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Persistence properties for the Fokas-Olver-Rosenau-Qiao equation in weighted L^p spaces

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Abstract
In this paper, we mainly study persistence properties for a generalized Camassa-Holm equation with cubic nonlinearity, and we prove the persistence properties in weighted spaces of the solution to the equation, provided that the initial potential satisfies a certain sign condition. Our results extend the work of Brandolese (Int. Math. Res. Not. 22:5161-5181, 2012) on persistence properties to the Fokas-Olver-Rosenau-Qiao equation. In contrast to the Camassa-Holm equation with quadratic nonlinearity, the effect of cubic nonlinearity of the Fokas-Olver-Rosenau-Qiao equation on the persistence properties is rather delicate.

MSC: 35G25; 35L05; 35Q50

Keywords: nonuniform dependence; persistence properties; cubic nonlinearity

1 Introduction

The present paper focuses on the Cauchy problem of the integrable modified Camassa-Holm equation with cubic nonlinearity

$$\begin{cases} m_t + (u^2 - u_x^2)m_x + 2u_x m^2 + \gamma u_x = 0, & m = u - u_{xx}, t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where γ is a constant. Equation (1.1) was independently proposed by Fokas [2], Fuchssteiner [3], and Olver and Rosenau [4] as a new generalization of an integrable system by applying the general method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified Korteweg-de Vries equation. Later, it was obtained by Qiao [5, 6] from the two-dimensional Euler equations, where the variables $u(t,x)$ and $m(t,x)$ represent, respectively, the velocity of the fluid and its potential density. Ivanov and Lyons [7] obtained a class of soliton solutions of the integrable hierarchy, which has been put forward in a series of works by Qiao [5, 6]. It was shown that Equation (1.1) admits the Lax pair and the Cauchy problem (1.1) may be solved by the inverse scattering transform method. The formation of singularities and the existence of peaked traveling-wave solutions for Equation (1.1) was investigated in [8]. The well-posedness, blow-up mechanism, and persistence properties are given in [9]. Using the method of approximate solutions in conjunction with well-posedness estimate, Himonas and Mantzavinos [10] proved that the solution map of the Cauchy problem for this modified Camassa-Holm equation is not uniformly continuous in Sobolev spaces H^s with $s > 5/2$ and called this equation the Fokas-Olver-Rosenau-Qiao equation. Recently, the authors in [11] showed that the local structure of the initial profile can affect the singularity formation and seek initial datum with a sign-changing momentum density m that can generate finite-time blow-up. It was also found that Equation (1.1) is related to the short-pulse equation derived by Schäfer and Wayne [12],

$$v_{xt} = \frac{1}{3}\left(v^3\right)_{xx} + \gamma v, \tag{1.2}$$

which is a model for the propagation of ultra-short light pulses in silica optical fibers [12] and is also an approximation of nonlinear wave packets in dispersive media in the limit of few cycles on the ultra-short pulse scale [13].

The original Camassa-Holm (CH) equation

$$m_t + um_x + 2u_x m + \gamma u_x = 0, \quad m = u - u_{xx}, \tag{1.3}$$

can itself be derived from the Korteweg-de Vries equation by tri-Hamiltonian duality. The Camassa-Holm equation arises in a variety of different contexts. In 1981, it was originally derived as a bi-Hamiltonian equation with infinitely many conservation laws by Fokas and Fuchssteiner [14]. It has been widely studied since 1993 when Camassa and Holm [15] proposed it as a model for the unidirectional propagation of shallow water waves over a flat bed. The Camassa-Holm equation also has a bi-Hamiltonian structure [14, 16] and is completely integrable [15, 17, 18], and it possesses infinitely many conservation laws and is solvable by its corresponding inverse scattering transform [19, 20]. The stability of smooth solitons was considered in [21], and the orbital stability of the peaked solitons was proved in [22]. It is worth pointing out that solutions of this type are not mere abstractions: the peakons replicate a feature that is characteristic for the waves of greatest height - waves of largest amplitude that are exact solutions of the governing equations for irrotational water waves (see [23] and references therein). An explicit interaction of the peaked solitons was given in [24]. It has been shown that this problem is locally well posed for initial data $u_0 \in H^s$ with $s > \frac{3}{2}$ [25–27]. Moreover, the Camassa-Holm equation not only has global strong solutions, but also admits finite-time blow-up solutions [25, 27–30], and the blow-up occurs in the form of breaking waves, namely, the solution remains bounded, but its slope becomes unbounded in finite time. On the other hand, it also has global weak solutions in H^1 (see [31–34]). The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models peculiar wave breaking phenomena [29, 35].

Clearly, the nonlinearity in the CH equation is quadratic. Two integrable Camassa-Holm-type equations with cubic nonlinearity have been discovered, Equation (1.1) and the Novikov equation [36]

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx}, \tag{1.4}$$

which was recently discovered by Novikov in a symmetry classification of nonlocal PDEs with quadratic or cubic nonlinearity [36]. The perturbative symmetry approach [37] yields necessary conditions for a PDE to admit infinitely many symmetries. Using this approach, Novikov was able to isolate Equation (1.4) and find its first few symmetries, and he subsequently found a scalar Lax pair for it, proving that the equation is integrable. By using the prolongation algebra method Hone and Wang [38] gave a matrix Lax pair and many conserved densities and a bi-Hamiltonian structure of the Novikov equation and showed how it was related by a reciprocal transformation to a negative flow in the Sawada-Kotera hierarchy. Then in [39], the authors calculated explicit formulas for multipeakon solutions of the Novikov equation. Recently, the well-posedness and persistence properties of the solution for (1.4) were established in [40–42]. In [43, 44], we discuss the Cauchy problem for a generalized b-equation with higher-order nonlinearities and peakons in critical Besov spaces and weighted L^p spaces, which includes the famous Camassa-Holm and Novikov equations as particular cases.

The spacial decay rates for the strong solutions to the Camassa-Holm and Novikov equations were established, provided that the corresponding initial datum decays at infinity [1, 45, 46]. This kind of property is the so-called persistence property. Motivated by the recent work [1] on the nonlinear Camassa-Holm equation in weighted Sobolev spaces, the other aim of this paper is to establish the persistence properties for the modified Camassa-Holm equation (1.1) in weighted L^p spaces. However, there are high nonlinearity and regularity in (1.1), which makes the proof of several required nonlinear estimates very difficult.

In the present paper, we intend to find a large class of weight functions ϕ such that

$$\sup_{t\in[0,T)} \left(\left\|u(t)\phi\right\|_p + \left\|\partial_x u(t)\phi\right\|_p\right) < \infty,$$

where $\|\cdot\|_p$ denotes the usual L_p norm. This way we obtain a persistence result on solutions u to Equation (1.1) in the weight L_p spaces $L_{p,\phi} := L_p(\mathbb{R}, \phi^p\, dx)$. As a consequence and an application, we determine the spatial asymptotic behavior of certain solutions to Equation (1.1). Our results generalize the work of [1] on persistence and nonpersistence of solutions to Equation (1.1) in $L_{p,\phi}$. We will work with moderate weight functions that appear with regularity in the theory of time-frequency analysis [47, 48] and have led to optimal results for the Camassa-Holm equation in [1], and we first give the definition for admissible weight function. (The predefined terminologies like v-moderate, submultiplicative, and so on are given in Section 2. For more details, we refer the reader to [1, 49].)

Definition 1.1 An admissible weight function for Equation (1.1) is a locally absolutely continuous function $\phi : \mathbb{R} \to \mathbb{R}$ such that, for some $A > 0$ and almost all $x \in \mathbb{R}$, $|\phi'(x)| \le A|\phi(x)|$, and that is v-moderate for some submultiplicative weight function v satisfying $\inf_{\mathbb{R}} v > 0$ and

$$\int_{\mathbb{R}} \frac{v(x)}{e^{|x|}}\, dx < \infty. \tag{1.5}$$

We can now state our main result on admissible weights.

Theorem 1.1 *Let $T > 0$, $s > 5/2$, and $2 \le p \le \infty$. Let also $u \in C([0,T], H^s(\mathbb{R}))$ be a strong solution of the Cauchy problem for Equation* (1.1) *such that $u|_{t=0} = u_0$ satisfies*

$$u_0\phi \in L^p(\mathbb{R}) \quad \textit{and} \quad (\partial_x u_0)\phi \in L^p(\mathbb{R}),$$

where ϕ is an admissible weight function for Equation (1.1). *Then, for all $t \in [0, T]$, we have the estimate*

$$\|u(t)\phi\|_p + \|(\partial_x u(t))\phi\|_p \le (\|u_0\phi\|_p + \|(\partial_x u_0)\phi\|_p) \exp\{C(M^2 + |\gamma|)t\}$$

for some constant $C > 0$ depending only on ν, ϕ (through the constants A, C_0, $\inf_{\mathbb{R}} \nu$, and $\int_{\mathbb{R}} \frac{\nu(x)}{e^{|x|}}\,dx < \infty$), and

$$M \equiv \sup_{t\in[0,T]} (\|u(t)\|_\infty + \|\partial_x u(t)\|_\infty + \|\partial_x^2 u(t)\|_\infty) < \infty.$$

The basic example of the application of Theorem 1.1 is obtained by taking the standard weights $\phi = \phi_{a,b,c,d}(x) = e^{a|x|^b}(1 + |x|)^c \log(e + |x|)^d$ with the following conditions:

$$a \ge 0, \qquad c, d \in \mathbb{R}, \qquad 0 \le b \le 1, \qquad ab < 1.$$

The restriction $ab < 1$ guarantees the validity of condition (1.5) for a multiplicative function $\nu(x) \ge 1$. Thus, we have the following two special persistence properties.

Remark 1.1 (1) Take $\phi = \phi_{0,0,c,0}$ with $c > 0$, and choose $p = \infty$. In this case, Theorem 1.2 states that the condition

$$|u_0(x)| + |\partial_x u_0(x)| \le C(1 + |x|)^{-c}$$

implies the uniform algebraic decay in $[0, T]$:

$$|u(x,t)| + |\partial_x u(x,t)| \le C(1 + |x|)^{-c}.$$

Thus, we obtain the algebraic decay rates of strong solutions to Equation (1.1). By the way, we already know that this result holds for the CH equation [46].

(2) Choose $\phi = \phi_{a,1,0,0}$ if $x \ge 0$ and $\phi(x) = 1$ if $x \le 0$ with $0 \le a < 1$. It is easy to see that such a weight satisfies the admissibility conditions of Definition 1.1. Let further $p = \infty$ in Theorem 1.1, then we deduce that Equation (1.1) preserve the pointwise decay $O(e^{-ax})$ as $x \to +\infty$ for any $t > 0$. Similarly, we have persistence of the decay $O(e^{-ax})$ as $x \to -\infty$. A corresponding result on persistence of strong solutions of the CH and Novikov equations and of Equation (1.1) can be found in [9, 45, 50], respectively.

Clearly, the limit case $\phi = \phi_{1,1,c,d}$ is not covered by Theorem 1.1. In the following theorem, however, we may choose the weight $\phi = \phi_{1,1,c,d}$ with $c < 0$, $d \in \mathbb{R}$, and $\frac{1}{|c|} < p \le \infty$, or more generally when $(1 + |\cdot|)^c \log(e + |\cdot|)^d \in L^p(\mathbb{R})$. See Theorem 1.2, which covers the case of such fast growing weights. In other words, we want to establish a variant of Theorem 1.1 that can be applied to some ν-moderate weights ϕ for which condition (1.5) does not hold. Instead of assuming (1.5), we now put the weaker condition

$$\nu e^{-|\cdot|} \in L^p(\mathbb{R}), \tag{1.6}$$

where $2 \le p \le \infty$.

Theorem 1.2 *Let* $2 \le p \le \infty$, *and let* ϕ *be a* v*-moderate weight function as in Definition* 1.1 *satisfying condition* (1.6) *instead of* (1.5). *Let also* $u|_{t=0} = u_0$ *satisfy*

$$u_0\phi \in L^p(\mathbb{R}), \qquad u_0\phi^{\frac{1}{3}} \in L^3(\mathbb{R}), \qquad \gamma\phi \in L^1(\mathbb{R})$$

and

$$(\partial_x u_0)\phi \in L^p(\mathbb{R}), \qquad (\partial_x u_0)\phi^{\frac{1}{3}} \in L^3(\mathbb{R}).$$

Let also $u \in C([0,T], H^s(\mathbb{R}))$ *with* $s > 5/2$ *be the strong solution of the Cauchy problem for Equation* (1.1) *emanating from* u_0. *Then,*

$$\sup_{t\in[0,T]} \left(\left\|u(t)\phi\right\|_{L^p} + \left\|\left(\partial_x u(t)\right)\phi\right\|_{L^p}\right)$$

and

$$\sup_{t\in[0,T]} \left(\left\|u(t)\phi^{\frac{1}{3}}\right\|_{L^3} + \left\|\left(\partial_x u(t)\right)\phi^{\frac{1}{3}}\right\|_{L^3}\right)$$

are finite.

Remark 1.2 Choosing $\phi(x) = \phi_{1,1,0,0}(x) = e^{|x|}$ and $p = \infty$ in Theorem 1.2, it follows that if $|u_0(x)|$ and $|\partial_x u_0(x)|$ are both bounded by $ce^{-|x|}$, then the strong solution satisfies

$$\left|u(x,t)\right| + \left|\partial_x u(x,t)\right| \le Ce^{-|x|} \tag{1.7}$$

uniformly in $[0,T]$. Thus, Theorems 1.1 and 1.2 generalize the main result of [9] on persistence properties of strong solutions to Equation (1.1).

2 Analysis of the Equation (1.1) in weighted spaces

In this section, we shall discuss the persistence properties for a generalized Camassa-Holm equation (1.1) in weighted L^p spaces. For the convenience of the readers, we present some standard definitions. In general, a weight function is simply a nonnegative function. A weight function $v : \mathbb{R}^n \to \mathbb{R}$ is called submultiplicative if

$$v(x+y) \le v(x)v(y) \quad \text{for all } x, y \in \mathbb{R}^n.$$

Given a submultiplicative function v, a positive function ϕ is v-moderate if and only if

$$\exists C_0 > 0: \quad \phi(x+y) \le C_0 v(x)\phi(y) \quad \text{for all } x, y \in \mathbb{R}^n.$$

If ϕ is v-moderate for some submultiplicative function v, then we say that ϕ is moderate. This is the usual terminology in time-frequency analysis papers [47]. Let us recall the most standard examples of such weights. Let

$$\phi(x) = \phi_{a,b,c,d}(x) = e^{a|x|^b}\left(1+|x|\right)^c \log\left(e+|x|\right)^d. \tag{2.1}$$

We have (see [1]) the following conditions:

(i) For $a, c, d \geq 0$ and $0 \leq b \leq 1$, such a weight is submultiplicative.
(ii) If $a, c, d \in \mathbb{R}$ and $0 \leq b \leq 1$, then ϕ is moderate. More precisely, $\phi_{a,b,c,d}$ is $\phi_{\alpha,\beta,\gamma,\delta}$-moderate for $|a| \leq \alpha$, $|b| \leq \beta$, $|c| \leq \gamma$, and $|d| \leq \delta$.

The elementary properties of submultiplicative and moderate weights can be found in [1]. Now, we prove Theorem 1.1.

Proof of Theorem 1.1 In fact, we can rewrite the Cauchy problem (1.1) as follows:

$$\begin{cases} u_t + (u^2 - \frac{1}{3}u_x^2)u_x = -A(u) - B(u), \\ u(x,0) = u_0(x), \end{cases} \tag{2.2}$$

where $A(u) = \frac{1}{3}(1-\partial_x)^{-1}u_x^3 = \frac{1}{3}G * (u_x^3)$, $B(u) = \partial_x(1-\partial_x^2)^{-1}(\frac{2}{3}u^3 + uu_x^2 + \gamma u) = \partial_x G * (\frac{2}{3}u^3 + uu_x^2 + \gamma u)$ with kernel $G(x) = \frac{1}{2}e^{-|x|}$.

On the other hand, from the assumption $u \in C([0,T], H^s)$, $s > 5/2$, we get

$$M \equiv \sup_{t\in[0,T]} \left(\left\| u(t) \right\|_\infty + \left\| \partial_x u(t) \right\|_\infty + \left\| \partial_{xx} u(t) \right\|_\infty \right) < \infty.$$

For any $N \in \mathbb{Z}^+$, let us consider the N-truncations of $\phi(x)$: $f(x) = f_N(x) = \max\{\phi, N\}$. Then $f : \mathbb{R} \to \mathbb{R}$ is a locally absolutely continuous function such that

$$\|f\|_\infty \leq N, \qquad \left|f'(x)\right| \leq A\left|f(x)\right| \quad \text{a.e. on } \mathbb{R}.$$

In addition, if $C_1 = \max\{C_0, \alpha^{-1}\}$, where $\alpha = \inf_{x\in\mathbb{R}} \nu(x) > 0$, then

$$f(x+y) \leq C_1 \nu(x) f(y), \quad \forall x, y \in \mathbb{R}.$$

Moreover, as shown in [1], the N-truncations f of a ν-moderate weight ϕ are uniformly ν-moderate with respect to N.

We start by considering the case $2 \leq p < \infty$. Multiplying Equation (2.2) by $f|uf|^{p-2}(uf)$ and integrate to obtain

$$\begin{aligned} &\int_{\mathbb{R}} |uf|^{p-2}(uf)(\partial_t uf)\,dx + \int_{\mathbb{R}} |uf|^{p-2}(uf)\left(u^2 - \frac{1}{3}u_x^2\right)u_x f\,dx \\ &\quad + \int_{\mathbb{R}} |uf|^{p-2}(uf)f \cdot \bigl(A(u) + B(u)\bigr)\,dx = 0. \end{aligned} \tag{2.3}$$

Note that the estimates

$$\begin{aligned} &\int_{\mathbb{R}} |uf|^{p-2}(uf)(\partial_t uf)\,dx = \frac{1}{p}\frac{d}{dt}\|uf\|_{L^p}^p = \|uf\|_{L^p}^{p-1}\frac{d}{dt}\|uf\|_{L^p}, \\ &\left| \int_{\mathbb{R}} (uf)^p (u\partial_x u)\,dx \right| \leq \|u\partial_x u\|_{L^\infty} \|uf\|_{L^p}^p \leq M^2 \|uf\|_{L^p}^p, \end{aligned}$$

and

$$\left| \int_{\mathbb{R}} (uf)^{p-1}(\partial_x u)^3 f\,dx \right| \leq \|uf\|_{L^p}^{p-1} \left\| (\partial_x u)^3 f \right\|_{L^p} \leq M^2 \|uf\|_{L^p}^{p-1} \left\| (\partial_x u) f \right\|_{L^p}$$

are true. Moreover, we get

$$\begin{aligned}
&\left|\int_{\mathbb{R}} |uf|^{p-2}(uf)\big[f\cdot\big(A(u)+B(u)\big)\big]\,dx\right| \\
&\quad\le \|uf\|_{L^p}^{p-1}\big\|f\cdot\big(A(u)+B(u)\big)\big\|_{L^p} \\
&\quad\le c\|uf\|_{L^p}^{p-1}\left\{\|G\nu\|_{L^1}\big\|fu_x^3\big\|_{L^p}+\big\|(\partial_x G)\nu\big\|_{L^1}\left\|f\left(\frac{2}{3}u^3+uu_x^2+\gamma u\right)\right\|_{L^p}\right\} \\
&\quad\le C\|uf\|_{L^p}^{p-1}\big[\big(M^2+|\gamma|\big)\|uf\|_{L^p}+M^2\|f\partial_x u\|_{L^p}\big].
\end{aligned}$$

In the first inequality, we used Hölder's inequality, in the second inequality, we applied Propositions 3.1 and 3.2 in [1], and in the last one, we used condition (1.5). Here, C depends only on ν and ϕ. From (2.3) we can obtain

$$\frac{d}{dt}\|uf\|_{L^p}\le C_1\big(M^2+|\gamma|\big)\|uf\|_{L^p}+C_2M^2\big\|(\partial_x u)f\big\|_{L^p}. \tag{2.4}$$

Next, we will give estimates on u_xf. Differentiating (2.2) with respect to the x-variable and then multiplying by f produce the equation

$$\partial_t\big[(\partial_x u)f\big]+u^2f\partial_x^2u+2\big[(\partial_x u)f\big](u\partial_x u)-(\partial_x u)^2f\partial_x^2u+f\big[\partial_x\big(A(u)+B(u)\big)\big]=0.$$

Multiply this equation by $|f\partial_x u|^{p-2}(f\partial_x u)$ with $p\in\mathbb{Z}^+$, integrate the result in the x-variable, and note that

$$\begin{aligned}
&\int_{\mathbb{R}}|f\partial_x u|^{p-2}(f\partial_x u)\partial_t\big[(\partial_x u)f\big]\,dx=\|f\partial_x u\|_{L^p}^{p-1}\frac{d}{dt}\|f\partial_x u\|_{L^p}, \\
&\int_{\mathbb{R}}|f\partial_x u|^{p-2}(f\partial_x u)(\partial_x u)^2f\partial_x^2u\,dx \\
&\quad\le\|f\partial_x u\|_{L^p}^{p-1}\big\|(\partial_x u)^2f\partial_x^2u\big\|_{L^p}\le M^2\|f\partial_x u\|_{L^p}^{p-1}\big\|(\partial_x u)f\big\|_{L^p},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\mathbb{R}}|f\partial_x u|^{p-2}(f\partial_x u)\big[f\partial_x\big(A(u)+B(u)\big)\big]\,dx\right| \\
&\quad\le\|f\partial_x u\|_{L^p}^{p-1}\big\|f\partial_x\big(A(u)+B(u)\big)\big\|_{L^p} \\
&\quad\le C\|f\partial_x u\|_{L^p}^{p-1}\big[\big(M^2+|\gamma|\big)\|uf\|_{L^p}+M^2\|f\partial_x u\|_{L^p}\big].
\end{aligned}$$

In the third inequality, we applied the pointwise bound $|\partial_x G(x)|\le\frac{1}{2}e^{-|x|}$ and the condition

$$\begin{aligned}
&\left|\int_{\mathbb{R}}|f\partial_x u|^{p-2}(f\partial_x u)u^2f\partial_x^2u\,dx\right| \\
&\quad=\left|\int_{\mathbb{R}}|f\partial_x u|^{p-2}(f\partial_x u)u^2\big[\partial_x(f\partial_x u)-(\partial_x u)(\partial_x f)\big]\,dx\right| \\
&\quad=\left|\int_{\mathbb{R}}u^2\partial_x\left(\frac{|f\partial_x u|^p}{p}\right)-\int_{\mathbb{R}}|f\partial_x u|^{p-2}(f\partial_x u)u^2(\partial_x u)(\partial_x f)\,dx\right| \\
&\quad\le 2/pM^2\|f\partial_x u\|_{L^p}^p+AM^2\|f\partial_x u\|_{L^p}^p.
\end{aligned}$$

In the last inequality, we used $|\partial_x f(x)| \le Af(x)$ for a.e. x. Thus, we get

$$\frac{d}{dt}\|f\partial_x u\|_{L^p} \le C_3\bigl(M^2 + |\gamma|\bigr)\|uf\|_{L^p} + C_4M^2\bigl\|(\partial_x u)f\bigr\|_{L^p}. \tag{2.5}$$

Now, combining inequalities (2.4) and (2.5) and then integrating yield

$$\begin{aligned}&\bigl\|u(t)f\bigr\|_{L^p} + \bigl\|(\partial_x u)(t)f\bigr\|_{L^p}\\ &\quad\le \bigl(\|u_0f\|_{L^p} + \bigl\|(\partial_x u_0)f\bigr\|_{L^p}\bigr)\exp\bigl\{C\bigl(M^2 + |\gamma|\bigr)t\bigr\} \quad \text{for all } t \in [0,T].\end{aligned}$$

Since $f(x) = f_N(x) \uparrow \phi(x)$ as $N \to \infty$ for a.e. $x \in \mathbb{R}$, recalling that $u_0\phi \in L^p(\mathbb{R})$ and $\partial_x u_0\phi \in L^p(\mathbb{R})$, we get

$$\begin{aligned}&\bigl\|u(t)\phi\bigr\|_{L^p} + \bigl\|(\partial_x u)(t)\phi\bigr\|_{L^p}\\ &\quad\le \bigl(\|u_0\phi\|_{L^p} + \bigl\|(\partial_x u_0)\phi\bigr\|_{L^p}\bigr)\exp\bigl\{C\bigl(M^2 + |\gamma|\bigr)t\bigr\} \quad \text{for all } t \in [0,T].\end{aligned}$$

Finally, we treat the case $p = \infty$. We have $u_0, \partial_x u_0, \partial_x^2 u_0 \in L^2 \cap L^\infty$ and $f(x) = f_N(x) \in L^\infty$. Hence, we have

$$\begin{aligned}&\bigl\|u(t)f\bigr\|_{L^q} + \bigl\|(\partial_x u)(t)f\bigr\|_{L^q}\\ &\quad\le \bigl(\|u_0f\|_{L^q} + \bigl\|(\partial_x u_0)f\bigr\|_{L^q}\bigr)\exp\bigl\{C\bigl(M^2 + |\gamma|\bigr)t\bigr\}, \quad q \in [2,\infty).\end{aligned} \tag{2.6}$$

The last factor in the right-hand side is independent of q. Since $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \cap L^2$, this implies that

$$\bigl\|u(t)f\bigr\|_{L^\infty} + \bigl\|(\partial_x u)(t)f\bigr\|_{L^\infty} \le \bigl(\|u_0f\|_{L^\infty} + \bigl\|(\partial_x u_0)f\bigr\|_{L^\infty}\bigr)\exp\bigl\{C\bigl(M^2 + |\gamma|\bigr)t\bigr\}.$$

The last factor in the right-hand side is independent of N. Now taking $N \to \infty$ implies that estimate (2.6) remains valid for $p = \infty$. □

Proof of Theorem 1.2 Arguing as in the proof of Theorem 1.1, we can easily get

$$\frac{d}{dt}\|uf\|_{L^p} \le M^2\|uf\|_{L^p} + \bigl\|f\bigl(A(u) + B(u)\bigr)\bigr\|_{L^p} \quad \text{for } p < \infty \tag{2.7}$$

and

$$\frac{d}{dt}\|f\partial_x u\|_{L^p} \le CM^2\bigl\|(\partial_x u)f\bigr\|_{L^p} + \bigl\|f\partial_x\bigl(A(u) + B(u)\bigr)\bigr\|_{L^p} \quad \text{for } p < \infty, \tag{2.8}$$

where $A(u) = \frac{1}{3}G * (u_x^3)$, $B(u) = \partial_x G * (\frac{2}{3}u^3 + uu_x^2 + |\gamma|u)$, and $f(x) = f_N(x) = \min\{\phi(x), N\}$.

Next, we estimate $\|f(A(u) + B(u))\|_{L^p}$ and $\|f\partial_x(A(u) + B(u))\|_{L^p}$. Note that $\phi^{\frac{1}{3}}$ is a $v^{\frac{1}{3}}$-moderate weight such that $(\phi^{\frac{1}{3}})'(x) \le \frac{A}{3}\phi^{\frac{1}{3}}(x)$. Moreover, $\inf_{\mathbb{R}} v^{\frac{1}{3}} > 0$. By condition (1.6), $v^{\frac{1}{3}}e^{-|x|/3} \in L^{3p}(\mathbb{R})$; hence, Hölder's inequality implies that $v^{\frac{1}{3}}e^{-|x|} \in L^1(\mathbb{R})$. Then Theorem 1.2 applies with $p = 3$ to the weight $\phi^{\frac{1}{3}}$, yielding

$$\bigl\|u(t)\phi^{\frac{1}{3}}\bigr\|_{L^3} + \bigl\|(\partial_x u)(t)\phi^{\frac{1}{3}}\bigr\|_{L^3} \le \bigl(\bigl\|u_0\phi^{\frac{1}{3}}\bigr\|_{L^3} + \bigl\|(\partial_x u_0)\phi^{\frac{1}{3}}\bigr\|_{L^3}\bigr)\exp\bigl\{C\bigl(M^2 + |\gamma|\bigr)t\bigr\}.$$

Therefore,

$$\begin{aligned}
&\left\|f\left(A(u)+B(u)\right)\right\|_{L^p}\\
&\quad\le C\left(\|Gv\|_{L^p}\left\|\phi u_x^3\right\|_{L^1}+\left\|(\partial_x G)v\right\|_{L^p}\left\|\phi\left(\frac{2}{3}u^3+uu_x^2+\gamma u\right)\right\|_{L^1}\right)\\
&\quad\le C\left(\left\|\phi^{\frac13}u_x\right\|_{L^3}^3+\left\|\phi^{\frac13}u\right\|_{L^3}^3+\left\|\phi u(\partial_x u)^2\right\|_{L^1}+\|\gamma\phi u\|_{L^1}\right)\\
&\quad\le C\left(\left\|\phi^{\frac13}u_x\right\|_{L^3}^3+\left\|\phi^{\frac13}u\right\|_{L^3}^3+\left\|\phi^{\frac13}u\right\|_{L^3}\left\|\phi^{\frac23}(\partial_x u)^2\right\|_{L^{\frac32}}+\|\gamma\phi\|_{L^1}^{2/3}\left\|\phi^{\frac13}u\right\|_{L^3}\right)\\
&\quad\le C\left(\left\|\phi^{\frac13}u_x\right\|_{L^3}^3+\left\|\phi^{\frac13}u\right\|_{L^3}^3+\left\|\phi^{\frac13}u\right\|_{L^3}\left\|\phi^{\frac13}(\partial_x u)\right\|_{L^3}^2+\left\|\phi^{\frac13}u\right\|_{L^3}\right)\\
&\quad\le C_0\exp\left\{3C\left(M^2+|\gamma|\right)t\right\},
\end{aligned}$$

where the constants C_0 depending only on ϕ, $|\gamma|$, and the initial data.

Similarly, recalling that $\partial_x G\le\frac12 e^{-|x|}$ and $\partial_x^2 G=G-\delta$, we have

$$\begin{aligned}
&\left\|f\left(\partial_x\left[A(u)+B(u)\right]\right)\right\|_{L^p}\\
&\quad\le c\left\|f\left(\partial_x G*\left(u_x^3\right)\right)\right\|_{L^p}+c\left\|f\left(\frac23u^3+uu_x^2+\gamma u\right)\right\|_{L^p}+c\left\|fG*\left(\frac23u^3+uu_x^2+\gamma u\right)\right\|_{L^p}\\
&\quad\le C_1\exp\left\{3C\left(M^2+|\gamma|\right)t\right\}+C\left(M^2+|\gamma|\right)\left(\|uf\|_{L^p}+\left\|(\partial_x u)f\right\|_{L^p}\right).
\end{aligned}$$

Plugging the last two estimates into (2.7) and (2.8), and summing, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|u(t)f\right\|_{L^p}+\left\|(\partial_x u)(t)f\right\|_{L^p}\right)\\
&\quad\le K_1\left(M^2+|\gamma|\right)\left(\|u_0f\|_{L^p}+\left\|\left(\partial_x u_0\right)f\right\|_{L^p}\right)+2K_0\exp\left\{3C\left(M^2+|\gamma|\right)t\right\}.
\end{aligned}$$

Integrating and finally letting $N\to\infty$ yield the conclusion in the case $2\le p<\infty$. The constants throughout the proof are independent of p. Therefore, for $p=\infty$, we can rely on the result established for finite exponents q and then let $q\to\infty$. The remaining argument is fully similar to that of Theorem 1.1. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

SZ participated in the design of the study and drafted the manuscript. MX and FZ carried out the theoretical studies and helped to draft the manuscript. All authors have read and approved the final manuscript.

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