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# Iterative solutions for fractional nonlocal boundary value problems involving integral conditions

Hai-E Zhang\*

\*Correspondence:  
haiezhang@126.com  
Department of Basic Science,  
Tangshan College, Tangshan, Hebei  
063000, People's Republic of China

## Abstract

This paper is concerned with the following nonlinear fractional boundary value problem:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = u'(0) = 0, & D_{0+}^{\beta} u(1) = \int_0^1 D_{0+}^{\beta} u(t) dA(t), \end{cases}$$

where  $2 < \alpha \leq 3$ ,  $0 < \beta \leq 1$  are real numbers and  $\int_0^1 D_{0+}^{\beta} u(t) dA(t)$  denotes a Riemann-Stieltjes integral. By means of monotone iterative technique and some inequalities associated with the Green function, not only the existence of nontrivial solutions or positive solutions is obtained but also iterative schemes for approximating the solutions are established, which start off with simple functions, which are feasible for computational purposes. An example is also included to illustrate the main results.

**MSC:** 34B15

**Keywords:** fractional; monotone iterative technique; Riemann-Stieltjes integrals

## 1 Introduction

Differential equations with fractional order are a generalization of ordinary differential equations to non-integer order. This generalization is not merely one of the mathematical curiosities but rather has interesting applications in various fields of sciences and engineering such as control, porous media, electromagnetic, and other branches of science. The need for fractional order differential equations stems in part from the fact that many phenomena cannot be modeled by differential equations with integer derivatives. For details, see [1–4].

In recent years, there has grown a vast literature devoted to the applications of fixed point theorems focused on solutions or positive solutions of boundary value problems (BVPs for short) for fractional differential equations, for example, we mention only [5–16]. Note that Wang *et al.* [16] studied a fractional BVP with changing sign nonlinearity,

$$\begin{cases} D_{0+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = u'(0) = u(1) = 0, \end{cases} \quad (1.1)$$

where  $2 < \alpha \leq 3$ . By means of the Guo-Krasnoselskii fixed point theorem, the existence of positive solutions was established. In [6], Xu and Fei investigated multiple positive solutions of the singular BVP

$$D_{0+}^{\alpha} u(t) + \lambda f(t, u(t)) + e(t) = 0, \quad t \in (0, 1)$$

subject to the three-point boundary conditions

$$u(0) = 0, \quad D_{0+}^{\beta} u(1) = aD_{0+}^{\beta} u(\xi),$$

where  $1 < \alpha \leq 2, 0 < \beta \leq 1$ . Their tool was Schauder’s fixed point theorem.

Recently, iterative methods have been successfully employed to prove the existence of solutions or positive solutions of nonlinear BVPs. However, only few papers considered the computational methods of solutions for fractional differential equations; see [8, 9, 17–19]. In particular, it should be pointed out that Jiang and Zhong [8] obtained the existence of nontrivial sign-changing solutions to fractional differential equations with integral boundary conditions, the main tool used was monotone iterative method. Sun and Zhao [9] studied the fractional differential equation with integral boundary conditions

$$\begin{cases} D_{0+}^{\alpha} u(t) + q(t)f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = u'(0) = 0, & u(1) = \int_0^1 g(s)u(s) ds. \end{cases} \tag{1.2}$$

The existence result of monotone positive solutions was obtained.

Motivated greatly by the above mentioned excellent works, in this paper we investigate the existence and iteration of nontrivial solutions and positive solutions for the following BVP:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = u'(0) = 0, & D_{0+}^{\beta} u(1) = \int_0^1 D_{0+}^{\beta} u(t) dA(t), \end{cases} \tag{1.3}$$

where  $2 < \alpha \leq 3, 0 < \beta \leq 1$  are real numbers and  $A$  is of bounded variation, the nonlinear term  $f(t, u)$  may change sign on some set and  $\int_0^1 D_{0+}^{\beta} u(t) dA(t)$  denotes a Riemann-Stieltjes integral with a signed measure. This includes both the multi-point and a Riemann integral in a single framework. For more comments on the Stieltjes integral and its importance, we refer the reader to the papers by Webb and Infante [20–22] and their other related works. To the best of our knowledge, the results are new and BVP (1.3) is studied in this form for the first time. The proofs of our main results are based on the monotone iterative technique, which we present now.

**Theorem 1.1** [23] *Let  $K$  be a normal cone of a Banach space  $E$  and  $u_0 \leq v_0$ . Suppose that*

- (1)  $T : [u_0, v_0] \rightarrow E$  is completely continuous;
- (2)  $T$  is monotone increasing on  $[u_0, v_0]$ ;
- (3)  $u_0$  is a lower solution of  $T$ , that is,  $u_0 \leq Tu_0$ ;
- (4)  $v_0$  is an upper solution of  $T$ , that is,  $Tv_0 \leq v_0$ .

*Then the iterative sequences  $u_n = Tu_{n-1}, v_n = Tv_{n-1}, n = 1, 2, 3, \dots$ , satisfy  $u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0$ , and converge to, respectively,  $u, v \in [u_0, v_0]$ , which are fixed points of  $T$ .*

## 2 Preliminary lemmas

For convenience of the reader, we present here some necessary definitions.

**Definition 2.1** [24] Let  $X$  be the Banach space with norm  $\| \cdot \|$ . A nonempty closed set  $K \subset X$  is called a cone if  $K$  satisfies the following conditions: (i) if  $x, y \in K$ , then  $x + y \in K$ ; (ii) if  $x \in K$ , then  $\mu x \in K$ , for any  $\mu \geq 0$ ; (iii) if  $0 \neq x \in K$ , then  $-x \notin K$ . Let  $x_1, x_2 \in X$ . We write  $x_1 \preceq x_2$ , if  $x_2 - x_1 \in K$ . The cone  $K$  is called normal, if there exists  $\rho > 0$  such that

$$\|x_1 + x_2\| \geq \rho, \quad \forall x_1, x_2 \in K, \|x_1\| = \|x_2\| = 1.$$

We call the set  $[x_1, x_2] = \{x \in X, x_1 \preceq x \preceq x_2\}$  an order interval in  $X$ . The operator  $T : [x_1, x_2] \rightarrow X$  is called increasing if  $T\bar{x} \preceq T\tilde{x}$  for any  $\bar{x}, \tilde{x} \in [x_1, x_2]$  with  $\bar{x} \preceq \tilde{x}$ .

**Definition 2.2** [3] The Riemann-Liouville fractional integral of order  $\alpha > 0$  of a function  $y : (0, \infty) \rightarrow R$  is given by

$$I_{0+}^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

provided the right side is pointwise defined on  $[0, \infty)$ .

**Definition 2.3** [3] The Riemann-Liouville fractional derivative of order  $\alpha > 0$  of a function  $y : (0, \infty) \rightarrow R$  is given by

$$D_{0+}^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} y(s) ds,$$

where  $n = [\alpha] + 1$ ,  $[\alpha]$  denotes the integer part of number  $\alpha$ , provided the right side is pointwise defined on  $[0, \infty)$ .

From the definitions of Riemann-Liouville's derivative, we can obtain the statements.

**Lemma 2.4** [3] *Let  $\alpha > 0$ , if we assume  $u \in C(0, 1) \cap L^1(0, 1)$ , then the fractional differential equation*

$$D_{0+}^\alpha u(t) = 0$$

*has  $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_N t^{\alpha-N}$ , from some  $C_i \in R, i = 1, 2, \dots, N$ , as a unique solution, where  $N$  is the smallest integer greater than or equal to  $\alpha$ .*

**Lemma 2.5** [3] *Let  $\alpha > 0$ , if we assume  $u \in C(0, 1) \cap L^1(0, 1)$ , then the fractional differential equation*

$$I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_N t^{\alpha-N},$$

*from some  $C_i \in R, i = 1, 2, \dots, N$ , where  $N$  is the smallest integer greater than or equal to  $\alpha$ .*

**Lemma 2.6** Assume that  $y \in C(0,1) \cap L^1(0,1)$ ,  $\alpha, \beta$  are two positive constants with  $\alpha - \beta \geq 0$ . Then

$$D_{0+}^\beta I_{0+}^\alpha y(t) = I_{0+}^{\alpha-\beta} y(t).$$

*Proof* Let  $h(t) = \int_0^t (t-s)^{\alpha-1} y(s) ds$ . Then by Definition 2.3,

$$D_{0+}^\beta h(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-1-\beta} h(\tau) d\tau.$$

On the other hand, let  $\frac{\tau-s}{t-s} = u$ , then

$$\begin{aligned} \int_0^t (t-\tau)^{n-1-\beta} h(\tau) d\tau &= \int_0^t (t-\tau)^{n-1-\beta} \left( \int_0^\tau (\tau-s)^{\alpha-1} y(s) ds \right) d\tau \\ &= \int_0^t ds \int_s^t (t-\tau)^{n-1-\beta} (\tau-s)^{\alpha-1} y(s) d\tau \\ &= \int_0^t (t-s)^{\alpha-\beta+(n-1)} y(s) ds \int_0^1 u^{\alpha-1} (1-u)^{(n-\beta)-1} du \\ &= B(\alpha, n-\beta) \int_0^t (t-s)^{\alpha-\beta+(n-1)} y(s) ds. \end{aligned}$$

Therefore, we have

$$\begin{aligned} D_{0+}^\beta h(t) &= \frac{B(\alpha, n-\beta)}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t (t-s)^{\alpha-\beta+(n-1)} y(s) ds \\ &= \frac{B(\alpha, n-\beta)}{\Gamma(n-\beta)} \int_0^t [\alpha-\beta+(n-1)][\alpha-\beta+(n-2)] \cdots [\alpha-\beta] \\ &\quad \times (t-s)^{\alpha-\beta-1} y(s) ds \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s) ds. \end{aligned} \quad \square$$

In the remainder of the paper, we always assume that  $2 < \alpha \leq 3$  and  $0 \leq \beta \leq 1$ , so we also easily get  $0 < \alpha - \beta - 1 \leq 2$ .

**Lemma 2.7** Let  $y \in C(0,1) \cap L^1[0,1]$  and  $\delta := \int_0^1 t^{\alpha-\beta-1} dA(t) < 1$ , the unique solution of

$$\begin{cases} D_{0+}^\alpha u(t) + y(t) = 0, & t \in (0,1), \\ u(0) = u'(0) = 0, & D_{0+}^\beta u(1) = \int_0^1 D_{0+}^\beta u(t) dA(t) \end{cases} \quad (2.1)$$

is  $u(t) = \int_0^1 G(t,s)y(s) ds$ , in which

$$G(t,s) = K(t,s) + \frac{t^{\alpha-1}}{1-\delta} \int_0^1 H(t,s) dA(t),$$

where

$$K(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \leq t \leq s \leq 1 \end{cases}$$

and

$$H(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1}, & 0 \leq s \leq t \leq 1, \\ [t(1-s)]^{\alpha-\beta-1}, & 0 \leq t \leq s \leq 1. \end{cases}$$

*Proof* As deduced from Lemma 2.5, we have

$$u(t) = -I_{0+}^\alpha y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3}.$$

Consequently, the solution of (2.1) is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3}.$$

By  $u(0) = u'(0) = 0$ , there is  $C_2 = C_3 = 0$ . Moreover,  $D_{0+}^\beta u(1) = \int_0^1 D_{0+}^\beta u(t) dA(t)$  and Lemma 2.6 yield

$$C_1 = \frac{1}{\Gamma(\alpha)(1-\delta)} \left[ \int_0^1 (1-s)^{\alpha-\beta-1} y(s) ds - \int_0^1 \left( \int_0^t (t-s)^{\alpha-\beta-1} y(s) ds \right) dA(t) \right].$$

Therefore, the solution of (2.1) is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\delta)} \\ &\quad \times \left[ \int_0^1 (1-s)^{\alpha-\beta-1} y(s) ds - \int_0^1 \left( \int_0^t (t-s)^{\alpha-\beta-1} y(s) ds \right) dA(t) \right] \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s) ds \\ &\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\delta)} \left( \int_0^1 \left[ \int_0^1 t^{\alpha-\beta-1} (1-s)^{\alpha-\beta-1} y(s) ds \right] dA(t) \right. \\ &\quad \left. - \int_0^1 \left[ \int_0^t (t-s)^{\alpha-\beta-1} y(s) ds \right] dA(t) \right) \\ &= \int_0^1 \left[ K(t, s) + \frac{t^{\alpha-1}}{1-\delta} \int_0^1 H(t, s) dA(t) \right] y(s) ds. \quad \square \end{aligned}$$

**Lemma 2.8** For any  $t, s \in [0, 1]$ ,  $H(t, s)$  has the following property:

$$\min\{\alpha - \beta - 1, 1\} t^{\alpha-\beta-1} (1-t)(1-s)^{\alpha-\beta-1} s \leq \Gamma(\alpha) H(t, s) \leq \max\{\alpha - \beta - 1, 1\} (1-s)^{\alpha-\beta-1}.$$

*Proof* Case 1:  $0 < \alpha - \beta - 1 \leq 1$ . If  $s \leq t$ , then

$$\begin{aligned} \Gamma(\alpha) H(t, s) &= [t(1-s)]^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1} \\ &= (\alpha - \beta - 1) \int_{t-s}^{t(1-s)} x^{\alpha-\beta-2} dx \\ &\geq (\alpha - \beta - 1) [t(1-s)]^{\alpha-\beta-2} [t(1-s) - (t-s)] \\ &\geq (\alpha - \beta - 1) [t(1-s)]^{\alpha-\beta-1} s(1-t) \\ &= (\alpha - \beta - 1) t^{\alpha-\beta-1} (1-t)(1-s)^{\alpha-\beta-1} s \end{aligned}$$

and

$$\begin{aligned} \Gamma(\alpha)H(t,s) &= [t(1-s)]^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1} \\ &= [t(1-s)]^{\alpha-\beta-2} \left[ t(1-s) - (t-s) \cdot \frac{(t-s)^{\alpha-\beta-2}}{[t(1-s)]^{\alpha-\beta-2}} \right] \\ &= [t(1-s)]^{\alpha-\beta-2} \left[ t(1-s) - (t-s) \cdot \frac{(1-\frac{s}{t})^{\alpha-\beta-2}}{(1-s)^{\alpha-\beta-2}} \right] \\ &\leq [t(1-s)]^{\alpha-\beta-2} [t(1-s) - (t-s)] \\ &\leq t^{\alpha-\beta-2}(1-t)(1-s)^{\alpha-\beta-2}s \\ &\leq s^{\alpha-\beta-2}(1-s)(1-s)^{\alpha-\beta-2}s \\ &= s^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} \leq (1-s)^{\alpha-\beta-1}. \end{aligned}$$

If  $s \geq t$ , then

$$\Gamma(\alpha)H(t,s) = [t(1-s)]^{\alpha-\beta-1} \leq s^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} \leq (1-s)^{\alpha-\beta-1}$$

and

$$\begin{aligned} \Gamma(\alpha)H(t,s) &= [t(1-s)]^{\alpha-\beta-1} \geq t^{\alpha-\beta-1}(1-t)(1-s)^{\alpha-\beta-1}s \\ &\geq (\alpha - \beta - 1)t^{\alpha-\beta-1}(1-t)(1-s)^{\alpha-\beta-1}s. \end{aligned}$$

Case 2:  $1 < \alpha - \beta - 1 \leq 2$ . We have

$$(1-t)t^{\alpha-\beta-1}s(1-s)^{\alpha-\beta-1} \leq \Gamma(\alpha)H(t,s) \leq (\alpha - \beta - 1)(1-s)^{\alpha-\beta-1}.$$

The proof is similar to Lemma 2.3 in [16], so it is omitted. □

**Lemma 2.9** *The function  $K(t,s)$  satisfies:*

- (i)  $\Gamma(\alpha)K(t,s) \leq t^{\alpha-1}(1-s)^{\alpha-\beta-1}$ ,  $t,s \in [0,1]$ ;
- (ii)  $\Gamma(\alpha)K(t,s) \geq t^{\alpha-1}\beta s(1-s)^{\alpha-\beta-1}$ ,  $t,s \in [0,1]$ .

*Proof* Since (i) holds obviously, we only show that (ii) is true. Here we need the fact  $1 - (1-s)^\beta \geq \beta s$ ,  $0 < \beta \leq 1$ ,  $s \in [0,1]$ . In fact,

$$[1 - (1-s)^\beta - \beta s]' = \beta[(1-s)^{\beta-1} - 1] \geq 0,$$

which implies  $1 - (1-s)^\beta - \beta s$  is nondecreasing in  $[0,s]$ , so  $1 - (1-s)^\beta \geq \beta s$ .

If  $s \leq t$ , then

$$\begin{aligned} \Gamma(\alpha)K(t,s) &= t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} \\ &\geq t^{\alpha-1}(1-s)^{\alpha-\beta-1} - t^{\alpha-1}(1-s)^{\alpha-1} \\ &= t^{\alpha-1}(1-s)^{\alpha-\beta-1} [1 - (1-s)^\beta] \\ &\geq t^{\alpha-1}(1-s)^{\alpha-\beta-1} \beta s. \end{aligned}$$

If  $t \leq s$ , then

$$\Gamma(\alpha)K(t, s) = t^{\alpha-1}(1-s)^{\alpha-\beta-1} \geq t^{\alpha-1}(1-s)^{\alpha-\beta-1}\beta s.$$

□

**Lemma 2.10** *Let*

$$\Phi_1(s) := \frac{1}{\Gamma(\alpha)} \left[ \beta + \frac{\min\{\alpha - \beta - 1, 1\} \int_0^1 t^{\alpha-\beta-1}(1-t) dA(t)}{1-\delta} \right] s(1-s)^{\alpha-\beta-1}$$

and

$$\Phi_2(s) := \frac{1}{\Gamma(\alpha)} \left[ 1 + \frac{\max\{\alpha - \beta - 1, 1\} \int_0^1 dA(t)}{1-\delta} \right] (1-s)^{\alpha-\beta-1}.$$

Then the function  $G(t, s)$  has the following property:

$$t^{\alpha-1}\Phi_1(s) \leq G(t, s) \leq t^{\alpha-1}\Phi_2(s), \quad t, s \in [0, 1].$$

*Proof* For any  $t, s \in [0, 1]$ , it follows from Lemmas 2.8 and 2.9 that

$$\begin{aligned} G(t, s) &= K(t, s) + \frac{t^{\alpha-1} \int_0^1 H(t, s) dA(t)}{1-\delta} \\ &\leq \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-s)^{\alpha-\beta-1} + \frac{t^{\alpha-1} \max\{\alpha - \beta - 1, 1\} \int_0^1 dA(t)}{\Gamma(\alpha)(1-\delta)} (1-s)^{\alpha-\beta-1} \\ &\leq t^{\alpha-1} \frac{1}{\Gamma(\alpha)} \left[ 1 + \frac{\max\{\alpha - \beta - 1, 1\} \int_0^1 dA(t)}{(1-\delta)} \right] (1-s)^{\alpha-\beta-1} \\ &= t^{\alpha-1}\Phi_2(s). \end{aligned}$$

On the other hand, for any  $t, s \in [0, 1]$ , we have

$$\begin{aligned} G(t, s) &= K(t, s) + \frac{t^{\alpha-1} \int_0^1 H(t, s) dA(t)}{1-\delta} \\ &\geq \frac{1}{\Gamma(\alpha)} t^{\alpha-1}\beta s(1-s)^{\alpha-\beta-1} + \frac{t^{\alpha-1} \min\{\alpha - \beta - 1, 1\} \int_0^1 t^{\alpha-\beta-1}(1-t) dA(t)}{\Gamma(\alpha)(1-\delta)} \\ &\quad \times s(1-s)^{\alpha-\beta-1} \\ &= t^{\alpha-1} \frac{1}{\Gamma(\alpha)} \left[ \beta + \frac{\min\{\alpha - \beta - 1, 1\} \int_0^1 t^{\alpha-\beta-1}(1-t) dA(t)}{1-\delta} \right] s(1-s)^{\alpha-\beta-1} \\ &= t^{\alpha-1}\Phi_1(s). \end{aligned}$$

□

**Lemma 2.11** *The function  $G(t, s)$  is continuous and satisfies*

$$\begin{aligned} |G(t_2, s) - G(t_1, s)| &\leq \max_{0 \leq s \leq 1} |G(t_2, s) - G(t_1, s)| \\ &\leq (\alpha - 1) \left[ \frac{2}{\Gamma(\alpha)} + \frac{\max_{0 \leq s \leq 1} \int_0^1 H(t, s) dA(t)}{1-\delta} \right] (t_2 - t_1), \end{aligned}$$

for  $0 \leq t_1 \leq t_2 \leq 1$ .

*Proof* For  $0 \leq t_1 \leq t_2 \leq 1$ , we have

$$\begin{aligned} |G(t_2, s) - G(t_1, s)| &= \left| K(t_2, s) - K(t_1, s) + \frac{(t_2^{\alpha-1} - t_1^{\alpha-1}) \int_0^1 H(t, s) dA(t)}{1 - \delta} \right| \\ &= \left| \frac{1}{\Gamma(\alpha)} (t_2^{\alpha-1} - t_1^{\alpha-1})(1-s)^{\alpha-\beta-1} + \frac{1}{\Gamma(\alpha)} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \right. \\ &\quad \left. + \frac{(t_2^{\alpha-1} - t_1^{\alpha-1}) \int_0^1 H(t, s) dA(t)}{1 - \delta} \right| \\ &\leq \left( \frac{1}{\Gamma(\alpha)} |t_2^{\alpha-1} - t_1^{\alpha-1}| + \frac{1}{\Gamma(\alpha)} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| \right. \\ &\quad \left. + \frac{|t_2^{\alpha-1} - t_1^{\alpha-1}| \max_{0 \leq s \leq 1} \int_0^1 H(t, s) dA(t)}{1 - \delta} \right). \end{aligned}$$

Note that, applying the mean value theorem, we arrive at  $t_2^{\alpha-1} - t_1^{\alpha-1} < (\alpha - 1)(t_2 - t_1)$  and  $(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} < (\alpha - 1)(t_2 - t_1)$ , which implies that

$$\begin{aligned} |G(t_2, s) - G(t_1, s)| &\leq \max_{0 \leq s \leq 1} |G(t_2, s) - G(t_1, s)| \\ &\leq (\alpha - 1) \left[ \frac{2}{\Gamma(\alpha)} + \frac{\max_{0 \leq s \leq 1} \int_0^1 H(t, s) dA(t)}{1 - \delta} \right] (t_2 - t_1), \end{aligned}$$

for  $0 \leq t_1 \leq t_2 \leq 1$ . □

### 3 Main results

Let  $E = C[0, 1]$  be equipped with the norm  $\|u\| = \|u\|_\infty$ , where  $\|u\|_\infty$  is the usual supremum norm in  $C[0, 1]$ .

Define the cone  $K$  by

$$K = \{u \in E : u(t) \geq 0, t \in [0, 1]\}.$$

Then it is easy to verify that  $K$  is a cone in  $E$ . For  $u, v \in C[0, 1]$ ,  $u \leq v$  if and only if  $u(t) \leq v(t)$ ,  $t \in [0, 1]$ . Moreover, the cone  $K$  is normal on account of  $\|u + v\| \geq \|u\| = 1$  for any  $u, v \in K$  and  $\|u\| = \|v\| = 1$ .

Define the operator  $T : K \rightarrow K$  as follows:

$$(Tu)(t) = \int_0^1 G(t, s) f(s, u(s)) ds, \quad 0 \leq t \leq 1.$$

It is clear that the existence of a solution for BVP (1.3) is equivalent to the existence of a fixed points of  $T$  in  $K$ .

**Theorem 3.1** *Assume that there exist two real numbers  $a, b$  such that  $a < b$  and the following conditions are satisfied:*

(H<sub>1</sub>) *There exists a nonnegative function  $h \in C(0, 1) \cap L^1[0, 1]$  such that one of the following conditions is satisfied:*



- (i)  $a \geq 0, f : (0, 1) \times [0, b] \rightarrow R$  is continuous,  $|f(t, u)| \leq h(t), (t, u) \in (0, 1) \times [0, b]$  and increasing with respect to  $u$  in  $[0, b]$ ;
- (ii)  $b \leq 0, f : (0, 1) \times [a, 0] \rightarrow R$  is continuous,  $|f(t, u)| \leq h(t), (t, u) \in (0, 1) \times [a, 0]$  and increasing with respect to  $u$  in  $[a, 0]$ ;
- (iii)  $a < 0 < b, f : (0, 1) \times [a, b] \rightarrow R$  is continuous,  $|f(t, u)| \leq h(t), (t, u) \in (0, 1) \times [a, b]$  and increasing with respect to  $u$  in  $[a, b]$ .

(H<sub>2</sub>) The following inequalities hold:

$$\int_0^1 \Phi_1(s) \max\{f(s, as^{\alpha-1}), 0\} ds + \int_0^1 \Phi_2(s) \min\{f(s, as^{\alpha-1}), 0\} ds \geq a,$$

$$\int_0^1 \Phi_2(s) \max\{f(s, bs^{\alpha-1}), 0\} ds + \int_0^1 \Phi_1(s) \min\{f(s, bs^{\alpha-1}), 0\} ds \leq b.$$

(H<sub>3</sub>)  $f(t, 0) \neq 0$ , for  $0 \leq t \leq 1$ .

Then the problem (1.3) has two nontrivial solutions  $u^*, v^* \in C^2[0, 1] \cap C^3(0, 1)$  ( $u^*, v^*$  may be coincident) such that  $at^{\alpha-1} \leq u^* \leq v^* \leq bt^{\alpha-1}, t \in [0, 1]$ , and  $\lim_{n \rightarrow \infty} u_n = u^*, \lim_{n \rightarrow \infty} v_n = v^*$ , where the two iterative sequences are generated by

$$u_0(t) = at^{\alpha-1}, \quad u_{n+1} = \int_0^1 G(t, s)f(s, u_n(s)) ds, \quad n = 0, 1, 2, \dots,$$

$$v_0(t) = bt^{\alpha-1}, \quad v_{n+1} = \int_0^1 G(t, s)f(s, v_n(s)) ds, \quad n = 0, 1, 2, \dots$$

*Proof* Consider the order interval  $[u_0, v_0]$  in  $C[0, 1]$ , then  $T : [u_0, v_0] \rightarrow C[0, 1]$  is completely continuous. In fact, let  $u \in [u_0, v_0]$ , then  $at^{\alpha-1} \leq u \leq bt^{\alpha-1}, 0 \leq t \leq 1$ . By the assumption (H<sub>1</sub>), we have

$$\begin{aligned} \|Tu\| &= \max_{0 \leq s \leq 1} \int_0^1 G(t, s)|f(s, u(s))| ds \leq \int_0^1 t^{\alpha-1} \Phi_2(s)|f(s, u(s))| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \left[ 1 + \frac{\max\{\alpha - \beta - 1, 1\} \int_0^1 dA(t)}{1 - \delta} \right] \int_0^1 |f(s, u(s))| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \left[ 1 + \frac{\max\{\alpha - \beta - 1, 1\} \int_0^1 dA(t)}{1 - \delta} \right] \int_0^1 |h(s)| ds, \end{aligned}$$

which shows that  $T([u_0, v_0])$  is uniform bounded in  $C[0, 1]$ .

On the other hand, for any  $u \in [u_0, v_0]$  and  $0 \leq t_1 \leq t_2 \leq 1$ , by Lemma 2.11,

$$\begin{aligned} & |(Tu)(t_1) - (Tu)(t_2)| \\ & \leq \int_0^1 |G(t_1, s) - G(t_2, s)| |f(s, u(s))| ds \\ & \leq \max_{0 \leq s \leq 1} |G(t_1, s) - G(t_2, s)| \int_0^1 |f(s, u(s))| ds \\ & \leq (\alpha - 1) \left[ \frac{2}{\Gamma(\alpha)} + \frac{\max_{0 \leq s \leq 1} \int_0^1 H(t, s) dA(t)}{1 - \delta} \right] (t_2 - t_1) \int_0^1 |f(s, u(s))| ds \\ & \leq (\alpha - 1) \left[ \frac{2}{\Gamma(\alpha)} + \frac{\max_{0 \leq s \leq 1} \int_0^1 H(t, s) dA(t)}{1 - \delta} \right] (t_2 - t_1) \int_0^1 |h(s)| ds. \end{aligned}$$

This shows that the set  $T([u_0, v_0])$  is equicontinuous in  $C[0, 1]$ . Furthermore, for  $u \in [u_0, v_0]$ , the operator  $T$  is continuous obviously. By the Arzelá-Ascoli theorem, the operator  $T : [u_0, v_0] \rightarrow C[0, 1]$  is completely continuous.

Now, we divide our proof into the following steps.

Step 1. We assert that  $T$  is monotone increasing on  $[u_0, v_0]$ . By the assumption  $(H_1)$ , it is easy to know  $T$  is an increasing operator.

Step 2. We prove that  $u_0$  is a lower solution of  $T$ .

In view of the assumption  $(H_2)$ ,  $(H_3)$ , and Lemma 2.10, we have

$$\begin{aligned} (Tu_0)(t) &= \int_0^1 G(t, s)f(s, u_0(s)) \, ds = \int_0^1 G(t, s)f(s, as^{\alpha-1}) \, ds \\ &= \int_0^1 G(t, s) \max\{f(s, as^{\alpha-1}), 0\} \, ds + \int_0^1 G(t, s) \min\{f(s, as^{\alpha-1}), 0\} \, ds \\ &\geq t^{\alpha-1} \int_0^1 \Phi_1(s) \max\{f(s, as^{\alpha-1}), 0\} \, ds + t^{\alpha-1} \int_0^1 \Phi_2(s) \min\{f(s, as^{\alpha-1}), 0\} \, ds \\ &\geq at^{\alpha-1} = u_0(t). \end{aligned}$$

It shows

$$u_1 = \int_0^1 G(t, s)f(s, u_0(s)) \, ds = Tu_0 \geq u_0.$$

Step 3. We prove that  $v_0$  is an upper solution of  $T$ . We have

$$\begin{aligned} (Tv_0)(t) &= \int_0^1 G(t, s)f(s, v_0(s)) \, ds = \int_0^1 G(t, s)f(s, bs^{\alpha-1}) \, ds \\ &= \int_0^1 G(t, s) \max\{f(s, bs^{\alpha-1}), 0\} \, ds + \int_0^1 G(t, s) \min\{f(s, bs^{\alpha-1}), 0\} \, ds \\ &\leq t^{\alpha-1} \int_0^1 \Phi_2(s) \max\{f(s, bs^{\alpha-1}), 0\} \, ds + t^{\alpha-1} \int_0^1 \Phi_1(s) \min\{f(s, bs^{\alpha-1}), 0\} \, ds \\ &\leq bt^{\alpha-1} = v_0(t). \end{aligned}$$

It yields

$$v_1 = \int_0^1 G(t, s)f(s, v_0(s)) \, ds = Tv_0 \leq v_0.$$

Step 4. We claim that the BVP (1.3) has monotone nontrivial solutions.

In fact, if we construct sequences  $\{u_n\}_{n=1}^\infty$  and  $\{v_n\}_{n=1}^\infty$  as follows:

$$u_n = Tu_{n-1} \quad \text{and} \quad v_n = Tv_{n-1}, \quad n = 1, 2, 3, \dots,$$

then it follows from Theorem 1.1 that

$$u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0,$$

and  $\{u_n\}_{n=1}^\infty$  and  $\{v_n\}_{n=1}^\infty$  converge to, respectively,  $u^*$  and  $v^* \in [u_0, v_0]$ , which are monotone solutions of the BVP (1.3). Moreover, in view of  $f(t, 0) \neq 0$ , we know that the zero function is not a solution of BVP (1.3). Thus  $u^*$  and  $v^*$  are nontrivial.  $\square$

**Remark 3.2** We do not require that  $f : (0, 1) \times R \rightarrow R$  is continuous and increasing, we only require the local continuity and local monotonicity;  $f(t, u)$  in our case may change sign. If  $a > 0$ , then the two nontrivial solutions  $u^*, v^*$  ( $u^*, v^*$  may be coincident) satisfy  $0 < at^{\alpha-1} \leq u^* \leq v^* \leq bt^{\alpha-1} < b, 0 \leq t \leq 1$ , which implies that  $u^*, v^*$  are positive solutions of BVP (1.3).

**Corollary 3.3** *Assume that there exist two positive constants  $a, b$  such that  $a < b$  and the following conditions are satisfied:*

- (C<sub>1</sub>)  $f : (0, 1) \times [0, b] \rightarrow R^+$  is continuous and there exists a nonnegative function  $h \in C(0, 1) \cap L^1[0, 1]$  such that  $|f(t, u)| \leq h(t), (t, u) \in (0, 1) \times [0, b]$  and increasing with  $u$  in  $[0, b]$ ;
- (C<sub>2</sub>)  $\int_0^1 \Phi_1(s)f(s, as^{\alpha-1}) ds \geq a$  and  $\int_0^1 \Phi_2(s)f(s, bs^{\alpha-1}) ds \leq b$ ;
- (C<sub>3</sub>)  $f(t, 0) \neq 0, \text{ for } 0 \leq t \leq 1$ .

Then the problem (1.1) has two positive solutions  $u^*, v^* \in C^2[0, 1] \cap C^3(0, 1)$  such that  $0 < at^{\alpha-1} \leq u^* \leq v^* \leq bt^{\alpha-1}, t \in [0, 1]$  and  $\lim_{n \rightarrow \infty} u_n = u^*, \lim_{n \rightarrow \infty} v_n = v^*$  ( $u^*, v^*$  may be coincident), where the two iterative sequences are generated by

$$\begin{aligned}
 u_0(t) &= at^{\alpha-1}, & u_{n+1} &= \int_0^1 G(t, s)f(s, u_n(s)) ds, & n &= 0, 1, 2, \dots, \\
 v_0(t) &= bt^{\alpha-1}, & v_{n+1} &= \int_0^1 G(t, s)f(s, v_n(s)) ds, & n &= 0, 1, 2, \dots
 \end{aligned}$$

### 4 Example

In this section, we give an example to illustrate our main results.

Consider the BVP

$$\begin{cases}
 D_{0+}^\alpha u(t) + f(t, u(t)) = 0, & t \in (0, 1), \\
 u(0) = u'(0) = 0, & D_{0+}^\beta u(1) = \int_0^1 D_{0+}^\beta u(t) dA(t),
 \end{cases} \tag{4.1}$$

where  $\alpha = \frac{5}{2}, \beta = \frac{1}{2}, A(t) = t$ ,

$$f(t, u) = \begin{cases}
 4t^5 - \frac{1}{100}, & u \in (-\infty, 0), t \in (0, 1), \\
 4t^5 - \frac{1}{100} + 4u - u^2, & u \in [0, 2], t \in (0, 1), \\
 4t^5 + \frac{399}{100}, & u \in (2, +\infty), t \in (0, 1).
 \end{cases}$$

A simple calculation shows that

$$\Phi_1(s) := \frac{1}{\Gamma(\frac{5}{2})} \left[ \frac{1}{2} + \frac{\int_0^1 (1-t)t dt}{1 - \int_0^1 t dt} \right] s(1-s) \approx 0.6269s(1-s)$$

and

$$\Phi_2(s) := \frac{1}{\Gamma(\frac{5}{2})} \left[ 1 + \frac{\int_0^1 dt}{1 - \int_0^1 t dt} \right] (1-s) \approx 2.2568(1-s).$$

Let  $a = 0$ ,  $b = 2$ , so  $f(t, u)$  is continuous and nondecreasing with respect to  $u$ . Thus, the condition  $(H_1)$  holds and  $f(t, 0) = 4t^5 - \frac{1}{100}$ ,  $f(t, 2t^{\frac{5}{2}}) = 8t^{\frac{5}{2}} - \frac{1}{100}$ . By direct calculation, we have

$$\begin{aligned} & \int_0^1 \Phi_1(s) \max\{f(s, as^{\alpha-1}), 0\} ds + \int_0^1 \Phi_2(s) \min\{f(s, as^{\alpha-1}), 0\} ds \\ &= \int_{(\frac{1}{400})^{\frac{1}{5}}}^1 0.6269s(1-s) \left(4s^5 - \frac{1}{100}\right) ds + \int_0^{(\frac{1}{400})^{\frac{1}{5}}} 2.2568(1-s) \left(4s^5 - \frac{1}{100}\right) ds \\ &\approx 0.0508 \geq 0 \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 \Phi_2(s) \max\{f(s, bs^{\alpha-1}), 0\} ds + \int_0^1 \Phi_1(s) \min\{f(s, bs^{\alpha-1}), 0\} ds \\ &= \int_{(\frac{1}{800})^{\frac{2}{5}}}^1 2.2568(1-s) \left(8s^{\frac{5}{2}} - \frac{1}{100}\right) ds + \int_0^{(\frac{1}{800})^{\frac{2}{5}}} 0.6269s(1-s) \left(8s^{\frac{5}{2}} - \frac{1}{100}\right) ds \\ &\approx 0.725 \leq 2, \end{aligned}$$

which show that the condition  $(H_2)$  holds. So, it follows from Theorem 3.1 that BVP (4.1) has two nontrivial solutions  $u^*$  and  $v^*$  ( $u^*$ ,  $v^*$  may be coincident), such that  $0 \leq u^* \leq v^* \leq 2t^{\frac{3}{2}} \leq 2$ ,  $0 \leq t \leq 1$ , and which start off with the zero function or  $2t^{\frac{3}{2}}$ , which are very feasible for computational purposes.

#### Competing interests

The author declares that she has no competing interests.

#### Author's contributions

The author read and approved the final manuscript.

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