \\
&\quad \geq \frac{1}{2^\mu} \int_{\Omega_n \cap \Omega_0} H(x)\big|(s_0, w_0)\big|^\mu \, dx - \int_{\Omega_n \cap \Omega_0} H(x)\big|(s_n, w_n)\big|^\mu \, dx \\
&\quad \geq \left(\frac{\xi_1}{2^\mu} - \frac{1}{n}\right) \operatorname{meas}(\Omega_n \cap \Omega_0) > 0,
\end{aligned}$$

which is a contradiction with (3.7). Therefore, (3.3) holds. For the $\varepsilon$ given in (3.3), let

$$\Omega_{(u,v)} = \operatorname{meas}\big\{x \in \mathbb{R}^N : H(x)\big|(u,v)\big|^\mu \geq \varepsilon \big\|(u,v)\big\|^\mu\big\}, \quad \forall (u,v) \in \widetilde{E} \setminus \big\{(0,0)\big\}.$$

Then by (3.3)

$$\operatorname{meas}(\Omega_{(u,v)}) \geq \varepsilon, \quad \forall (u,v) \in \widetilde{E} \setminus \big\{(0,0)\big\}. \tag{3.10}$$

Combining ($H_3$) and (3.10), we have

$$B(u,v) \geq \int_{\mathbb{R}^N} c_1 H(x)\big|(u,v)\big|^\mu \, dx \geq \int_{\Omega_{(u,v)}} \varepsilon c_1 \big\|(u,v)\big\|^\mu \, dx \geq \varepsilon^2 c_1 \big\|(u,v)\big\|^\mu,$$

which implies that $B(u,v) \to \infty$ as $\|(u,v)\| \to \infty$ on any finite-dimensional space of $E$. The proof is completed. □

**Lemma 3.2** *Suppose that* ($H_1$)-($H_2$) *and* ($H_4$) *are satisfied. Then there exists a sequence* $\rho_k \to 0^+$ *as* $k \to \infty$ *such that* $a_k(\lambda) := \inf_{(u,v)\in Z_k\times Z_k, \|(u,v)\|=\rho_k} \varphi_\lambda(u,v) \geq 0$ *and* $d_k(\lambda) := \inf_{(u,v)\in Z_k\times Z_k, \|(u,v)\|\leq\rho_k} \varphi_\lambda(u,v) \to 0$ *as* $k \to \infty$ *uniformly for* $\lambda \in [1,2]$, *where* $Z_k = \overline{\bigoplus_{j=k+1}^\infty X_j} = \overline{\operatorname{span}\{e_k, \ldots\}}$ *for all* $k \in N$.

*Proof* Let

$$\alpha_k(r) := \sup_{(u,v)\in Z_k\times Z_k, \|(u,v)\|=1} \big\|(u,v)\big\|_r, \quad \forall k \in N, \tag{3.11}$$

where $\|(u,v)\|_r = (\int_{\mathbb{R}^N} |(u,v)|^r)^{\frac{1}{r}}$. Then $\alpha_k(r) \to 0$ as $k \to \infty$. Indeed, $\alpha_k(r)$ is convergent since $\alpha_k(r)$ are decreasing in $k$ and $\alpha_k(r) \geq 0$. Furthermore, for any $k$, there exists $(u_k, v_k) \in Z_k \times Z_k$ such that $\|(u_k, v_k)\| = 1$ and $\|(u_k, v_k)\|_r \geq \frac{\alpha_k(r)}{2}$.

For any $\varphi \in X$, $\varphi = \sum_{n=1}^\infty a_n e_n$, we get

$$\big|\langle u_k, \varphi\rangle_X\big| = \left|\left\langle u_k, \sum_{n=k+1}^\infty a_n e_n\right\rangle_X\right| \leq \|u_k\|_X \left\|\sum_{n=k+1}^\infty a_n e_n\right\|_X \leq \left\|\sum_{n=k+1}^\infty a_n e_n\right\|_X \to 0$$

as $k \to \infty$, which implies that $u_k \rightharpoonup 0$ in $X$. Since the embedding $X \hookrightarrow L^r(\mathbb{R}^N)$ is compact, $u_k \to 0$ in $L^r(\mathbb{R}^N)$ for $r \in [2, 2^*)$. The same argument implies that $v_k \to 0$ in $L^r(\mathbb{R}^N)$ for

$r \in [2, 2^*)$. Consequently,

$$\|(u_k, v_k)\|_r^r \le 2^{\frac{r}{2}} \left(\|u_k\|_r^r + \|v_k\|_r^r\right) \to \quad \text{as } k \to \infty,$$

that is,

$$\alpha_k(r) \to 0 \quad \text{as } k \to \infty. \tag{3.12}$$

By $(H_2)$ we have

$$\begin{aligned} |F(x,u,v)| &= |F(x,u,v) - F(x,0,0)| \\ &\le \int_0^1 |F_u(x,tu,tv)||u|\,dt + \int_0^1 |F_v(x,tu,tv)||v|\,dt \\ &\le C\left(|(u,v)|^2 + |(u,v)|^p + |(u,v)|^q\right). \end{aligned} \tag{3.13}$$

Therefore, by (3.2), (3.11)-(3.13), and the Hölder inequality we get

$$\begin{aligned} \varphi_\lambda(u,v) &= \frac{1}{2}\|(u,v)\|^2 - \lambda \int_{\mathbb{R}^N} H(x)F(x,u,v)\,dx \\ &\ge \frac{1}{2}\|(u,v)\|^2 - 2C\left(\|(u,v)\|_2^2 + \|H(x)\|_{\frac{2}{2-p}} \|(u,v)\|_2^p + \|H(x)\|_{\frac{2}{2-q}} \|(u,v)\|_2^q\right) \\ &\ge \frac{1}{2}\|(u,v)\|^2 - 2C\left(\alpha_k^2(2)\|(u,v)\|^2 + \alpha_k^p(2)\|(u,v)\|^p + \alpha_k^q(2)\|(u,v)\|^q\right). \end{aligned} \tag{3.14}$$

By (3.13) there exist a positive integer $k_1$ such that

$$2C\alpha_k^2(2) \le \frac{1}{8}, \quad \forall k \ge k_1. \tag{3.15}$$

Then, by (3.14), we have

$$\varphi_\lambda(u,v) \ge \frac{3}{8}\|(u,v)\|^2 - 2C\left(\alpha_k^p(2)\|(u,v)\|^p + \alpha_k^q(2)\|(u,v)\|^q\right). \tag{3.16}$$

Let

$$\rho_k = \max\left\{\left(16C\alpha_k^p(2)\right)^{\frac{1}{2-p}}, \left(16C\alpha_k^q(2)\right)^{\frac{1}{2-q}}\right\}. \tag{3.17}$$

Obviously, $\rho_k \to 0$ as $k \to \infty$ since $p, q \in (1,2)$. By (3.16) and (3.17) direct computation shows that

$$a_k(\lambda) := \inf_{(u,v) \in Z_k \times Z_k, \|(u,v)\| = \rho_k} \varphi_\lambda(u,v) \ge \frac{\rho_k^2}{8} > 0, \quad \forall k \ge k_1.$$

Moreover, by (3.14), for any $(u,v) \in Z_k \times Z_k$ with $\|(u,v)\| = \rho_k$, we have

$$\varphi_\lambda(u,v) \ge -2C\left(\alpha_k^2(2)\|(u,v)\|^2 + \alpha_k^p(2)\|(u,v)\|^p + \alpha_k^q(2)\|(u,v)\|^q\right).$$

Therefore,

$$0 \geq \inf_{(u,v)\in Z_k\times Z_k, \|(u,v)\|\leq\rho_k} \varphi_\lambda(u,v) \geq -2C\left(\alpha_k^2(2)\left\|(u,v)\right\|^2 + \alpha_k^p(2)\left\|(u,v)\right\|^p + \alpha_k^q(2)\left\|(u,v)\right\|^q\right).$$

Since $\alpha_k(2) \to 0$ as $k \to \infty$, we have

$$d_k(\lambda) := \inf_{(u,v)\in Z_k\times Z_k, \|(u,v)\|\leq\rho_k} \varphi_\lambda(u,v) \to 0 \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

The proof is completed. □

**Lemma 3.3** *Suppose that* $(\mathrm{H}_1)$-$(\mathrm{H}_4)$ *hold. Then for the positive integer* $k_1$ *and the sequence* $\{\rho_k\}$ *obtained in Lemma* 3.2, *for all* $k \geq k_1$, *there exist* $0 < r_k < \rho_k$ *such that*

$$b_k(\lambda) := \max_{(u,v)\in Y_k\times Y_k, \|(u,v)\|=r_k} \varphi_\lambda(u,v) < 0 \quad \textit{for all } \lambda \in [1,2], \forall k \geq k_1,$$

*where* $Y_k = \bigoplus_{j=1}^k X_j = \operatorname{span}\{e_1,\ldots e_k\}$ *for all* $k \in N$.

*Proof* Note that $Y_k \times Y_k$ is a finite-dimensional subspace of $E$. Then by (3.3) there exists a constant $\varepsilon_k$ such that

$$\operatorname{meas}\left(\Omega_{(u,v)}^k\right) \geq \varepsilon_k, \quad \forall (u,v) \in Y_k \times Y_k \setminus \left\{(0,0)\right\}, \tag{3.18}$$

where

$$\Omega_{(u,v)}^k = \operatorname{meas}\left\{x \in \mathbb{R}^N : H(x)\left|(u,v)\right|^\mu \geq \varepsilon_k\left\|(u,v)\right\|^\mu\right\}, \quad \forall (u,v) \in Y_k \times Y_k \setminus \left\{(0,0)\right\}.$$

Combining (3.2), $(\mathrm{H}_3)$, $(\mathrm{H}_4)$, and (3.18), for any $k \in N$ and $\lambda \in [1,2]$, we have

$$\begin{aligned}
\varphi_\lambda(u,v) &= \frac{1}{2}\left\|(u,v)\right\|^2 - \lambda\int_{\mathbb{R}^N} H(x)F(x,u,v)\,dx \\
&\leq \frac{1}{2}\left\|(u,v)\right\|^2 - c_1\int_{\mathbb{R}^N} H(x)\left|(u,v)\right|^\mu\,dx \\
&\leq \frac{1}{2}\left\|(u,v)\right\|^2 - c_1\int_{\Omega_{(u,v)}^k} H(x)\left|(u,v)\right|^\mu\,dx \\
&\leq \frac{1}{2}\left\|(u,v)\right\|^2 - c_1\varepsilon_k\left\|(u,v)\right\|^\mu \operatorname{meas}\left(\Omega_{(u,v)}^k\right) \\
&\leq \frac{1}{2}\left\|(u,v)\right\|^2 - c_1\varepsilon_k^2\left\|(u,v)\right\|^\mu.
\end{aligned} \tag{3.19}$$

For $\|(u,v)\| = r_k < \rho_k$ small enough, we have

$$b_k(\lambda) := \max_{(u,v)\in Y_k\times Y_k, \|(u,v)\|=r_k} \varphi_\lambda(u,v) < 0 \quad \text{for all } \lambda \in [1,2], \forall k \geq k_1,$$

since $\mu \in (1,2)$. The proof is completed. □

Now we give the proof of Theorem 1.1.

*Proof* Obviously, condition $(C_1)$ in Theorem 2.1 holds. By Lemmas 3.1-3.3 conditions $(C_2)$ and $(C_3)$ in Theorem 2.1 are also satisfied. Furthermore, by Theorem 2.1, there exist $\lambda_n \to 1$ and $(u(\lambda_n), v(\lambda_n)) \in Y_n \times Y_n$ such that

$$\begin{aligned} &\varphi'_{\lambda_n}|_{Y_n\times Y_n}\big(u(\lambda_n), v(\lambda_n)\big) = 0, \\ &\varphi_{\lambda_n}\big(u(\lambda_n), v(\lambda_n)\big) \to c_k \in \big[d_k(2), b_k(1)\big] \quad \text{as } n \to \infty. \end{aligned} \tag{3.20}$$

For simplicity, in what follows, we always set $(u_n, v_n) = (u(\lambda_n), v(\lambda_n))$ for all $n \in \mathbb{N}$.

Now we claim that the sequence $\{(u_n, v_n)\}$ obtained in (3.19) is bounded in $E$. Indeed, by $(H_2)$, $(H_4)$, (3.2), (3.20), and the Hölder inequality we have

$$\begin{aligned} \big\|(u_n, v_n)\big\|^2 &\le 2\varphi_{\lambda_n}\big((u_n, v_n)\big) + 2\lambda_n \int_{\mathbb{R}^N} H(x)F(x, u_n, v_n)\,dx \\ &\le C_0 + C\big(\big\|(u_n, v_n)\big\|^2 + \big\|(u, v)\big\|_p^p + \big\|(u, v)\big\|_q^q\big) \end{aligned} \tag{3.21}$$

for some $C_0 > 0$. Since $p, q \in (1, 2)$, (3.21) implies that $\{(u_n, v_n)\}$ is bounded in $E$.

Finally, we show that $\{(u_n, v_n)\}$ possesses a strong convergent sequence in $E$. Indeed, since $\{(u_n, v_n)\}$ is bounded, there exists $(u_0, v_0) \in E$ such that

$$\begin{aligned} &(u_n, v_n) \rightharpoonup (u_0, v_0) \quad \text{in } E, \\ &(u_n, v_n) \to (u_0, v_0) \quad \text{in } L^p\big(\mathbb{R}^N\big) \times L^p\big(\mathbb{R}^N\big), p \in \big[2, 2^*\big), \\ &(u_n, v_n) \to (u_0, v_0) \quad \text{a.e. on } \mathbb{R}^N. \end{aligned}$$

By (2.2) we easily get

$$\begin{aligned} \big\|(u_n, v_n) - (u_0, v_0)\big\|^2 = {}&\big\langle \varphi'_{\lambda_n}(u_n, v_n) - \varphi'_1(u_0, v_0), (u_n, v_n) - (u_0, v_0)\big\rangle \\ &+ \int_{\mathbb{R}^N} H(x)\big(\lambda_n F_u(x, u_n, v_n) - F_u(x, u_0, v_0)\big)(u_n - u_0)\,dx \\ &+ \int_{\mathbb{R}^N} H(x)\big(\lambda_n F_v(x, u_n, v_n) - F_v(x, u_0, v_0)\big)(v_n - v_0)\,dx. \end{aligned} \tag{3.22}$$

Clearly,

$$\big\langle \varphi'_{\lambda_n}(u_n, v_n) - \varphi'_1(u_0, v_0), (u_n, v_n) - (u_0, v_0)\big\rangle \to 0. \tag{3.23}$$

Denote

$$\begin{aligned} M &:= \int_{\mathbb{R}^N} H(x)\big(\lambda_n F_u(x, u_n, v_n) - F_u(x, u_0, v_0)\big)(u_n - u_0)\,dx, \\ N &:= \int_{\mathbb{R}^N} H(x)\big(\lambda_n F_v(x, u_n, v_n) - F_v(x, u_0, v_0)\big)(v_n - v_0)\,dx. \end{aligned}$$

Then by $(H_2)$, $(H_4)$, and the Hölder and Minkowski inequalities we have

$$\begin{aligned} M \le C\|u_n - u_0\|_2 \bigg(\int_{\mathbb{R}^N} H^2(x)\big(2\big|(u_n, v_n)\big| + 2\big|(u_n, v_n)\big|^{p-1} + 2\big|(u_n, v_n)\big|^{q-1} \\ + \big|(u_0, v_0)\big| + \big|(u_0, v_0)\big|^{p-1} + \big|(u_0, v_0)\big|^{q-1}\big)^2\,dx\bigg)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\le C\|u_n - u_0\|_2 \biggl( \int_{\mathbb{R}^N} H^2(x)\bigl(2\bigl|(u_n,v_n)\bigr| + \bigl|(u_0,v_0)\bigr|\bigr)^2 + H^2(x)\bigl(2\bigl|(u_n,v_n)\bigr|^{p-1} \\
&\quad + \bigl|(u_0,v_0)\bigr|^{p-1}\bigr)^2 + H^2(x)\bigl(2\bigl|(u_n,v_n)\bigr|^{q-1} + \bigl|(u_0,v_0)\bigr|^{q-1}\bigr)^2\,dx \biggr)^{\frac{1}{2}} \\
&\le C\|u_n - u_0\|_2 \biggl( \int_{\mathbb{R}^N} H^2(x)\bigl(4\bigl|(u_n,v_n)\bigr|^2 + \bigl|(u_0,v_0)\bigr|^2\bigr) + H^2(x)\bigl(4\bigl|(u_n,v_n)\bigr|^{2p-2} \\
&\quad + \bigl|(u_0,v_0)\bigr|^{2p-2}\bigr) + H^2(x)\bigl(4\bigl|(u_n,v_n)\bigr|^{2q-2} + \bigl|(u_0,v_0)\bigr|^{2q-2}\bigr)\,dx \biggr)^{\frac{1}{2}} \\
&\le C\|u_n - u_0\|_2 \biggl[ \biggl( \int_{\mathbb{R}^N} 4H^2(x)\bigl|(u_n,v_n)\bigr|^2\,dx \biggr)^{\frac{1}{2}} + \biggl( \int_{\mathbb{R}^N} H^2(x)\bigl|(u_0,v_0)\bigr|^2\,dx \biggr)^{\frac{1}{2}} \\
&\quad + \biggl( \int_{\mathbb{R}^N} 4H^2(x)\bigl|(u_n,v_n)\bigr|^{2p-2} \biggr)^{\frac{1}{2}} + \biggl( \int_{\mathbb{R}^N} H^2(x)\bigl|(u_0,v_0)\bigr|^{2p-2}\,dx \biggr)^{\frac{1}{2}} \\
&\quad + \biggl( \int_{\mathbb{R}^N} 4H^2(x)\bigl|(u_n,v_n)\bigr|^{2q-2}\,dx \biggr)^{\frac{1}{2}}\,dx + \biggl( \int_{\mathbb{R}^N} H^2(x)\bigl|(u_0,v_0)\bigr|^{2q-2}\,dx \biggr)^{\frac{1}{2}} \biggr] \\
&\le C\|u_n - u_0\|_2 \bigl[ 2\|H\|_\infty \bigl\|(u_n,v_n)\bigr\|_2^2 + \|H\|_\infty \bigl\|(u_0,v_0)\bigr\|_2^2 + 2\|H\|_{\frac{2}{2-p}} \bigl\|(u_n,v_n)\bigr\|_2^{p-1} \\
&\quad + \|H\|_{\frac{2}{2-p}} \bigl\|(u_0,v_0)\bigr\|_2^{p-1} + 2\|H\|_{\frac{2}{2-q}} \bigl\|(u_n,v_n)\bigr\|_2^{q-1} + \|H\|_{\frac{2}{2-q}} \bigl\|(u_0,v_0)\bigr\|_2^{q-1} \bigr] \\
&\le C\|u_n - u_0\|_2 \bigl( \bigl\|(u_n,v_n)\bigr\|_2^2 + \bigl\|(u_0,v_0)\bigr\|_2^2 + \bigl\|(u_n,v_n)\bigr\|_2^{p-1} + \bigl\|(u_0,v_0)\bigr\|_2^{p-1} \\
&\quad + \bigl\|(u_n,v_n)\bigr\|_2^{q-1} + \bigl\|(u_0,v_0)\bigr\|_2^{q-1} \bigr).
\end{aligned} \tag{3.24}$$

Since $(u_n, v_n) \to (u_0, v_0)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$, for any $p \in [2, 2^*)$, we obtain

$$\int_{\mathbb{R}^N} H(x)\bigl(\lambda_n F_u(x,u_n,v_n) - F_u(x,u_0,v_0)\bigr)(u_n - u_0)\,dx \to 0 \quad \text{as } n \to \infty. \tag{3.25}$$

Similarly, we can also obtain

$$\int_{\mathbb{R}^N} H(x)\bigl(\lambda_n F_v(x,u_n,v_n) - F_v(x,u_0,v_0)\bigr)(v_n - v_0)\,dx \to 0 \quad \text{as } n \to \infty. \tag{3.26}$$

Therefore, by (3.22)-(3.26) we get $\|(u_n, v_n) - (u_0, v_0)\| \to 0$ as $n \to \infty$.

Now from the last assertion of Theorem 2.1 we know that $I = \varphi_1$ has infinitely many nontrivial critical points. Therefore, system (1.1) possesses infinitely many small negative-energy solutions. The proof is completed. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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**References**

1. De Figueiredo, DG, Felmer, PL: On superquadratic elliptic systems. Trans. Am. Math. Soc. **223**, 99-116 (1994)

2. De Figueiredo, DG, Mitidieri, E: A maximum principle for an elliptic system and applications to semilinear problem. SIAM J. Math. Anal. **17**, 836-849 (1986)
3. Lazer, AC, McKenna, PJ: On steady-state solutions of a system of reaction-diffusion equations from biology. Nonlinear Anal. TMA **6**, 523-530 (1982)
4. Silva, EA: Existence and multiplicity of solutions for semilinear elliptic systems. NoDEA Nonlinear Differ. Equ. Appl. **1**, 339-363 (1994)
5. Qu, Z, Tang, C: Existence and multiplicity results for some elliptic systems at resonance. Nonlinear Anal. **71**, 2660-2666 (2009)
6. Duan, S, Wu, X: The existence of solutions for a class of semilinear elliptic systems. Nonlinear Anal. **73**, 2842-2854 (2010)
7. Pomponio, A: Asymptotically linear cooperative elliptic system: existence and multiplicity. Nonlinear Anal. **52**, 989-1003 (2003)
8. Costa, DG, Magalhães, CA: A unified approach to a class of strong indefinite functions. J. Differ. Equ. **125**, 521-547 (1996)
9. Costa, DG, Magalhães, CA: A variational method to subquadratic perturbations of elliptic systems. J. Differ. Equ. **111**, 103-122 (1994)
10. Cao, D, Tang, Z: Solutions with prescribed number of nodes to superlinear elliptic systems. Nonlinear Anal. **55**, 702-722 (2003)
11. Zhang, Z, Shi, Y, Xue, Y: Existence of entire solutions for semilinear elliptic systems under the Keller-Osserman condition. Electron. J. Differ. Equ. **2011**, 39 (2011)
12. Wu, X: High energy solutions of systems of Kirchhoff-type equations in $\mathbb{R}^N$. J. Math. Phys. **53**, 063508 (2012)
13. Zhou, F, Wu, K, Wu, X: High energy solutions of systems of Kirchhoff-type equations on $\mathbb{R}^N$. Comput. Math. Appl. **66**, 1299-1305 (2013)
14. Li, G, Tang, X: Nehari-type state solutions for Schrödinger equations including critical exponent. Appl. Math. Lett. **37**, 101-106 (2014)
15. Maia, LA, Montefusco, E, Pellacci, B: Positive solutions for a weakly coupled nonlinear Schrödinger system. J. Differ. Equ. **229**, 743-767 (2006)
16. Tang, X: Infinitely many solutions for semilinear Schrodinger equations with sign-changing potential and nonlinearity. J. Math. Anal. Appl. **401**, 407-415 (2013)
17. Huang, W, Tang, X: The existence of infinitely many solutions for the nonlinear Schrödinger-Maxwell equations. Results Math. **65**, 223-234 (2014)
18. Sun, J, Chen, H, Yang, L: Positive solutions of asymptotically linear Schrödinger-Poisson systems with a radial potential vanishing at infinity. Nonlinear Anal. **74**, 413-423 (2011)
19. Liu, H, Chen, H, Yang, X: Multiple solutions for superlinear Schrödinger-Poisson systems with sign-changing potential and nonlinearity. Comput. Math. Appl. **68**, 1982-1990 (2014)
20. Zhang, J, Tang, X: Infinitely many solutions of quasilinear Schrödinger equation with sign-changing potential. J. Math. Anal. Appl. **420**, 1762-1775 (2014)
21. Xu, L, Chen, H: Existence and multiplicity of solutions for fourth-order elliptic equations of Kirchhoff type via genus theory. Bound. Value Probl. **2014**, 212 (2014)
22. Xu, L, Chen, H: Existence of infinitely many solutions for generalized Schrödinger-Poisson system. Bound. Value Probl. **2014**, 196 (2014)
23. Qin, D, Tang, X: New conditions on solutions for periodic Schrödinger equation with spectrum zero. Taiwan. J. Math. **19**(4), 977-993 (2015)
24. Zou, W, Schechter, M: Critical Point Theory and Its Applications. Springer, New York (2006)
25. Xu, L, Chen, H: Multiplicity of small negative-energy solutions for a class of nonlinear Schrödinger-Poisson systems. Appl. Math. Comput. **243**, 817-824 (2014)