Liu *Boundary Value Problems* (2015) 2015:210
DOI 10.1186/s13661-015-0471-9

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Large-time behavior of the strong solution to nonhomogeneous incompressible MHD system with general initial data

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Abstract
This paper investigates the large-time behavior of strong solutions to the nonhomogeneous incompressible magnetohydrodynamic equations on a bounded domain in $\mathbb{R}^2$. Based on uniform estimates, we prove that the velocity, the magnetic field, and their derivatives converge to zero in L^2 norm as time goes to infinity without any additional assumption on the initial data and external force by a pure energy method.

MSC: 76W05; 76N10; 35B40; 35B45

Keywords: magnetohydrodynamic equations; strong solution; uniform estimate; large-time behavior

1 Introduction

Magnetohydrodynamics (MHD) studies the theory of the macroscopic interaction of electrically conducting fluids with a magnetic field. MHD has a very broad range of applications, such as, the intensely heated and ionized fluids in an electromagnetic field in astrophysics, geophysics, high-speed aerodynamics, and plasma physics. In this paper, we are concerned with the nonhomogeneous incompressible MHD equations in a domain $\Omega \subset \mathbb{R}^2$ as follows (see, *e.g.*, [1–4]):

$$\rho_t + \mathrm{u} \cdot \nabla \rho = 0, \tag{1.1}$$

$$\rho \mathrm{u}_t + \rho \mathrm{u} \cdot \nabla \mathrm{u} + \nabla P = -\frac{1}{2}\nabla |\mathrm{H}|^2 + \mathrm{H} \cdot \nabla \mathrm{H} + \mu \Delta \mathrm{u} + \rho \mathrm{f}, \tag{1.2}$$

$$\mathrm{H}_t - \nu \Delta \mathrm{H} + \mathrm{u} \cdot \nabla \mathrm{H} - \mathrm{H} \cdot \nabla \mathrm{u} = 0, \tag{1.3}$$

$$\operatorname{div} \mathrm{H} = 0, \qquad \operatorname{div} \mathrm{u} = 0, \tag{1.4}$$

where $t \geq 0$ is time and $x \in \Omega$ is the spatial coordinate. The unknown functions are the density $\rho(x,t) \geq 0$, velocity $\mathrm{u} = (\mathrm{u}^1(x,t), \mathrm{u}^2(x,t)) \in \mathbb{R}^2$, the pressure $P(x,t)$, and the magnetic field $\mathrm{H} = (\mathrm{H}^1(x,t), \mathrm{H}^2(x,t)) \in \mathbb{R}^2$, respectively. f stands for the external force. The positive constants μ and ν denote the viscosity of fluid and the relative strengths of advection and diffusion of H, respectively. Throughout this paper, Ω is assumed to be a bounded domain in $\mathbb{R}^2$ with smooth boundary. Without loss of generality, we will take $\mu = \nu = 1$ for simplicity.

In this paper, we are interested in the initial-boundary value problem of (1.1)-(1.4) subject to the following initial conditions:

$$(\rho, \mathrm{u}, \mathrm{H})(x,0) = (\rho_0, \mathrm{u}_0, \mathrm{H}_0)(x) \quad \text{and} \quad \underline{\rho} \le \rho_0 \le \bar{\rho} \quad \text{for all } x \in \Omega, \tag{1.5}$$

and boundary conditions:

$$\mathrm{u}|_{\partial\Omega} = (\mathrm{H} \cdot n)|_{\partial\Omega} = \nabla \times \mathrm{H}|_{\partial\Omega} = 0, \tag{1.6}$$

where $\bar{\rho}$ and $\underline{\rho}$ are two fixed positive constants. n is the unit outward normal to $\partial\Omega$.

System (1.1)-(1.4) has drawn many attentions of engineers and applied mathematicians due to its important physical background and mathematical feature. The Cauchy problem to (1.1)-(1.4) has been much studied in the literature, including the existence, uniqueness, and regularity of solutions of the system. Zhang [5] established local classical solutions of (1.1)-(1.4) and proved that as the viscosity (μ) and resistivity (ν) went to zero, the solution of (1.1)-(1.4) converged to the solution of ideal MHD system (*i.e.* $\mu = \nu = 0$). Gerbeau and Le Bris [3] and Desjardins and Le Bris [6] considered the global existence of weak solutions of finite energy in the whole space or in the torus. Global existence of strong solutions with small initial data in some Besov spaces was considered by Abidi and Paicu [7]. For the initial-boundary problem of (1.1)-(1.4), Huang and Wang [8] recently proved the unique global strong solution with initial vacuum in dimension two without external force f. For more related results, we refer the reader to [1, 9–12] and the references therein.

For homogeneous incompressible MHD ($\rho \equiv$ const in (1.1)-(1.4)), we first emphasize that the unique global classical solutions for every initial data $(\mathrm{u}_0, \mathrm{H}_0) \in H^m$ with $m \ge 2$ have been established in [13, 14]. Furthermore, some sufficient conditions for smoothness were presented for weak solutions to the MHD equations in [15, 16]. For Cauchy problem, the authors in [17] studied the long time behavior of solutions to the MHD equations in two and three dimensions with some smallness conditions. In [18], the authors considered the asymptotic behavior of the strong solutions to MHD equations in a half space. We also notice that if the partial derivatives of the viscosity and resistivity are zero, the global regularity issue has been established in [19].

However, the large-time asymptotic behavior of strong solutions to (1.1)-(1.6) has not been well understood, especially for the case with general initial data and external force. The main purpose of this paper is to investigate the influence of the magnetic fields, viscosity, and boundary effects on the behavior of the solution of (1.1)-(1.6). We show that the velocity, the magnetic field, and their derivatives will converge to zero as time tends to infinity. The main result of this paper is stated as follows.

Theorem 1.1 *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^2$. Assume that the external force and the initial data satisfy*

$$\begin{cases} \mathrm{f} \in L^2(0,\infty; H^1(\Omega)), \quad \mathrm{f}_t \in L^2(0,\infty; L^2(\Omega)), \\ (\rho_0, \mathrm{u}_0, \mathrm{H}_0) \in H^2, \quad \underline{\rho} \le \rho_0 \le \bar{\rho} \textit{ for all } x \in \Omega, \end{cases} \tag{1.7}$$

then there exists a unique global solution $(\rho, \mathrm{u}, \mathrm{H})$ *to* (1.1)-(1.6) *satisfying*

$$\begin{cases} \rho \in C([0,\infty); H^2(\Omega)) \quad \textit{with } \underline{\rho} \le \rho \le \bar{\rho}, \\ (\mathrm{u}, \mathrm{H}) \in C([0,\infty); H^2(\Omega)) \cap L^2(0,\infty; H^3(\Omega)). \end{cases} \tag{1.8}$$

In particular, the following large-time behavior holds:

$$\lim_{t\to\infty} \|\mathrm{u}\|_{H^1} = 0, \qquad \lim_{t\to\infty} \|\mathrm{H}\|_{H^1} = 0, \tag{1.9}$$

$$\lim_{t\to\infty} \left(\|\mathrm{u}_t\|_{L^2} + \|\mathrm{H}_t\|_{L^2}\right) = 0. \tag{1.10}$$

Remark 1.1 If $B = 0$ (no magnetic field), then the system (1.1)-(1.4) becomes the classical Navier-Stokes system. The global existence and large-time behavior of the strong solution to the nonhomogeneous incompressible Navier-Stokes equations has been proved by Zhao [20].

Remark 1.2 We should point out that Huang and Wang in [8] have established a global existence of the strong solution to (1.1)-(1.4) with initial vacuum and $\mathrm{f} = 0$. Theorem 1.1 is still an interesting result, because we get the large-time behavior.

Our second main result gives an exponential decay rate for the solution on taking the external force $f = 0$, which is stated as follows.

Theorem 1.2 *Assume the conditions in Theorem* 1.1 *and take the external force* $f = 0$ *in* (1.2). *Then there exists a unique global solution* $(\rho, \mathrm{u}, \mathrm{H})$ *to* (1.1)-(1.6) *satisfying* (1.8). *In particular, one has the following exponential decay*:

$$\|\mathrm{u}\|_{L^2}^2 + \|\mathrm{H}\|_{L^2}^2 \le \exp\{-C_1 t\}, \tag{1.11}$$

where C_1 *is a given positive constant independent of* t.

Remark 1.3 Theorem 1.2 establishes exponential decay. Indeed, if we imposed some conditions on the external force f but would not take $f = 0$, we can also give an algebraic rate by performing energy estimates with weights in time.

We will prove Theorem 1.1 via the pure energy method, which is based on the uniform estimate for the local solution. This approach is motivated by the previous work on the nonhomogeneous incompressible Navier-Stokes equations due to Zhao [20]. Due to the strong coupling between the velocity, magnetic field and density equations, the nonhomogeneous problem (1.1)-(1.4) under consideration is much more involved. It should be pointed out that the analysis for (1.1)-(1.4) is very complicated and more efforts should be made as regards the estimates involving these coupling terms and space-time-dependent external force.

Next, we give a brief outline of the proof. To obtain the large-time behavior, we must establish the uniform estimate. As for the uniform estimate, it is difficult to estimate the higher-order derivative of u and H due to the lack of the spatial derivatives of the solution at the boundary. So it is divided into two steps: first of all applying the standard energy estimate on the temporal derivatives of the solution, and then spatial derivatives by applying regularity theory of Stokes equation repeatedly. With all the estimates in hand, we can establish the global existence by a standard method. Finally, the proof of the large-time behavior is mainly based on the following fact: if $f(t) \in W^{1,1}(0,\infty)$, then $f(t) \to 0$ as $t \to \infty$. By combining these uniform estimates in Lemmas 3.1-3.3 and the fact above we prove the desired large-time behavior of the solution in Section 4.

The rest of the paper is organized as follows. In Section 2, we state some well-known inequalities, and in Section 3 we deduce some uniform estimates and the time-dependent estimate of the higher-order derivative. Finally, we will complete the proof of Theorem 1.1 in Section 4.

2 Auxiliary lemmas

In this section, we state some inequalities which will be frequently used in this paper. First of all, we recall some Sobolev embedding inequalities (see [21, 22]).

Lemma 2.1 *The following inequalities hold*:

$$(1)\quad \|f\|_{L^p} \le C\|f\|_{H^1}, \quad \forall 1 \le p < \infty, \forall f \in H^1, \tag{2.1}$$

$$(2)\quad \|f\|_{L^\infty} \le C\|f\|_{W^{1,p}}, \quad \forall 2 < p < \infty, \forall f \in W^{1,p}, \tag{2.2}$$

$$(3)\quad \|f\|_{L^4}^2 \le C\|f\|_{L^2}\|\nabla f\|_{L^2}, \quad \forall f \in H_0^1, \tag{2.3}$$

$$(4)\quad \|f\|_{L^p} \le C\|\nabla f\|_{L^2}, \quad \forall 1 \le p < \infty, \forall f \in H_0^1, \tag{2.4}$$

$$(5)\quad \|f\|_{L^4}^2 \le C\big(\|f\|_{L^2}^2 + \|f\|_{L^2}\|\nabla f\|_{L^2}\big), \quad \forall f \in H^1. \tag{2.5}$$

Next, we state the following Sobolev embedding (see [23, 24]), which will be used to deal with the estimate of H.

Lemma 2.2 *Let* $v \in H^1(\Omega)$ *be a vector function satisfying* $v|_{\partial\Omega} = 0$ *or* $v \cdot n|_{\partial\Omega} = 0$, *then*

$$\|v\|_{L^2} \le C\|\nabla v\|_{L^2}. \tag{2.6}$$

3 Some *a priori* estimates

The local existence has been established by Chen *et al.* [9]. Hence, it only remains to establish some necessary *a priori* bounds for the strong solutions $(\rho, \mathrm{u}, \mathrm{H})$ to the initial-boundary value problem (1.1)-(1.4).

Let $T > 0$ be a fixed time and $(\rho, \mathrm{u}, \mathrm{H})$ be the strong solution to (1.1)-(1.4) defined on $\Omega \times (0, T]$. Throughout this paper, we will denote by C the various generic positive constants, which may depend on the initial data and $\bar{\rho}$, $\underline{\rho}$, f, and Ω but are independent of t. A special dependence will be pointed out explicitly in this paper if necessary.

3.1 Uniform estimate

First, by the method of characteristics and standard energy we have the following uniform estimates:

$$\underline{\rho} \le \rho(x,t) \le \bar{\rho} \tag{3.1}$$

and

$$\sup_{0\le t\le\infty}\int\left(|\mathrm{u}|^2+\frac{1}{2}|\mathrm{H}|^2\right)\mathrm{d}x+\int_0^\infty\left(\|\nabla \mathrm{u}\|_{L^2}^2+\|\nabla \mathrm{H}\|_{L^2}^2\right)\mathrm{d}t\le C. \tag{3.2}$$

The next lemma is the crucial estimate in this paper. Higher-order estimates of the density, velocity and magnetic field can be obtained in a standard way provided that $\|\mathrm{u}\|_{H^1}$ and $\|\mathrm{H}\|_{H^1}$ are uniformly bounded with respect to time.

Lemma 3.1 *Let* $(\rho,\mathrm{u},\mathrm{H})$ *be a smooth solution of* (1.1)-(1.6) *on* $\Omega\times(0,\infty)$. *Then there exists a constant C such that*

$$\sup_{0\le t\le\infty}\left(\|\mathrm{u}\|_{H^1}^2+\|\mathrm{H}\|_{H^1}^2\right)+\int_0^\infty\left(\left\|\rho^{1/2}\mathrm{u}_t\right\|_{L^2}^2+\|\mathrm{H}_t\|_{L^2}^2+\|\mathrm{H}\|_{H^2}^2+\|\mathrm{u}\|_{H^2}^2\right)\mathrm{d}t\le C. \tag{3.3}$$

Proof Multiplying (1.2) by u_t and integrating by parts over Ω, one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathrm{u}\|_{L^2}^2+\left\|\rho^{1/2}\mathrm{u}_t\right\|_{L^2}^2=\int(\mathrm{H}\cdot\nabla\mathrm{H}\cdot\mathrm{u}_t-\rho\mathrm{u}\cdot\nabla\mathrm{u}\cdot\mathrm{u}_t+\rho\mathrm{f}\cdot u_t)\,\mathrm{d}x. \tag{3.4}$$

Similarly, it follows from (1.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathrm{H}\|_{L^2}^2+\left(\|\mathrm{H}_t\|_{L^2}^2+\|\triangle\mathrm{H}\|_{L^2}\right)=\int|\mathrm{H}_t-\Delta\mathrm{H}|^2\,\mathrm{d}x=\int|\mathrm{H}\cdot\nabla\mathrm{u}-\mathrm{u}\cdot\nabla\mathrm{H}|^2\,\mathrm{d}x. \tag{3.5}$$

Putting (3.4) and (3.5) together leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\mathrm{u}\|_{L^2}^2+\|\nabla\mathrm{H}\|_{L^2}^2\right)+\left(\left\|\rho^{1/2}\mathrm{u}_t\right\|_{L^2}^2+\|\mathrm{H}_t\|_{L^2}^2+\|\triangle\mathrm{H}\|_{L^2}^2\right)\\&\quad\le\int|\mathrm{H}||\nabla\mathrm{H}||\mathrm{u}_t|\,\mathrm{d}x+\int\rho|\mathrm{u}||\nabla\mathrm{u}||\mathrm{u}_t|\,\mathrm{d}x+2\int|\mathrm{H}|^2|\nabla\mathrm{u}|^2\,\mathrm{d}x\\&\qquad+2\int|\mathrm{u}|^2|\nabla\mathrm{H}|^2\,\mathrm{d}x+\int\rho|\mathrm{f}||\mathrm{u}_t|\,\mathrm{d}x=\sum_{i=1}^5 I_i.\end{aligned} \tag{3.6}$$

Next, we will estimate all the terms on the right-hand side of (3.6) term by term. Using the Young inequality and (3.1), we have

$$\begin{aligned}I_1&\le\varepsilon\left\|\rho^{1/2}\mathrm{u}_t\right\|_{L^2}^2+C_1\|\mathrm{H}\|_{L^4}^2\|\nabla\mathrm{H}\|_{L^4}^2\\&\le\varepsilon\left\|\rho^{1/2}\mathrm{u}_t\right\|_{L^2}^2+\varepsilon\|\Delta\mathrm{H}\|_{L^2}+C(\varepsilon)\|\nabla\mathrm{H}\|_{L^2}^4,\end{aligned}$$

where we use the fact

$$\begin{aligned}\|\mathrm{H}\|_{L^4}^2\|\nabla\mathrm{H}\|_{L^4}^2&\le C\left(\|\mathrm{H}\|_{L^2}^2+\|\mathrm{H}\|_{L^2}\|\nabla\mathrm{H}\|_{L^2}\right)\left(\|\nabla\mathrm{H}\|_{L^2}\|\Delta\mathrm{H}\|_{L^2}+\|\nabla\mathrm{H}\|_{L^2}^2\right)\\&\le\|\mathrm{H}\|_{L^2}^2\|\nabla\mathrm{H}\|_{L^2}\|\Delta\mathrm{H}\|_{L^2}+\|\mathrm{H}\|_{L^2}^2\|\nabla\mathrm{H}\|_{L^2}^2\\&\quad+\|\mathrm{H}\|_{L^2}\|\nabla\mathrm{H}\|_{L^2}^2\|\Delta\mathrm{H}\|_{L^2}+\|\mathrm{H}\|_{L^2}^2\|\nabla\mathrm{H}\|_{L^2}^3\\&\le\frac{\varepsilon}{C_1}\|\Delta\mathrm{H}\|_{L^2}+C\|\nabla\mathrm{H}\|_{L^2}^4,\end{aligned} \tag{3.7}$$

due to (3.2), (2.5), and (2.6). Similarly, we have

$$\begin{aligned} I_2 &\le \varepsilon \left\| \rho^{1/2} \mathrm{u}_t \right\|_{L^2}^2 + C(\varepsilon) \|\mathrm{u}\|_{L^4}^2 \|\nabla \mathrm{u}\|_{L^4}^2 \\ &\le \varepsilon \left\| \rho^{1/2} \mathrm{u}_t \right\|_{L^2}^2 + \varepsilon \left\| \nabla^2 u \right\|_{L^2} + C \|\nabla \mathrm{u}\|_{L^2}^4. \end{aligned}$$

Next, we turn to estimating $\|\nabla^2 u\|_{L^2}$. From (1.2), we know that u satisfies the following Stokes equations:

$$\begin{cases} -\triangle \mathrm{u} + \nabla P = -\rho \mathrm{u}_t - \rho \mathrm{u} \cdot \nabla \mathrm{u} + \frac{1}{2} \nabla |\mathrm{H}|^2 - \mathrm{H} \cdot \nabla \mathrm{H} + \rho \mathrm{f}, \\ \operatorname{div} \mathrm{u} = 0, \quad \text{in } \Omega, \\ \mathrm{u} = 0, \quad \text{on } \partial\Omega. \end{cases}$$

By the well-known regularity theory on the Stokes equations (see [23]) and using (2.5), (3.2), and (3.7), we have

$$\begin{aligned} \|\mathrm{u}\|_{H^2} &\le C \big(\|\rho \mathrm{u}_t\|_{L^2} + \|\rho \mathrm{u} \cdot \nabla \mathrm{u}\|_{L^2} + \|\nabla |\mathrm{H}|^2\|_{L^2} + \|\mathrm{H} \cdot \nabla \mathrm{H}\|_{L^2} + \|\mathrm{f}\|_{L^2} \big) \\ &\le C \big(\left\| \rho^{1/2} \mathrm{u}_t \right\|_{L^2} + \|\mathrm{u}\|_{L^4} \|\nabla \mathrm{u}\|_{L^4} + \|\mathrm{H}\|_{L^4} \|\nabla \mathrm{H}\|_{L^4} + \|\mathrm{f}\|_{L^2} \big) \\ &\le C \big(\left\| \rho^{1/2} \mathrm{u}_t \right\|_{L^2} + \|\nabla \mathrm{u}\|_{L^2} \left\| \nabla^2 \mathrm{u} \right\|_{L^2}^{1/2} \\ &\quad + \|\nabla \mathrm{u}\|_{L^2}^2 + \|\nabla \mathrm{H}\|_{L^2} \|\Delta \mathrm{H}\|_{L^2}^{1/2} + \|\nabla \mathrm{H}\|_{L^2}^2 + \|\mathrm{f}\|_{L^2} \big) \\ &\le C \big(\left\| \rho^{1/2} \mathrm{u}_t \right\|_{L^2} + \|\nabla \mathrm{u}\|_{L^2}^2 + \|\nabla \mathrm{H}\|_{L^2}^2 + \|\mathrm{f}\|_{L^2} \big) + \frac{1}{2} \left\| \nabla^2 \mathrm{u} \right\|_{L^2}^{1/2} + \|\Delta \mathrm{H}\|_{L^2}^{1/2}, \end{aligned}$$

which immediately leads to

$$\|\mathrm{u}\|_{H^2} \le C \big(\left\| \rho^{1/2} \mathrm{u}_t \right\|_{L^2} + \|\nabla \mathrm{u}\|_{L^2}^2 + \|\nabla \mathrm{H}\|_{L^2}^2 + \|\mathrm{f}\|_{L^2} \big) + 2 \|\Delta \mathrm{H}\|_{L^2}.$$

Substituting the above inequality into I_2, we have

$$I_2 \le C\varepsilon \left\| \rho^{1/2} \mathrm{u}_t \right\|_{L^2}^2 + \varepsilon \|\Delta \mathrm{H}\|_{L^2}^2 + C \|\nabla \mathrm{u}\|_{L^2}^4 + C \|\nabla \mathrm{u}\|_{L^2}^2 \|\nabla \mathrm{H}\|_{L^2}^2 + C \|\mathrm{f}\|_{L^2}^2.$$

Similarly, for I_3 and I_4, we have

$$I_3 \le \varepsilon \left\| \rho^{1/2} \mathrm{u}_t \right\|_{L^2}^2 + \varepsilon \|\Delta \mathrm{H}\|_{L^2}^2 + C \|\nabla \mathrm{H}\|_{L^2}^2 \|\nabla \mathrm{u}\|_{L^2}^2 + C \|\nabla \mathrm{u}\|_{L^2}^4 + C \|\nabla \mathrm{H}\|_{L^2}^4$$

and

$$\begin{aligned} I_4 &\le C \|\mathrm{u}\|_{L^4}^2 \|\nabla \mathrm{H}\|_{L^4}^2 \le C \|\mathrm{u}\|_{L^2} \|\nabla \mathrm{u}\|_{L^2} \big(\|\nabla \mathrm{H}\|_{L^2}^2 + \|\nabla \mathrm{H}\|_{L^2} \|\Delta \mathrm{H}\|_{L^2} \big) \\ &\le C \|\nabla \mathrm{H}\|_{L^2}^2 \|\nabla \mathrm{u}\|_{L^2}^2 + \varepsilon \|\Delta \mathrm{H}\|_{L^2}^2, \end{aligned}$$

due to (3.2).

Using the Young inequality immediately leads to

$$I_5 \le \varepsilon \left\| \rho^{1/2} \mathrm{u}_t \right\|_{L^2}^2 + C(\varepsilon) \|\mathrm{f}\|_{L^2}^2.$$

Taking ε small enough and substituting I_1-I_5 into (3.6), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla \mathrm{u}\|_{L^2}^2 + \|\nabla \mathrm{H}\|_{L^2}^2\big) + \big(\big\|\rho^{1/2}\mathrm{u}_t\big\|_{L^2}^2 + \|\mathrm{H}_t\|_{L^2}^2 + \|\triangle \mathrm{H}\|_{L^2}^2\big)$$
$$\le C\big(\|\nabla \mathrm{H}\|_{L^2}^2 + \|\nabla \mathrm{u}\|_{L^2}^2\big)^2 + \|\mathrm{f}\|_{L^2}^2,$$

which, together with Gronwall's inequality, immediately leads to

$$\sup_{0\le t\le\infty}\big(\|\nabla \mathrm{u}\|_{L^2}^2 + \|\nabla \mathrm{H}\|_{L^2}^2\big) + \int_0^\infty \big(\big\|\rho^{1/2}\mathrm{u}_t\big\|_{L^2}^2 + \|\mathrm{H}_t\|_{L^2}^2 + \big\|\nabla^2 \mathrm{H}\big\|_{L^2}^2\big)\,\mathrm{d}t \le C. \tag{3.8}$$

We get the desired estimate (3.3) by (3.2) and (3.8). □

The following lemma is concerned with the L^2-estimate of $\rho^{1/2}\mathrm{u}_t$ and H_t.

Lemma 3.2 *Let* $(\rho, \mathrm{u}, \mathrm{H})$ *be a strong solution of* (1.1)-(1.6) *on* $\Omega \times (0,\infty)$. *Then there exists a constant C such that*

$$\sup_{0\le t\le\infty}\big(\big\|\rho^{1/2}\mathrm{u}_t\big\|_{L^2}^2 + \|\mathrm{H}_t\|_{L^2}^2\big) + \int_0^\infty \|\nabla \mathrm{u}_t\|_{L^2}^2 + \|\nabla \mathrm{H}_t\|_{L^2}^2\,\mathrm{d}t \le C. \tag{3.9}$$

Proof Differentiating the momentum equations (1.2) with respect to t yields

$$\rho\mathrm{u}_{tt} + \rho\mathrm{u}\cdot\nabla\mathrm{u}_t + \rho\mathrm{u}_t\cdot\nabla\mathrm{u} + \rho_t(\mathrm{u}_t + \mathrm{u}\cdot\nabla\mathrm{u}) + \nabla P_t$$
$$= \triangle\mathrm{u}_t + \left(\mathrm{H}\cdot\nabla\mathrm{H} - \frac{1}{2}\nabla|\mathrm{H}|^2\right)_t + (\rho\mathrm{f})_t.$$

Multiplying the equation above with u_t and integrating by parts over Ω, one gets

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\rho^{1/2}\mathrm{u}_t\big\|_{L^2}^2 + \|\nabla\mathrm{u}_t\|_{L^2}^2$$
$$= -\int \rho_t|\mathrm{u}_t|^2\,\mathrm{d}x - \int \rho(\mathrm{u}_t\cdot\nabla\mathrm{u})\cdot\mathrm{u}_t\,\mathrm{d}x - \int \rho_t(\mathrm{u}\cdot\nabla\mathrm{u})\cdot\mathrm{u}_t\,\mathrm{d}x$$
$$+ \int (\mathrm{H}_t\cdot\nabla\mathrm{H} + \mathrm{H}\cdot\nabla\mathrm{H}_t + \rho_t\mathrm{f} + \rho\mathrm{f}_t)\cdot\mathrm{u}_t\,\mathrm{d}x. \tag{3.10}$$

Differentiating (1.3) with respect to t and multiplying the resulting equation by H_t, we obtain after integrating by parts

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{H}_t\|_{L^2}^2 + \|\nabla\mathrm{H}_t\|_{L^2}^2$$
$$= -\int \mathrm{u}_t\cdot\nabla\mathrm{H}\cdot\mathrm{H}_t\,\mathrm{d}x + \int \mathrm{H}_t\cdot\nabla\mathrm{u}\cdot\mathrm{H}_t + \mathrm{H}\cdot\nabla\mathrm{u}_t\cdot\mathrm{H}_t\,\mathrm{d}x. \tag{3.11}$$

Putting (3.10) and (3.11) together leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\mathrm{H}_t\|_{L^2}^2 + \big\|\rho^{1/2}\mathrm{u}_t\big\|_{L^2}^2\big) + \|\nabla\mathrm{u}_t\|_{L^2}^2 + \|\nabla\mathrm{H}_t\|_{L^2}^2$$
$$= -\int \rho_t|\mathrm{u}_t|^2\,\mathrm{d}x - \int \rho(\mathrm{u}_t\cdot\nabla\mathrm{u})\cdot\mathrm{u}_t\,\mathrm{d}x - \int \rho_t(\mathrm{u}\cdot\nabla\mathrm{u})\cdot\mathrm{u}_t\,\mathrm{d}x$$

$$
\begin{aligned}
&+\int \rho_t \mathrm{f}\cdot \mathrm{u}_t\,\mathrm{d}x+\int \rho \mathrm{f}_t\cdot \mathrm{u}_t\,\mathrm{d}x\\
&+\int \mathrm{H}_t\cdot\nabla \mathrm{H}\cdot \mathrm{u}_t-\mathrm{u}_t\cdot\nabla \mathrm{H}\cdot \mathrm{H}_t\,\mathrm{d}x+\int \mathrm{H}_t\cdot\nabla \mathrm{u}\cdot \mathrm{H}_t\,\mathrm{d}x\\
&+\int \mathrm{H}\cdot\nabla \mathrm{u}_t\cdot \mathrm{H}_t+\mathrm{H}\cdot\nabla \mathrm{H}_t\cdot \mathrm{u}_t\,\mathrm{d}x\\
\triangleq{}&\sum_{i=1}^{8}R_i. \qquad (3.12)
\end{aligned}
$$

We now estimate each term on the right-hand side of (3.12). First, using the Hölder and Young inequalities, we obtain

$$
\begin{aligned}
|R_1| &= \left|\int \rho \mathrm{u}\cdot\nabla|\mathrm{u}_t|^2\,\mathrm{d}x\right| \le C\|\nabla \mathrm{u}_t\|_{L^2}\|\mathrm{u}\|_{L^4}\|\mathrm{u}_t\|_{L^4}\\
&\le C\|\nabla \mathrm{u}_t\|_{L^2}\|\mathrm{u}\|_{L^2}^{1/2}\|\nabla \mathrm{u}\|_{L^2}^{1/2}\|\mathrm{u}_t\|_{L^2}^{1/2}\|\nabla \mathrm{u}_t\|_{L^2}^{1/2}\\
&\le C\|\mathrm{u}_t\|_{L^2}^{1/2}\|\nabla \mathrm{u}_t\|_{L^2}^{3/2}\le \varepsilon\|\nabla \mathrm{u}_t\|_{L^2}^2+C(\varepsilon)\left\|\rho^{1/2}\mathrm{u}_t\right\|_{L^2}^2,\\
|R_2| &\le C\int |\mathrm{u}_t|^2|\nabla \mathrm{u}|\,\mathrm{d}x\le C\|\nabla \mathrm{u}\|_{L^2}\|\mathrm{u}_t\|_{L^4}^2\\
&\le C\|\mathrm{u}_t\|_{L^2}\|\nabla \mathrm{u}_t\|_{L^2}\le \varepsilon\|\nabla \mathrm{u}_t\|_{L^2}^2+C(\varepsilon)\left\|\rho^{1/2}\mathrm{u}_t\right\|_{L^2}^2,
\end{aligned}
$$

where (3.1), (3.2), and (3.3) are all used.

The estimate of R_3 is given as follows:

$$
\begin{aligned}
|R_3| &= \left|\int \rho \mathrm{u}\cdot\nabla(\mathrm{u}\cdot\nabla \mathrm{u}\cdot \mathrm{u}_t)\,\mathrm{d}x\right|\\
&\le C\int |\mathrm{u}||\nabla \mathrm{u}|^2|\mathrm{u}_t|+|\mathrm{u}|^2|\nabla^2 \mathrm{u}||\mathrm{u}_t|+|\mathrm{u}|^2|\nabla \mathrm{u}||\nabla \mathrm{u}_t|\,\mathrm{d}x\triangleq\sum_{i=1}^{3}J_i.
\end{aligned}
$$

Using (2.4) and (3.3), we can make the reduction

$$
\begin{aligned}
|J_1| &\le C\int |\mathrm{u}||\nabla \mathrm{u}|^2|\mathrm{u}_t|\,\mathrm{d}x\le C\|\nabla \mathrm{u}\|_{L^4}^2\|\mathrm{u}\|_{L^4}\|\mathrm{u}_t\|_{L^4}\\
&\le C\bigl(\|\nabla \mathrm{u}\|_{L^2}^2+\|\nabla \mathrm{u}\|_{L^2}\left\|\nabla^2 \mathrm{u}\right\|_{L^2}\bigr)\|\nabla \mathrm{u}\|_{L^2}\|\nabla \mathrm{u}_t\|_{L^2}\\
&\le C\bigl(1+\left\|\nabla^2 \mathrm{u}\right\|_{L^2}\bigr)\|\nabla \mathrm{u}\|_{L^2}\|\nabla \mathrm{u}_t\|_{L^2}\\
&\le \varepsilon\|\nabla \mathrm{u}_t\|_{L^2}^2+C(\varepsilon)\|\nabla \mathrm{u}\|_{L^2}^2+C(\varepsilon)\left\|\nabla^2 \mathrm{u}\right\|_{L^2}^2,\\
|J_2| &\le C\left\|\nabla^2 \mathrm{u}\right\|_{L^2}\|\mathrm{u}\|_{L^8}^2\|\mathrm{u}_t\|_{L^4}\\
&\le C\left\|\nabla^2 \mathrm{u}\right\|_{L^2}\|\nabla \mathrm{u}\|_{L^2}^2\|\nabla \mathrm{u}_t\|_{L^2}\\
&\le \varepsilon\|\nabla \mathrm{u}_t\|_{L^2}^2+C(\varepsilon)\left\|\nabla^2 \mathrm{u}\right\|_{L^2}^2,\\
|J_3| &\le C\|\nabla \mathrm{u}\|_{L^4}\|\mathrm{u}\|_{L^8}^2\|\nabla \mathrm{u}_t\|_{L^2}\le \varepsilon\|\nabla \mathrm{u}_t\|_{L^2}^2+C(\varepsilon)\left\|\nabla^2 \mathrm{u}\right\|_{L^2}^2+C(\varepsilon)\|\nabla \mathrm{u}\|_{L^2}^2.
\end{aligned}
$$

From the estimate of J_1-J_3, we obtain

$$
|R_3|\le 3\varepsilon\|\nabla \mathrm{u}_t\|_{L^2}^2+C\left\|\nabla^2 \mathrm{u}\right\|_{L^2}^2+C\|\nabla \mathrm{u}\|_{L^2}^2.
$$

For R_4-R_7, we have

$$\begin{aligned}
|R_4| &= \left| \int \rho \mathrm{u} \cdot \nabla(\mathrm{f} \cdot \mathrm{u}_t)\,\mathrm{d}x \right| \\
&\le C\|\mathrm{u}\|_{L^4}\|\mathrm{u}_t\|_{L^4}\|\nabla \mathrm{f}\|_{L^2} + C\|\mathrm{u}\|_{L^4}\|\nabla \mathrm{u}_t\|_{L^2}\|\mathrm{f}\|_{L^4} \\
&\le C\|\nabla \mathrm{u}_t\|_{L^2}\|\nabla \mathrm{f}\|_{L^2} + C\|\mathrm{f}\|_{H^1}\|\nabla \mathrm{u}_t\|_{L^2} \\
&\le \varepsilon\|\nabla \mathrm{u}_t\|_{L^2}^2 + C(\varepsilon)\|\mathrm{f}\|_{H^1}^2, \\
|R_5| &\le \left\|\rho^{1/2}\mathrm{u}_t\right\|_{L^2}^2 + \|\mathrm{f}_t\|_{L^2}^2, \\
|R_6| &\le C\int |\mathrm{H}_t||\nabla \mathrm{u}||\mathrm{u}_t|\,\mathrm{d}x \\
&\le C\|\nabla \mathrm{u}\|_{L^4}\|\mathrm{H}_t\|_{L^2}\|\mathrm{u}_t\|_{L^4} \\
&\le C\|\nabla \mathrm{u}\|_{H^1}^{1/2}\|\mathrm{H}_t\|_{L^2}\|\nabla \mathrm{u}_t\|_{L^2} \\
&\le \varepsilon\|\nabla \mathrm{u}_t\|_{L^2}^2 + C(\varepsilon)\|\nabla \mathrm{u}\|_{H^1}\|\mathrm{H}_t\|_{L^2}^2 \\
&\le \varepsilon\|\nabla \mathrm{u}_t\|_{L^2}^2 + C(\varepsilon)\|\mathrm{H}_t\|_{L^2}^4 + C(\varepsilon)\|\mathrm{u}\|_{H^2}^2, \\
|R_7| &\le C\int |\mathrm{H}_t|^2|\nabla \mathrm{u}|\,\mathrm{d}x \le C\|\mathrm{H}_t\|_{L^4}^2\|\nabla \mathrm{u}\|_{L^2} \\
&\le C\big(\|\mathrm{H}_t\|_{L^2}^2 + \|\mathrm{H}_t\|_{L^2}\|\nabla \mathrm{H}_t\|_{L^2}\big) \\
&\le \varepsilon\|\nabla \mathrm{H}_t\|_{L^2}^2 + C(\varepsilon)\|\mathrm{H}_t\|_{L^2}^2.
\end{aligned}$$

It is easy to prove that $R_8 = 0$. Taking ε small enough and substituting R_1-R_8 into (3.12), one obtains

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\mathrm{H}_t\|_{L^2}^2 + \left\|\rho^{1/2}\mathrm{u}_t\right\|_{L^2}^2\big) + \big(\|\nabla \mathrm{u}_t\|_{L^2}^2 + \|\nabla \mathrm{H}_t\|_{L^2}^2\big) \\
&\quad \le C\|\nabla \mathrm{u}\|_{L^2}^2 + C\|\mathrm{u}\|_{H^2}^2 + C\|\mathrm{H}_t\|_{L^2}^2 + C\|\mathrm{H}_t\|_{L^2}^4 + C\left\|\rho^{1/2}\mathrm{u}_t\right\|_{L^2}^2 + C\|\nabla \mathrm{f}\|_{L^2}^2 + C\|\mathrm{f}_t\|_{L^2}^2,
\end{aligned}$$

which, together with Gronwall's inequality and (3.3), immediately leads to the desired estimate (3.9). □

3.2 Time-dependent estimate for higher derivative

It suffices to prove the large-time behavior with the help of the uniform estimates in Lemmas 3.1 and 3.2. Lemma 3.3 below deals with the higher-order estimates of the solutions which are needed to guarantee the extension of a local classical solution to a global one.

Lemma 3.3 *Let* $(\rho, \mathrm{u}, \mathrm{H})$ *be a strong solution of* (1.1)-(1.6) *on* $\Omega \times (0, T)$. *Then there exists a constant* $C(T)$ *such that*

$$\sup_{0\le t\le T}\big(\|\rho\|_{H^2} + \|\mathrm{u}\|_{H^2} + \|\mathrm{H}\|_{H^2}\big) + \int_0^T \|\mathrm{u}\|_{H^3}^2 + \|\mathrm{H}\|_{H^3}^2\,\mathrm{d}t \le C(T). \tag{3.13}$$

Proof The proof of this lemma is standard; details are omitted. □

4 Proof of Theorems 1.1 and 1.2

With all the *a priori* estimates in Section 3 at hand, the existence and uniqueness of the strong solutions to (1.1)-(1.6) can be done by standard continuous arguments. Thus, this section is mainly devoted to the proof of the large-time behavior as (1.9)-(1.10) by the uniform estimates in Section 3. The proof is split into three steps as follows.

Proof of Theorem 1.1

Step 1: Decay of u and H.

Set

$$A_1(t) = \int \rho|\mathrm{u}|^2\,\mathrm{d}x, \qquad B_1(t) = \int |\mathrm{H}|^2\,\mathrm{d}x.$$

From Lemmas 2.1-2.2 and the energy inequality, we have

$$\int_0^t A_1(s)\,\mathrm{d}s \le C\int_0^t \|\mathrm{u}\|_{L^2}^2\,\mathrm{d}s \le C\int_0^t \|\nabla\mathrm{u}\|_{L^2}^2\,\mathrm{d}s \le C, \tag{4.1}$$

$$\int_0^t B_1(s)\,\mathrm{d}s \le C\int_0^t \|\nabla\mathrm{H}\|_{L^2}^2\,\mathrm{d}s \le C. \tag{4.2}$$

Next, we need to prove that $\frac{\mathrm{d}}{\mathrm{d}t}A_1$ and $\frac{\mathrm{d}}{\mathrm{d}t}B_1$ are in $L^1(0,\infty)$. From (1.2)-(1.3) and using integration by parts over Ω, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A_1(t) &= \int \rho_t|\mathrm{u}|^2 + 2\rho\mathrm{u}_t\cdot\mathrm{u}\,\mathrm{d}x\\ &= -2\int|\nabla\mathrm{u}|^2\,\mathrm{d}x + 2\int H\cdot\nabla H\cdot u\,\mathrm{d}x + 2\int\rho\mathrm{f}\cdot\mathrm{u}\,\mathrm{d}x, \end{aligned}\tag{4.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}B_1(t) = 2\int \mathrm{H}_t\cdot\mathrm{H}\,\mathrm{d}x = -2\int|\nabla\mathrm{H}|^2\,\mathrm{d}x + \int \mathrm{H}\cdot\nabla\mathrm{u}\cdot\mathrm{H}\,\mathrm{d}x. \tag{4.4}$$

For $A_1(t)$, we have by (4.3) and the lemmas in Section 3

$$\begin{aligned}\int_0^t\left|\frac{\mathrm{d}}{\mathrm{d}s}\big[A_1(s)\big]\right|\mathrm{d}s &\le C\int_0^t\|\nabla\mathrm{u}\|_{L^2}^2\,\mathrm{d}s + C\int_0^t\left|\int\mathrm{H}\cdot\nabla\mathrm{H}\cdot\mathrm{u}\,\mathrm{d}x\right|\mathrm{d}s\\ &\quad + C\int_0^t\left|\int\rho\mathrm{f}\cdot\mathrm{u}\,\mathrm{d}x\right|\mathrm{d}s\\ &\le C + C\int_0^t\left|\int\mathrm{H}\cdot\nabla\mathrm{H}\cdot\mathrm{u}\,\mathrm{d}x\right|\mathrm{d}s\\ &\le C,\end{aligned}\tag{4.5}$$

where we use the following estimate:

$$\begin{aligned}&\int_0^t\left|\int\mathrm{H}\cdot\nabla\mathrm{H}\cdot\mathrm{u}\,\mathrm{d}x\right|\mathrm{d}s\\ &\quad\le C\int_0^t\|\nabla\mathrm{H}\|_{L^2}\|\mathrm{H}\|_{L^4}\|\mathrm{u}\|_{L^4}\,\mathrm{d}s\\ &\quad\le C\int_0^t\|\nabla\mathrm{H}\|_{L^2}\big(\|\mathrm{H}\|_{L^2} + \|\mathrm{H}\|_{L^2}^{1/2}\|\nabla\mathrm{H}\|_{L^2}^{1/2}\big)\|\nabla\mathrm{u}\|_{L^2}\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_0^t \|\nabla \mathrm{H}\|_{L^2}^2 \|\nabla \mathrm{u}\|_{L^2}\,\mathrm{d}s \\
&\leq C\int_0^t \|\nabla \mathrm{H}\|_{L^2}^2\,\mathrm{d}s \leq C,
\end{aligned}
$$

due to (2.4), (2.6), and (3.3).

Similarly, for $B_1(t)$ we have

$$
\begin{aligned}
\int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}t}\big[B_1(s)\big]\right|\mathrm{d}s &\leq C\int_0^t \|\nabla \mathrm{H}\|_{L^2}^2\,\mathrm{d}s + \int_0^t\int |\mathrm{H}|^2|\nabla \mathrm{u}|\,\mathrm{d}x\,\mathrm{d}s \\
&\leq C + C\int_0^t\int |\mathrm{H}|^2|\nabla \mathrm{u}|\,\mathrm{d}x\,\mathrm{d}s \\
&\leq C.
\end{aligned}
\tag{4.6}
$$

From (4.1), (4.2), (4.5), (4.6), we prove that $A_1(t), B_1(t) \in W^{1,1}(0,\infty)$. Therefore, we conclude that

$$
\lim_{t\to\infty}\big(\|\mathrm{u}\|_{L^2} + \|\mathrm{H}\|_{L^2}\big) = 0,
$$

due to $\rho > \underline{\rho}$.

Step 2: Decay of $\|\nabla \mathrm{u}\|_{L^2}$ and $\|\nabla \mathrm{H}\|_{L^2}$.

Set

$$
A_2(t) = \int |\nabla \mathrm{u}|^2\,\mathrm{d}x, \qquad B_2(t) = \int |\nabla \mathrm{H}|^2\,\mathrm{d}x.
$$

By (3.2), we see that A_2 and B_2 are in $L^1(0,\infty)$. Next, it remains to prove that $\frac{\mathrm{d}}{\mathrm{d}t}A_2$ and $\frac{\mathrm{d}}{\mathrm{d}t}B_2$ are in $L^1(0,\infty)$. Using the Young inequality, (3.2) and (3.9), we have

$$
\begin{aligned}
\int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}t}A_2(s)\right|\mathrm{d}s &\leq C\int_0^t \|\nabla \mathrm{u}\|_{L^2}\|\nabla \mathrm{u}_t\|_{L^2}\,\mathrm{d}s \\
&\leq C\int_0^t \|\nabla \mathrm{u}\|_{L^2}^2 + \|\nabla \mathrm{u}_t\|_{L^2}^2\,\mathrm{d}s \leq C, \qquad (4.7) \\
\int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}t}B_2(s)\right|\mathrm{d}s &\leq C\int_0^t \|\nabla \mathrm{H}\|_{L^2}\|\nabla \mathrm{H}_t\|_{L^2}\,\mathrm{d} \\
&\leq C\int_0^t \|\nabla \mathrm{H}\|_{L^2}^2 + \|\nabla \mathrm{H}_t\|_{L^2}^2\,\mathrm{d}s \leq C. \qquad (4.8)
\end{aligned}
$$

Therefore, $A_2(t), B_2(t) \in W^{1,1}(0,\infty)$. We conclude that

$$
\lim_{t\to\infty}\big(\|\nabla \mathrm{u}\|_{L^2} + \|\nabla \mathrm{H}\|_{L^2}\big) = 0.
$$

Step 3: Decay of $\|\mathrm{u}_t\|_{L^2}$ and $\|\mathrm{H}_t\|_{L^2}$.

Set

$$
A_3(t) = \int \big|\rho^{1/2}\mathrm{u}_t\big|^2\,\mathrm{d}x, \qquad B_3(t) = \int |\mathrm{H}_t|^2\,\mathrm{d}x.
$$

From Lemma 3.1, we see that $A_3(t)$ and $B_3(t)$ are in $L^1(0,\infty)$. By the Poincaré inequality and (3.10)-(3.11), we have

$$\begin{aligned}\int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s}A_3(s)\right| \mathrm{d}s &\le \int_0^t \|\nabla \mathrm{u}_t\|_{L^2}^2 \,\mathrm{d}s + \int_0^t |R_1| + |R_2| + |R_3| + |R_4| + |R_5| \,\mathrm{d}s \\ &\quad + \int_0^t \int |\mathrm{H}_t||\nabla \mathrm{H}||\mathrm{u}_t| + |\mathrm{H}||\nabla \mathrm{H}_t||\mathrm{u}_t| \,\mathrm{d}x\,\mathrm{d}s \\ &\le C + C\int_0^t \|\mathrm{H}_t\|_{L^4}\|\mathrm{u}_t\|_{L^4}\|\nabla \mathrm{H}\|_{L^2} + \|\nabla \mathrm{H}_t\|_{L^2}\|\mathrm{H}\|_{L^4}\|\mathrm{u}_t\|_{L^4} \,\mathrm{d}s \\ &\le C \end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s}B_3(s)\right| \mathrm{d}s &\le C + \int_0^t |R_6| + |R_7| \,\mathrm{d}s + \int_0^t \int \mathrm{H}\cdot \nabla u_t \cdot \mathrm{H}_t \,\mathrm{d}x\,\mathrm{d}s \\ &\le C + \int_0^t \|\nabla \mathrm{u}_t\|_{L^2}\|\mathrm{H}\|_{L^4}\|\mathrm{H}_t\|_{L^4} \,\mathrm{d}s \\ &\le C. \end{aligned} \tag{4.10}$$

Therefore, $A_3(t), B_3(t) \in W^{1,1}(0,\infty)$, which, together with $\rho > \underline{\rho}$, leads to

$$\lim_{t\to\infty} \left(\|\mathrm{u}_t\|_{L^2} + \|\mathrm{H}_t\|_{L^2}\right) = 0.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 Indeed, for the case $f = 0$, all the estimates in Section 3 are still valid. Thus, the existence and uniqueness follow the same method. Here, we address the decay rate. First, the continuity equation (1.1) and the fact $\operatorname{div} u = 0$ lead to, for all $p \in [0,\infty]$,

$$\|\rho\|_{L^p} = \|\rho_0\|_{L^p}. \tag{4.11}$$

Particularly, one has

$$\underline{\rho} \le \|\rho\|_{L^\infty} = \|\rho_0\|_{L^\infty} \le \bar{\rho}. \tag{4.12}$$

Multiplying (1.2) and (1.3) by u and H, respectively, then adding the two resulting equations, one has after integrating by parts over Ω

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathrm{u}|^2 + |\mathrm{H}|^2 \,\mathrm{d}x + 2\left(\|\nabla \mathrm{u}\|_{L^2}^2 + \|\nabla \mathrm{H}\|_{L^2}^2\right) \le 0. \tag{4.13}$$

Because of the boundary condition $\mathrm{u}|_{\partial\Omega} = 0$ and $\mathrm{H}\cdot n|_{\partial\Omega} = 0$, using the Poincaré inequality, (2.4), and (2.6), we have

$$\left(\|\mathrm{u}\|_{L^2}^2 + \|\mathrm{H}\|_{L^2}^2\right) \le C\left(\|\nabla \mathrm{u}\|_{L^2}^2 + \|\nabla \mathrm{H}\|_{L^2}^2\right). \tag{4.14}$$

From (4.12), (4.13), and (4.14), it is easy to prove that there exists a $\theta_0 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathrm{u}|^2 + |\mathrm{H}|^2\,\mathrm{d}x + \theta_0\bigl(\bigl\|\rho^{1/2}\mathrm{u}\bigr\|_{L^2}^2 + \|\mathrm{H}\|_{L^2}^2\bigr) \le 0,$$

which immediately implies

$$\bigl\|\rho^{1/2}\mathrm{u}\bigr\|_{L^2}^2 + \|\mathrm{H}\|_{L^2}^2 \le \exp\{-\theta_0 t\}. \tag{4.15}$$

Notice that ρ is bounded from below, and the decay rate estimate (1.11) is established. Thus, Theorem 1.2 is completed. □

Competing interests
The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgements
The author would like to thank the referees for their comments. This work was supported by the Foundation of Education Department of Liaoning Province (No. L2013006) and Doctor Startup Fund of Liaoning Province (No. 20131040).

Received: 25 May 2015 Accepted: 4 November 2015 Published online: 17 November 2015

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