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Existence and global behavior of positive solutions for some eigenvalue problems

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Abstract

In this paper, we study the existence and nonexistence of positive bounded solutions of the integral equation $u = \lambda V(af(u))$, where λ is a positive parameter, a is a nontrivial nonnegative measurable function with bounded potential and V belongs to a class of positive kernels that contains in particular the potential kernel of the classical Laplacian $V = (-\Delta)^{-1}$ or $V = (\frac{\partial}{\partial t} - \Delta)^{-1}$ or the inverse of the polyharmonic Laplacian $(-\Delta)^m$, $m \geq 2$.

1 Introduction

Let Ω be a smooth domain of $\mathbb{R}^n$ ($n \geq 1$) and $f : [0,\infty) \to (0,\infty)$ be a nondecreasing continuous function. In this paper, we are interested in the existence of a positive solution of the following integral equation:

$$u = \lambda V\bigl(af(u)\bigr), \tag{1.1}$$

where λ is a positive parameter, a is a nontrivial nonnegative measurable function satisfying an appropriate condition and V belongs to a class of positive kernels that contains in particular the potential kernel of the classical Laplacian $V = (-\Delta)^{-1}$ or $V = (\frac{\partial}{\partial t} - \Delta)^{-1}$ or the inverse of the polyharmonic Laplacian $(-\Delta)^m$, $m \geq 1$.

We note that if V is the inverse of a differential operator $-L$ (*i.e.* $V = (-L)^{-1}$) and u is a solution of (1.1), then u satisfies

$$-Lu = \lambda a(x)f(u) \quad \text{in } \Omega \tag{1.2}$$

under some appropriate Dirichlet conditions on the boundary of Ω.

For $\lambda = 1$, (1.2) has been extensively studied either for nonincreasing positive nonlinearities (see [1–5]) or for sublinear positive nonlinearities (see [3, 6–9]). In fact existence, uniqueness and global behavior of the solution are discussed in the above works.

Here we are interested in the general equation (1.1). In fact, we will show that there exists $\lambda^\star > 0$ such that (1.1) has a positive solution for $0 < \lambda < \lambda^\star$ under the following hypotheses on the functions a and f:

(H_1) $f : [0,\infty) \to (0,\infty)$ is a nondecreasing continuous function.

(H_2) a is a nontrivial nonnegative measurable function on Ω such that the potential function Va is bounded in Ω.

More precisely, let $\|Va\|_\infty = \sup_{x\in\Omega} Va(x)$; $g(t) = \frac{t}{f(t)}$ for $t > 0$ and $\beta = \sup_{t>0} g(t) \in (0, \infty]$. So, we prove the following result.

Theorem 1.1 *Assume* (H_1)-(H_2). *Then we have*:
- (i) *If* $g(t) < \beta$ *for each* $t > 0$, *then* (1.1) *has a positive solution* u *for each* $0 < \lambda < \frac{\beta}{\|Va\|_\infty}$.
- (ii) *If there exists* $t_0 > 0$ *such that* $g(t_0) = \beta$, *then equation* (1.1) *has a positive solution* u *for each* $0 < \lambda \le \frac{\beta}{\|Va\|_\infty}$.

Moreover, in the two cases there exists $C > 0$ *such that for each* $x \in \Omega$,

$$\frac{1}{C} Va(x) \le u(x) \le C Va(x).$$

Remark 1.2 (see [4] and [10]) Let Ω be a smooth bounded domain and let $\delta(x)$ denote the Euclidean distance from $x \in \Omega$ to the boundary $\partial\Omega$. Then for each $\lambda < 2$, the function $a(x) = \frac{1}{(\delta(x))^\lambda}$ satisfies (H_2).

Next, we give a nonexistence result under some restrictions on the function a. So we assume that the kernel V is self-adjoint, namely for each nonnegative measurable functions h, k in Ω we have $\int_\Omega Vh(x)k(x)\,dx = \int_\Omega h(x)Vk(x)\,dx$. The functions f and a are required to satisfy the following hypotheses:

- (H_3) The function $g(t) = \frac{t}{f(t)}$ is nonnegative and bounded on $(0, \infty)$.
- (H_4) There exist $\lambda_1 > 0$ and a positive measurable function φ_1 such that $\lambda_1 V(a\varphi_1) = \varphi_1$ and $\int_\Omega a(x)\varphi_1(x)\,dx < \infty$.

Then we have the following nonexistence result.

Theorem 1.3 *Assume* (H_3)-(H_4). *Then for each* $\lambda > \lambda_1\|g\|_\infty$, (1.1) *has no positive bounded solution in* Ω.

Remark 1.4 The problem

$$\begin{cases} u'' = -\lambda(u+1), & 0 < x < 1, \\ u(0) = u(1) = 0 \end{cases}$$

has the positive bounded solution $u(x) = \frac{2}{\cos(\frac{\sqrt{\lambda}}{2})} \sin(\frac{\sqrt{\lambda}}{2}x)\sin(\frac{\sqrt{\lambda}}{2}(1-x))$ for $0 < \lambda < \pi^2 = \lambda_1$. Hence the nonexistence result in Theorem 1.3 is optimal.

Our paper is organized as follows. Section 2 is devoted to the proof of Theorems 1.1 and 1.3. The last section is devoted to the study of some examples and applications.

2 Proof of main results

Proof of Theorem 1.1 Assume that a satisfies (H_2) and let $0 < \lambda < \frac{\beta}{\|Va\|_\infty} = \frac{1}{\|Va\|_\infty} \sup_{t>0} \frac{t}{f(t)}$. Consider $0 < c_1 \le \lambda f(0)$ and $0 < c_2$ such that $0 < \lambda \le \frac{c_2}{f(c_2\|Va\|_\infty)}$. Then, from hypothesis (H_1), we deduce that $0 < c_1 \le c_2$. Consider the nonempty closed convex set

$$\Lambda = \{u \in B^+(\Omega) \text{ such that } c_1 Va \le u \le c_2 Va\},$$

where $B^+(\Omega)$ denotes the set of nonnegative Borel measurable functions in Ω. Let T_λ be the operator defined on Λ by

$$T_\lambda u = \lambda V\big(af(u)\big).$$

Since f is nondecreasing on $[0,\infty)$, T_λ is nondecreasing on Λ. Moreover, for each $u \in \Lambda$, we have

$$c_1 Va \le \lambda V\big(f(0)a\big) \le \lambda V\big(af(u)\big) \le \lambda f\big(c_2\|Va\|_\infty\big)Va \le c_2 Va.$$

So $T_\lambda\Lambda \subset \Lambda$. Consider the sequence $(u_k)_{k\ge0}$ defined by $u_0 = c_1 Va$ and $u_{k+1} = T_\lambda u_k$ for $k \ge 0$. Since $T_\lambda\Lambda \subset \Lambda$ and T_λ is nondecreasing, we find by induction that the sequence $(u_k)_{k\ge0}$ is nondecreasing and satisfies for each $k \ge 0$

$$c_1 Va \le u_k \le c_2 Va.$$

Hence, it follows from the monotone convergence theorem that the sequence $(u_k)_{k\ge0}$ converges to a function $u \in \Lambda$ satisfying the integral equation

$$u = \lambda V\big(af(u)\big). \tag{2.1}$$

This proves assertion (i) of the theorem.

Now, assume that there exists $t_0 > 0$ such that $g(t_0) = \beta$. Then for $\lambda = \frac{\beta}{\|Va\|_\infty}$, we take $c_2 = \frac{t_0}{\|Va\|_\infty}$ and $0 < c_1 < \lambda f(0) = \frac{t_0 f(0)}{\|Va\|_\infty f(t_0)}$, to adapt the previous proof. □

Proof of Theorem 1.3 Assume that (1.1) has a positive bounded solution u in Ω. Then we have

$$\begin{aligned}
\lambda_1 \int_\Omega \varphi_1(x)u(x)a(x)\,dx &= \lambda_1\lambda \int_\Omega V\big(af(u)\big)(x)\varphi_1(x)a(x)\,dx \\
&= \lambda\lambda_1 \int_\Omega a(x)f\big(u(x)\big)V(a\varphi_1)(x)\,dx \\
&= \lambda \int_\Omega a(x)f\big(u(x)\big)\varphi_1(x)\,dx \\
&\ge \lambda\left(\inf_{t>0}\frac{f(t)}{t}\right)\int_\Omega a(x)u(x)\varphi_1(x)\,dx.
\end{aligned}$$

This shows that $\lambda_1\|g\|_\infty \ge \lambda$. □

Example 2.1 As a consequence of Theorem 1.3, we deduce that the problem

$$\begin{cases} \Delta u = -\lambda e^u, & \text{in } B(0,1) \subset \mathbb{R}^3, \\ u_{/\partial B} = 0 \end{cases}$$

has no positive bounded solution for $\lambda > \frac{\pi^2}{e}$. Indeed, in this case we have $\lambda_1 = \pi^2$ and $\varphi_1(x) = c_0\frac{\sin(\pi|x|)}{|x|}$, $c_0 > 0$.

3 Examples and applications

3.1 Examples

Next, we give some examples where (H_1) and (H_2) are satisfied.

Example 3.1 Let $f(t) = t^p + 1$ for $p \geq 0$ and $t \geq 0$. Then we discuss three cases:

Case 1: $p \in [0,1)$. In this case $\beta = +\infty$ and the existence result for (1.1) holds for $\lambda \in (0,\infty)$.

Case 2: $p = 1$. Then $\beta = 1$ and the existence result for (1.1) holds for $\lambda \in (0, \frac{1}{\|Va\|_\infty})$.

Case 3: $p > 1$. In this case $g(t) = \frac{t}{1+t^p}$ and $\beta = g(\frac{1}{(p-1)^{\frac{1}{p}}}) = \frac{(p-1)^{1-\frac{1}{p}}}{p}$. So the existence result for (1.1) holds for $\lambda \in (0, \frac{(p-1)^{1-\frac{1}{p}}}{p\|Va\|_\infty}]$.

Example 3.2 Let $f(t) = e^t$. Then $\beta = \sup_{t>0} g(t) = g(1) = \frac{1}{e}$. In this case, the existence result for (1.1) holds for $\lambda \in (0, \frac{1}{e\|Va\|_\infty}]$.

3.2 Applications

Throughout the following first three applications, we assume that the function $f : [0,\infty) \to (0,\infty)$ is continuous and nondecreasing.

Application 3.1 *Let Ω be a smooth domain (bounded or unbounded) with compact boundary or $\Omega = \mathbb{R}^n_+ = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ $(n \geq 3)$ be the upper half space and let $G(x,y)$ be the Green function of the Laplacian $(-\Delta)$ on Ω with Dirichlet boundary conditions. Assume that a is a nontrivial nonnegative Borel measurable function such that its Green potential $Va(x) = \int_\Omega G(x,y)a(y)\,dy$ is continuous and satisfies $\lim_{x\to\partial^\infty\Omega} Va(x) = 0$, where $\partial^\infty\Omega = \partial\Omega$ if Ω is bounded and $\partial^\infty\Omega = \partial\Omega \cup \{\infty\}$ whenever Ω is unbounded. Then for each $0 < \lambda < \frac{1}{\|Va\|_\infty} \sup_{t>0} \frac{t}{f(t)}$, the problem*

$$\begin{cases} -\Delta u = \lambda a(x) f(u), & \text{in } \Omega, \\ u_{/\partial^\infty\Omega} = 0 \end{cases}$$

has a positive continuous solution u satisfying for each $x \in \Omega$

$$\frac{1}{C} Va(x) \leq u(x) \leq C Va(x) \quad \text{for some } C > 0.$$

Application 3.2 *Let m be a positive integer, $\Omega = B(0,1)$ be the unit ball in $\mathbb{R}^n$ $(n \geq 2)$ and $G_{m,n}(x,y)$ be the Green function of the polyharmonic Laplacian $(-\Delta)^m$ on $B(0,1)$ with Dirichlet boundary conditions (see* [11]*). Let a be a nontrivial nonnegative function on $B(0,1)$ such that its Green potential $Va(x) = \int_{B(0,1)} G_{m,n}(x,y)a(y)\,dy$ is continuous on $B(0,1)$ and $\lim_{|x|\to 1} \frac{Va(x)}{(1-|x|)^{m-1}} = 0$. Then, for each $0 < \lambda < \frac{1}{\|Va\|_\infty} \sup_{t>0} \frac{t}{f(t)}$, the problem*

$$\begin{cases} (-\Delta)^m u = \lambda a(x) f(u), & \text{in } \Omega, \\ \lim_{|x|\to 1} \frac{u(x)}{(1-|x|)^{m-1}} = 0 \end{cases}$$

has a positive continuous solution u satisfying for each $x \in B(0,1)$

$$\frac{1}{C} Va(x) \leq u(x) \leq C Va(x), \quad \text{for some } C > 0.$$

Application 3.3 *For $t > s$ and $x, y \in \mathbb{R}^n_+$, we denote by*

$$\Gamma(x,t,y,s) = \left(4\pi(t-s)\right)^{-\frac{n}{2}}\left[1-\exp\left(-\frac{x_n y_n}{(t-s)}\right)\right]\exp\left(-\frac{|x-y|^2}{4(t-s)}\right)$$

the Green function of the heat operator $\frac{\partial}{\partial t} - \Delta$ on $\mathbb{R}^n_+ \times (0,\infty)$ with Dirichlet boundary conditions. Put $P_t 1(x) = P1(x,t) = \int_{\mathbb{R}^n_+} \Gamma(x,t,y,0)\,dy = \frac{1}{\sqrt{\pi t}}\int_0^{x_n} \exp(-\frac{\xi^2}{4t})\,d\xi$. Consider a nontrivial nonnegative measurable function a on $\mathbb{R}^n_+ \times (0,\infty)$ such that the function $(x,t) \to \frac{a(x,t)}{P_t 1(x)}$ belongs to the parabolic Kato class $P^\infty(\mathbb{R}^n_+)$ introduced and studied in [12]. *Then from* [12] *the function $(x,t) \to Va(x,t) = \int_{\mathbb{R}^n_+\times(0,\infty)} \Gamma(x,t,y,s)a(y,s)\,dy\,ds$ is continuous and bounded in $\mathbb{R}^n_+ \times (0,\infty)$ and for each $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$ we have*

$$\lim_{\xi\to\partial\mathbb{R}^n_+} Va(\xi,t) = \lim_{r\to 0^+} Va(x,r) = 0.$$

Consequently, for $0 < \lambda < \frac{1}{\|Va\|_\infty}\sup_{t>0}\frac{t}{f(t)}$, the problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda a(x,t) f(u), & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ u(x,0) = 0 \quad \text{for } x \in \mathbb{R}^n_+, \\ u_{/\mathbb{R}^n_+\times(0,\infty)} = 0 \end{cases}$$

has a positive continuous solution u satisfying for each $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$

$$\frac{1}{C}Va(x,t) \le u(x,t) \le CVa(x,t) \quad \text{for some } C > 0.$$

Application 3.4 *Let Ω be a smooth bounded of $\mathbb{R}^n$ ($n \ge 2$), $f : [0,\infty) \to (0,\infty)$ be a continuous function such that $g(t) = \frac{t}{f(t)}$ is bounded and let $a(x) = \frac{1}{(\delta(x))^\lambda}$ with $\lambda < 2$. Let $\lambda_1 > 0$ be the first positive eigenvalue of the problem $-\Delta u = \lambda a(x)u$ in Ω with Dirichlet boundary conditions. From* [13], *λ_1 is a simple eigenvalue. Let φ_1 be the positive normalized ($\|\varphi_1\|_\infty = 1$) eigenfunction associated with λ_1. Then φ_1 satisfies the equation $\lambda V(a\varphi_1) = \varphi_1$. Moreover, it is well known that $\varphi_1(x) \approx \delta(x)$. Namely, there exists $C > 0$ such that $\frac{1}{C}\delta(x) \le \varphi_1(x) \le C\delta(x)$, for each $x \in \Omega$. So $\int_\Omega a(x)\varphi_1(x) \le C\int_\omega \frac{1}{(\delta(x))^{\lambda-1}}\,dx < \infty$. Hence from Theorem* 1.3, *the problem*

$$\begin{cases} -\Delta u = \lambda a(x) f(u), & \text{in } \Omega, \\ u_{/\partial\Omega} = 0 \end{cases}$$

has no positive bounded solution in Ω for each $\lambda > \lambda_1\|g\|_\infty$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Alsaedi, R, Mâagli, H, Zeddini, N: Exact behavior of the unique positive solution to some singular elliptic problem in exterior domains. Nonlinear Anal. **119**, 186-198 (2015)
2. Cheng, J, Zhang, Z: On the existence of positive solutions for a class of singular boundary value problems. Nonlinear Anal. TMA **44**, 645-655 (2001)
3. Lair, AV, Shaker, AW: Entire solution of a singular semilinear elliptic problem. J. Math. Anal. Appl. **200**, 498-505 (1996)
4. Mâagli, H, Zribi, M: On a new Kato class and singular solutions of a nonlinear elliptic equation in bounded domains of $\mathbb{R}^n$. Positivity **9**, 667-686 (2005)
5. Zeddini, N, Alsaedi, R, Mâagli, H: Exact boundary behavior of the unique positive solution to some singular elliptic problems. Nonlinear Anal. **89**, 146-156 (2013)
6. Alsaedi, R, Mâagli, H, Zeddini, N: Estimates on potential functions and boundary behavior of positive solutions for sublinear Dirichlet problems. Electron. J. Differ. Equ. **2014**, 08 (2014)
7. Brezis, H, Kamin, S: Sublinear elliptic equations in $\mathbb{R}^n$. Manuscr. Math. **74**, 87-106 (1992)
8. Bachar, I, Zeddini, N: On the existence of positive solutions for a class of semilinear elliptic equations. Nonlinear Anal. **52**, 1239-1247 (2003)
9. Ghergu, M, Rădulescu, VD: Sublinear singular elliptic problems with two parameters. J. Differ. Equ. **195**, 520-536 (2003)
10. Zeddini, N: Positive solutions for a singular nonlinear problem on a bounded domain in $\mathbb{R}^2$. Potential Anal. **18**, 97-118 (2003)
11. Bachar, I, Mâagli, H, Masmoudi, S, Zribi, M: Estimates for the Green function and singular solutions for polyharmonic nonlinear equations. Abstr. Appl. Anal. **12**, 715-741 (2003)
12. Mâagli, H, Masmoudi, S, Zribi, M: On a parabolic problem with nonlinear term in a half space and global behavior of solutions. J. Differ. Equ. **246**, 3417-3447 (2009)
13. Hansen, W: Valeurs propres pour l'opérateur de Schrödinger. In: Séminaire de Théorie du Potentiel, No. 9, Paris. Lecture Notes in Mathematics, vol. 1393, pp. 117-134 (1989)