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New results for positive solutions of singular fourth-order four-point p-Laplacian problem

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Abstract
The existence and uniqueness of positive solutions are obtained for singular fourth-order four-point boundary value problem with p-Laplace operator $[\varphi_p(u''(t))]'' = f(t,u(t))$, $0 < t < 1$, $u(0) = 0$, $u(1) = au(\xi)$, $u''(0) = 0$, $u''(1) = bu''(\eta)$, where $f(t,u)$ is singular at $t = 0,1$ and $u = 0$. A fixed point theorem for mappings that are decreasing with respect to a cone in a Banach space plays a key role in the proof.

MSC: 34B15; 34B16; 34B18

Keywords: p-Laplace operator; singular; fourth-order four-point boundary value problem; positive solution; cone

1 Introduction

In this paper, we investigate the existence and uniqueness of positive solutions for the singular fourth-order differential equation involving the p-Laplace operator

$$[\varphi_p(u''(t))]'' = f(t,u(t)), \quad t \in (0,1), \tag{1.1}$$

with the four-point boundary conditions

$$u(0) = 0, \qquad u(1) = au(\xi), \qquad u''(0) = 0, \qquad u''(1) = bu''(\eta), \tag{1.2}$$

where $\varphi_p(t) = |t|^{p-2}t$, $p > 1$, $0 < \xi, \eta < 1$, $0 \le a < 1/\xi$, $0 \le b^{p-1} < 1/\eta$, and $f(t,x)$ is singular at $t = 0,1$ and $x = 0$. Here by a positive solution u of SBVP (1.1)-(1.2) we mean a solution $u \in C^2[0,1]$ with $\varphi_p(u'') \in C^2(0,1) \cap C[0,1]$ satisfying $u(t) > 0$ on $(0,1)$.

It is well known that the bending of elastic beam can be described by some fourth-order boundary value problems. There are extensive studies on fourth-order boundary value problems with diverse boundary conditions by using different methods, for instance, [1–28] and the references therein.

Recently, in the case $0 \le a < 1$, $0 \le b < 1$, using the lower and upper solution method and the Schauder fixed-point theorem, Zhang and Liu [25] proved that the SBVP (1.1)-(1.2) has at least one positive solution under the following assumptions:

(H_1) $f \in C((0,1) \times (0,\infty), [0,\infty))$, and $f(t,x)$ is nonincreasing in x;

(H_2) For any constant $\lambda > 0$, $0 < \int_0^1 H(s,s)f(s,\lambda s(1-s))\,\mathrm{d}s < \infty$;
(H_3) There exist a continuous function $a(t)$ in $[0,1]$ and a fixed positive number k such that $a(t) \geq kt(1-t)$, $t \in [0,1]$, and

$$\int_0^1 G(t,r)\varphi_p^{-1}\left(\int_0^1 H(r,s)f\big(s,a(s)\big)\,\mathrm{d}s\right)\mathrm{d}r := b(t) \geq a(t), \quad t \in [0,1],$$
$$\int_0^1 G(t,r)\varphi_p^{-1}\left(\int_0^1 H(r,s)f\big(s,b(s)\big)\,\mathrm{d}s\right)\mathrm{d}r \geq a(t), \quad t \in [0,1],$$

where $G(t,s)$, $H(t,s)$ will be given in Section 2.

The purpose of this paper is to improve the existence results of [25]. Using a fixed point theorem for mappings that are decreasing with respect to a cone in a Banach space, we obtain the existence and uniqueness of positive solutions of SBVP (1.1)-(1.2). We note that, in our proofs, we just assume that (H_1) and (H_2) of [25] with (H_3) of [25] removed. Our study is motivated by the papers [11, 29].

In addition, we note that we also obtained the uniqueness of a positive solution for SBVP (1.1)-(1.2).

The rest of the paper is organized as follows. The fixed point theorem of Gatica *et al.* [29] and some definitions and lemmas are given in Section 2. The main results on the existence of positive solutions for SBVP (1.1)-(1.2) are presented in Section 3.

2 Preliminary

Let B be a Banach space. A nonempty closed set $K \subset B$ is called a *cone* if the following conditions are satisfied:

(i) $au + bv \in K$ for all $u, v \in K$ and all $a, b \geq 0$;
(ii) $u, -u \in K$ imply $u = 0$.

Given a cone K, a partial order $\preceq$ is induced on B as follows; $u \preceq v$ for $u, v \in B$ iff $v - u \in K$ (for clarity, we sometimes write $u \preceq v$ (w.r.t. K)). For $u, v \in B$ with $u \preceq v$, we denote by $\langle u, v\rangle$ the closed order interval between u and v, that is, $\langle u, v\rangle = \{w \in B : u \preceq w \preceq v\}$. A cone K is *normal* in B if there exists $\delta > 0$ such that $\|e_1 + e_2\| \geq \delta$ for all $e_1, e_2 \in K$ with $\|e_1\| = \|e_2\| = 1$.

Lemma 2.1 ([11, 29]) *Let B be a Banach space, K a normal cone in B, D a subset of K such that if $u, v \in D$ with $u \preceq v$, then $\langle u, v\rangle \subset D$, and let $T : D \to K$ be a continuous mapping that is compact on any closed order interval contained in D. Suppose that there exists $u_0 \in D$ such that T^2u_0 is defined and Tu_0 and T^2u_0 are order-comparable to u_0. Then T has a fixed point in D, provided that either*

(I) *$Tu_0 \preceq u_0$ and $T^2u_0 \preceq u_0$ or $u_0 \preceq Tu_0$ and $u_0 \preceq T^2u_0$, or*
(II) *the complete sequence of iterates $\{T^nu_0\}_{n=0}^{\infty}$ is defined and there exists $v_0 \in D$ such that $Tv_0 \in D$ and $v_0 \preceq T^nu_0$ for all $n \geq 0$.*

Let $G(t,s)$ denote the Green function for

$$-u'' = 0, \qquad u(0) = 0, \qquad u(1) = \alpha u(\xi).$$

Then by [18] the Green function $G(t,s)$ can be expressed as follows:

$$G(t,s)=\begin{cases} s\in[0,\xi]: \begin{cases} \frac{t}{1-a\xi}[(1-s)-a(\xi-s)], & t\le s,\\ \frac{s}{1-a\xi}[(1-t)-a(\xi-t)], & s\le t,\end{cases}\\ s\in[\xi,1]: \begin{cases} \frac{t}{1-a\xi}(1-s), & t\le s,\\ \frac{1}{1-a\xi}[s(1-t)+a\xi(t-s)], & s\le t.\end{cases}\end{cases}$$

Lemma 2.2 ([25]) *The Green function $G(t,s)$ has the following properties*:

$$t(1-t)G(s,s)\le G(t,s)\le G(s,s)\quad \textit{for } (t,s)\in[0,1]\times[0,1].$$

Let $B=C[0,1]$ denote the Banach space of continuous functions with norm

$$\|u\|=\sup_{t\in[0,1]}\bigl|u(t)\bigr|,\quad \forall u\in B,$$

and let $K=\{u\in B:u(t)\ge 0 \text{ on } [0,1]\}$ be the cone of nonnegative functions in B. It is easy to see that K is a normal cone in B. Now we define the subset $D\subset K$ as

$$D:=\bigl\{u\in K: \text{ there exists } \lambda(u)>0 \text{ such that } u(t)\ge\lambda t(1-t) \text{ on } [0,1]\bigr\}.$$

Moreover, define $T:D\to K$ by

$$(Tu)(t):=\int_0^1 G(t,\tau)\varphi_p^{-1}\biggl(\int_0^1 H(\tau,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s\biggr)\,\mathrm{d}\tau,\quad \forall u\in D,$$

where

$$H(t,s)=\begin{cases} s\in[0,\eta]: \begin{cases} \frac{t}{1-b^{p-1}\eta}[(1-s)-b^{p-1}(\eta-s)], & t\le s,\\ \frac{s}{1-b^{p-1}\eta}[(1-t)-b^{p-1}(\eta-t)], & s\le t,\end{cases}\\ s\in[\eta,1]: \begin{cases} \frac{t}{1-b^{p-1}\eta}(1-s), & t\le s,\\ \frac{1}{1-b^{p-1}\eta}[s(1-t)+b^{p-1}\eta(t-s)], & s\le t.\end{cases}\end{cases}$$

Then T is well defined. In fact, from Lemma 2.2 we have

$$t(1-t)H(s,s)\le H(t,s)\le H(s,s)\quad \text{for } (t,s)\in[0,1]\times[0,1].$$

It can be easily verified that

$$\bigl|H_t'(t,s)\bigr|\le\begin{cases}\frac{1+b}{1-b^{p-1}\eta}s, & 0\le s\le t,\\ \frac{1+b}{1-b^{p-1}\eta}(1-s), & 0\le t\le s.\end{cases}$$

It follows from conditions (H_1) and (H_2) that, for each $u\in K$,

$$\begin{aligned}\biggl|\int_0^1 H_\tau'(\tau,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s\biggr| &\le \int_0^1\bigl|H_\tau'(\tau,s)\bigr|f\bigl(s,\lambda s(1-s)\bigr)\,\mathrm{d}s\\ &\le\frac{1+b}{1-b^{p-1}\eta}\biggl[\int_0^\tau sf\bigl(s,\lambda s(1-s)\bigr)\,\mathrm{d}s\\ &\quad+\int_\tau^1(1-s)f\bigl(s,\lambda s(1-s)\bigr)\,\mathrm{d}s\biggr]:=\nu(\tau).\end{aligned}$$

By the Fubini theorem and (H_2) it is easy to show that $\nu(\tau) \in L^1[0,1]$, and hence $\int_0^1 H(\tau,s)f(s,u(s))\,\mathrm{d}s \in C[0,1]$. Thus, T is well defined, and also from Lemma 2.2 we have that, for all $u \in D$ and $t \in [0,1]$,

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,\tau)\varphi_p^{-1}\left(\int_0^1 H(\tau,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s\right)\mathrm{d}\tau \\ &\le \int_0^1 G(\tau,\tau)\varphi_p^{-1}\left(\int_0^1 H(\tau,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s\right)\mathrm{d}\tau.\end{aligned}$$

Hence,

$$\|Tu\| \le \int_0^1 G(\tau,\tau)\varphi_p^{-1}\left(\int_0^1 H(\tau,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s\right)\mathrm{d}\tau.$$

On the other hand, by Lemma 2.2, for all $u \in D$ and $t \in [0,1]$, we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,\tau)\varphi_p^{-1}\left(\int_0^1 H(\tau,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s\right)\mathrm{d}\tau \\ &\ge t(1-t)\int_0^1 G(\tau,\tau)\varphi_p^{-1}\left(\int_0^1 H(\tau,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s\right)\mathrm{d}\tau.\end{aligned}$$

Thus,

$$(Tu)(t) \ge t(1-t)\|Tu\|, \quad t \in [0,1]. \tag{2.1}$$

This implies that $Tu \in D$, that is, $T : D \to D$, and hence, it can be verified that $u \in D$ is a positive solution of SBVP (1.1)-(1.2) iff $Tu = u$.

3 Main results

In this section, we first establish an existence theorem of positive solutions for SBVP (1.1)-(1.2) by applying Lemma 2.1.

Before proceeding with our existence result for SBVP (1.1)-(1.2), we will define a sequence of functions that are modifications of f and have none of the singularities of f at $u = 0$. To this end, we define a sequence of functions $f_n : (0,1) \times [0,\infty) \to [0,\infty)$ by

$$f_n(t,u) = f\bigl(t, \max\bigl\{u, t(1-t)/n\bigr\}\bigr).$$

Note that, for $n = 1,2,\ldots$, f_n satisfies (H_1). Also, for $n = 1,2,\ldots$,

$$f_n(t,u) \le f(t,u), \quad (t,u) \in (0,1) \times (0,\infty), \tag{3.1}$$

$$f_n(t,u) \le f\bigl(t, t(1-t)/n\bigr), \quad (t,u) \in (0,1) \times [0,\infty). \tag{3.2}$$

We now state and prove our existence result for SBVP (1.1)-(1.2).

Theorem 3.1 *Assume that conditions* (H_1) *and* (H_2) *are satisfied. Then* SBVP (1.1)-(1.2) *has at least one positive solution* $u^* \in D$.

Proof We define the sequence of mappings $T_n : K \to K$ by

$$(T_n u)(t) := \int_0^1 G(t,\tau)\varphi_p^{-1}\left(\int_0^1 H(\tau,s)f_n\bigl(s,u(s)\bigr)\,\mathrm{d}s\right)\mathrm{d}\tau, \quad \forall u \in K.$$

Then by (3.2) and condition (H_2), T_n is well defined and bounded. We note that T_n is a continuous mapping by Lebesgue's dominated convergence theorem. Also, it is easy to show by (H_2) and the continuity of $G(t,s)$ that $\{(T_n u)(t) : u \in K\}$ is equicontinuous, and hence T_n is a compact mapping by the Arzelà-Ascoli theorem.

In addition, observe that for all n and $u \in K$, $T_n u$ satisfies the boundary conditions (1.2). Furthermore, for each n, since T_n satisfies (H_1), it follows that T_n is nonincreasing relative to the cone K. Also, it is clear that $0 \preceq T_n(0)$ and $0 \preceq T_n^2(0)$ for each n. Thus, by Lemma 2.1, for each n, there exists $u_n \in K$ such that $T_n u_n = u_n$. Hence, for each n, $u_n(t)$ satisfies the boundary conditions (1.2).

Now we claim that there exist $R > r > 0$ such that

$$r \le \|u_n\| \le R \quad \text{for all } n.$$

Firstly, we shall prove the right-hand side inequality. Assume to the contrary that the inequality is false. Then by passing to a subsequence and relabeling, without loss of generality, we may assume that

$$\lim_{n\to\infty} \|u_n\| = \infty \quad \text{and} \quad \|u_n\| \le \|u_{n+1}\| \quad \text{for all } n.$$

Similarly to the proof of (2.1), we can easily show that, for any $u \in K$,

$$(T_n u)(t) \ge t(1-t)\|T_n u\|, \quad t \in [0,1]. \tag{3.3}$$

Since $T_n u_n = u_n$ for each n, it follows that

$$u_n(t) \ge t(1-t)\|u_n\| \ge t(1-t)\|u_1\|, \quad t \in [0,1].$$

Then assumptions (H_1) and (H_2) and inequality (3.1) yield that, for any $0 \le t \le 1$ and n,

$$\begin{aligned} u_n(t) &= \int_0^1 G(t,\tau)\varphi_p^{-1}\left(\int_0^1 H(\tau,s)f_n\bigl(s,u_n(s)\bigr)\,\mathrm{d}s\right)\mathrm{d}\tau \\ &\le \int_0^1 G(\tau,\tau)\varphi_p^{-1}\left(\int_0^1 H(s,s)f\bigl(s,u_n(s)\bigr)\,\mathrm{d}s\right)\mathrm{d}\tau \\ &\le \int_0^1 G(\tau,\tau)\,\mathrm{d}\tau\cdot\varphi_p^{-1}\left(\int_0^1 H(s,s)f\bigl(s,s(1-s)\|u_1\|\bigr)\,\mathrm{d}s\right) := M. \end{aligned}$$

Thus, $\|u_n\| \le M$ for all n. This is a contradiction to $\lim_{n\to\infty}\|u_n\| = \infty$.

Next, we prove the left-hand side inequality. Assume to the contrary that the inequality is false. By passing to a subsequence and relabeling we may assume without loss of generality that $\lim_{n\to\infty}\|u_n\| = 0$. Then we have

$$\lim_{n\to\infty} u_n(t) = 0 \quad \text{uniformly on } [0,1]. \tag{3.4}$$

Since $f(t,u)$ is singular at $u=0$, by the finite covering theorem there exists $\delta>0$ such that, for $t\in[\xi/2,\xi]$ and $0<x<\delta$, we have

$$f(t,x)>\frac{2}{\xi(1-\xi)}\left[\int_{\xi/2}^{\xi}H(s,s)\,\mathrm{d}s\right]^{-1}.$$

By (3.4) there exists $n_0\geq 1$ such that, for any $n\geq n_0$,

$$0<u_n(t)<\delta/2,\qquad 0<t(1-t)/n<\delta/2,\quad t\in[\xi/2,\xi].$$

Hence, for any $n\geq n_0$, we have

$$\begin{aligned}
u_n(\xi)&=\int_0^1 G(\xi,\tau)\varphi_p^{-1}\left(\int_0^1 H(\tau,s)f_n\bigl(s,u_n(s)\bigr)\,\mathrm{d}s\right)\mathrm{d}\tau\\
&\geq\int_{\xi/2}^{\xi} G(\xi,\tau)\varphi_p^{-1}\left(\int_{\xi/2}^{\xi} H(\tau,s)f_n\bigl(s,u_n(s)\bigr)\,\mathrm{d}s\right)\mathrm{d}\tau\\
&\geq\int_{\xi/2}^{\xi} G(\xi,\tau)\varphi_p^{-1}\left(\tau(1-\tau)\int_{\xi/2}^{\xi} H(s,s)f(s,\delta/2)\,\mathrm{d}s\right)\mathrm{d}\tau\\
&\geq\int_{\xi/2}^{\xi} G(\xi,\tau)\varphi_p^{-1}\left(\frac{\xi(1-\xi)}{2}\int_{\xi/2}^{\xi} H(s,s)f(s,\delta/2)\,\mathrm{d}s\right)\mathrm{d}\tau\\
&\geq\int_{\xi/2}^{\xi} G(\xi,\tau)\varphi_p^{-1}(1)\,\mathrm{d}\tau=\int_{\xi/2}^{\xi}G(\xi,\tau)\,\mathrm{d}\tau=\frac{3\xi^2(1-\xi)}{8(1-a\xi)},
\end{aligned}$$

which implies $\|u_n\|\geq\frac{3\xi^2(1-\xi)}{8(1-a\xi)}$. This is a contradiction to $\lim_{n\to\infty}\|u_n\|=0$.

In summary, we conclude our claim. Furthermore, from (3.3) we have

$$rt(1-t)\preceq u_n\preceq R\quad(\text{w.r.t.}K), n=1,2,\ldots.\tag{3.5}$$

This implies that the sequence $\{u_n\}$ belongs to the closed order interval $\langle rt(1-t),R\rangle\subset D$. It is easy to see that the restriction of T to $\langle rt(1-t),R\rangle$ is a compact mapping. Hence, there exists a subsequence of $\{Tu_n\}$ that converges to some $u^*\in K$. Relabel the subsequence as the original sequence so that $\lim_{n\to\infty}\|Tu_n-u^*\|=0$.

Also, by (3.5) there exists n_0 such that, for all $n\geq n_0$,

$$t(1-t)/n\leq rt(1-t)\leq u_n(t)\quad\text{on }[0,1].$$

For all $n\geq n_0$ and $t\in[0,1]$, we have

$$\begin{aligned}
(Tu_n)(t)-u_n(t)&=(Tu_n)(t)-(T_nu_n)(t)\\
&=\int_0^1 G(t,\tau)\left[\varphi_p^{-1}\left(\int_0^1 H(\tau,s)f\bigl(s,u_n(s)\bigr)\,\mathrm{d}s\right)\right.\\
&\quad\left.-\varphi_p^{-1}\left(\int_0^1 H(\tau,s)f_n\bigl(s,u_n(s)\bigr)ds\right)\right]\mathrm{d}\tau\\
&=0.
\end{aligned}$$

This implies $\lim_{n\to\infty}\|Tu_n - u_n\| = 0$. It follows, in turn, that $\lim_{n\to\infty}\|u_n - u^*\| = 0$, and thus $u^* \in \langle rt(1-t), R\rangle \subset D$ and

$$u^* = \lim_{n\to\infty} Tu_n = T\Bigl(\lim_{n\to\infty} u_n\Bigr) = Tu^*.$$

In summary, we have $u^* \in D$ and $Tu^* = u^*$. This completes the proof of the theorem. □

Theorem 3.2 *Assume that conditions* (H_1) *and* (H_2) *are satisfied. Then* SBVP (1.1)-(1.2) *has exactly one positive solution* $u^* \in D$.

Proof The existence of positive solution to SBVP (1.1)-(1.2) immediately follows from Theorem 3.1. Thus, we only need to show the uniqueness.

Suppose that $u_1(t)$ and $u_2(t)$ are two positive solutions of SBVP (1.1)-(1.2). Then, by (H_1) and (H_2), $u_1(t)$ and $u_2(t)$ are both the solutions of the following boundary value problem:

$$\begin{aligned} &u''(t) + \varphi_p^{-1}\biggl(\int_0^1 H(t,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s\biggr) = 0, \quad 0 < t < 1, \\ &u(0) = 0, \qquad u(1) = au(\xi). \end{aligned}$$

Let $w(t) = u_1(t) - u_2(t)$ on $[0,1]$. Without loss of generality, we may assume that $w(1) \geq 0$. Now we show that $w(t) \equiv 0$ on $[0,1]$. There are two cases to consider.

Case 1. $w(1) > 0$. In this case, we have $w(t) \geq 0$ on $[0,1]$. Assume by contradiction that there exists $t_0 \in (0,1)$ such that $w(t_0) < 0$. Since $w(0) = 0$ and $w(1) > 0$, there exist $t_1, t_2 \in [0,1)$ with $t_1 < t_0 < t_2$ such that

$$w(t) < 0 \quad \text{on } (t_1,t_2), \qquad w(t_1) = w(t_2) = 0.$$

It follows that, for each $t \in (t_1,t_2)$,

$$\begin{aligned} w''(t) &= u_1''(t) - u_2''(t) \\ &= -\varphi_p^{-1}\biggl(\int_0^1 H(t,s)f\bigl(s,u_1(s)\bigr)\,\mathrm{d}s\biggr) + \varphi_p^{-1}\biggl(\int_0^1 H(t,s)f\bigl(s,u_2(s)\bigr)\,\mathrm{d}s\biggr) \leq 0. \end{aligned}$$

Hence, $w(t) \geq 0$ on $[t_1,t_2]$, which is a contradiction to $w(t) < 0$ on (t_1,t_2). Therefore, $w(t) \geq 0$ on $[0,1]$. Consequently, for each $t \in (0,1)$,

$$w''(t) = -\varphi_p^{-1}\biggl(\int_0^1 H(t,s)f\bigl(s,u_1(s)\bigr)\,\mathrm{d}s\biggr) + \varphi_p^{-1}\biggl(\int_0^1 H(t,s)f\bigl(s,u_2(s)\bigr)\,\mathrm{d}s\biggr) \geq 0.$$

Thus, $w(t)$ is concave upward on $[0,1]$. Since $w(1) > 0$ and $w(1) = \alpha w(\xi)$, we have $w(\xi) > 0$, and hence since $0 < \alpha < 1/\xi$, we have

$$w(1) < \frac{1}{\xi} w(\xi),$$

which is a contradiction to the upward concavity of $w(t)$ on $[0,1]$.

Case 2. $w(1) = 0$. In this case, we have $w(t) \equiv 0$ on $[0,1]$. Assume to the contrary that the conclusion is false. Then, there exists $t_0 \in (0,1)$ such that $w(t_0) \neq 0$. Without loss of generality, we may assume that $w(t_0) > 0$. Since $w(0) = w(1) = 0$, there exist $t_1, t_2 \in [0,1]$ with $t_1 < t_0 < t_2$ such that

$$w(t) > 0 \quad \text{on } (t_1, t_2), \qquad w(t_1) = w(t_2) = 0.$$

It follows that, for each $t \in (t_1, t_2)$,

$$w''(t) = -\varphi_p^{-1}\left(\int_0^1 H(t,s)f\bigl(s,u_1(s)\bigr)\,\mathrm{d}s\right) + \varphi_p^{-1}\left(\int_0^1 H(t,s)f\bigl(s,u_2(s)\bigr)\,\mathrm{d}s\right) \geq 0.$$

Since $w(t_1) = w(t_2) = 0$, we have that

$$w(t) \leq 0 \quad \text{for } t \in (t_1, t_2),$$

which is a contradiction to $w(t) > 0$ on (t_1, t_2).

In summary, $w(t) \equiv 0$ on $[0,1]$. This completes the proof of the theorem. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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