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Existence of periodic solutions for a class of second order Hamiltonian systems

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Abstract

By using the least action principle and the minimax methods, the existence of periodic solutions for a class of second order Hamiltonian systems is considered. The results obtained in this paper extend some previous results.

Keywords: periodic solutions; second order Hamiltonian systems; least action principle

1 Introduction and main results

Consider the second order Hamiltonian system

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)), \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0, \end{cases} \quad (1.1)$$

where $T > 0$ and $F : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$, continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1([0, T]; \mathbb{R}^+)$ such that

$$|F(t, x)| \leq a(|x|)b(t), \quad |\nabla F(t, x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

The corresponding functional $\varphi : H_T^1 \rightarrow \mathbb{R}$,

$$\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, u(t)) dt,$$

is continuously differentiable and weakly lower semi-continuous on H_T^1 , where H_T^1 is the usual Sobolev space with the norm

$$\|u\| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt \right)^{1/2}$$

for $u \in H_T^1$, and

$$\langle \varphi'(u), v \rangle = \int_0^T [(\dot{u}(t), \dot{v}) + (\nabla F(t, u(t)), v(t))] dt$$

for all $u, v \in H^1_T$. It is well known that the solutions of problem (1.1) correspond to the critical points of φ .

The existence of periodic solutions for problem (1.1) is obtained in [1–22] with many solvability conditions by using the least action principle and the minimax methods, such as the coercive type potential condition (see [2]), the convex type potential condition (see [5]), the periodic type potential conditions (see [16]), the even type potential condition (see [4]), the subquadratic potential condition in Rabinowitz’s sense (see [9]), the bounded nonlinearity condition (see [6]), the subadditive condition (see [11]), the sublinear nonlinearity condition (see [3, 13]), and the linear nonlinearity condition (see [7, 15, 19, 20]).

In particular, when the nonlinearity $\nabla F(t, x)$ is bounded, that is, there exists $g(t) \in L^1([0, T], \mathbb{R}^+)$ such that $|\nabla F(t, x)| \leq g(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, and that

$$\int_0^T F(t, x) dt \rightarrow \pm\infty \quad \text{as } |x| \rightarrow \infty,$$

Mawhin and Willem [6] proved that problem (1.1) has at least one periodic solution.

In [3, 13], Han and Tang generalized these results to the sublinear case:

$$|\nabla F(t, x)| \leq f(t)|x|^\alpha + g(t) \quad \text{for all } x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T] \tag{1.2}$$

and

$$|x|^{-2\alpha} \int_0^T F(t, x) dt \rightarrow \pm\infty \quad \text{as } |x| \rightarrow \infty, \tag{1.3}$$

where $f(t), g(t) \in L^1([0, T], \mathbb{R}^+)$ and $\alpha \in [0, 1)$.

Subsequently, when $\alpha = 1$ Zhao and Wu [19, 20] and Meng and Tang [7, 15] also proved the existence of periodic solutions for problem (1.1), *i.e.* $\nabla F(t, x)$ was linear:

$$|\nabla F(t, x)| \leq f(t)|x| + g(t) \quad \text{for all } x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T],$$

where $f(t), g(t) \in L^1([0, T], \mathbb{R}^+)$.

Recently, Wang and Zhang [21] used a control function $h(|x|)$ instead of $|x|^\alpha$ in (1.2) and (1.3) and got some new results, where h satisfied the following conditions:

(B) $h \in C([0, \infty), [0, \infty))$ and there exist constants $C_0 > 0, K_1 > 0, K_2 > 0, \alpha \in [0, 1)$ such that

- (i) $h(s) \leq h(t) \quad \forall s \leq t, s, t \in [0, \infty)$,
- (ii) $h(s + t) \leq C_0(h(s) + h(t)) \quad \forall s, t \in [0, \infty)$,
- (iii) $0 \leq h(s) \leq K_1 s^\alpha + K_2 \quad \forall s \in [0, \infty)$,
- (iv) $h(s) \rightarrow \infty$ as $s \rightarrow \infty$.

Motivated by the results mentioned above, we will consider the periodic solutions for problem (1.1). The following are our main results.

Theorem 1.1 *Suppose that $F(t, x) = F_1(t, x) + F_2(x)$, where F_1 and F_2 satisfy assumption (A) and the following conditions:*

- (1) *there exist $f, g \in L^1([0, T]; \mathbb{R}^+)$ such that*

$$|\nabla F_1(t, x)| \leq f(t)h(|x|) + g(t),$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, here h satisfies (B);

(2) there exist constants $r > 0$ and $\gamma \in [0, 2)$ such that

$$(\nabla F_2(x) - \nabla F_2(y), x - y) \geq -r|x - y|^\gamma,$$

for all $x, y \in \mathbb{R}^N$;

(3)

$$\liminf_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt > \frac{T^2 C_0^2}{8\pi^2} \int_0^T f^2(t) dt.$$

Then problem (1.1) has at least one periodic solution which minimizes φ on H_T^1 .

Theorem 1.2 Suppose that $F(t, x) = F_1(t, x) + F_2(x)$, where F_1 and F_2 satisfy assumption (A), (1), (2), and the following conditions:

(4) there exist $\delta \in [0, 2)$ and $\mu > 0$ such that

$$(\nabla F_2(x) - \nabla F_2(y), x - y) \leq \mu|x - y|^\delta,$$

for all $x, y \in \mathbb{R}^N$;

(5)

$$\limsup_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt < -\frac{3T^2 C_0^2}{8\pi^2} \int_0^T f^2(t) dt.$$

Then problem (1.1) has at least one periodic solution which minimizes φ on H_T^1 .

Theorem 1.3 Suppose that $F(t, x) = F_1(t, x) + F_2(x)$, where F_1 and F_2 satisfy assumption (A), (1), and the following conditions:

(6) there exists a constant $0 < r < 4\pi^2/T^2$, such that

$$(\nabla F_2(x) - \nabla F_2(y), x - y) \geq -r|x - y|^2,$$

for all $x, y \in \mathbb{R}^N$;

(7)

$$\liminf_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt > \frac{T^2}{2(4\pi^2 - rT^2)} \int_0^T f^2(t) dt.$$

Then problem (1.1) has at least one periodic solution which minimizes φ on H_T^1 .

Theorem 1.4 Suppose that $F = F_1 + F_2$, where F_1 and F_2 satisfy assumption (A), (1), and the following conditions:

(8) there exist $k \in L^1([0, T]; \mathbb{R}^+)$ and (λ, μ) -subconvex potential $G : \mathbb{R}^N \rightarrow \mathbb{R}$ with $\lambda > 1/2$ and $0 < \mu < 2\lambda^2$, such that

$$(\nabla F_2(t, x), y) \geq -k(t)G(x - y),$$

for all $x, y \in \mathbb{R}^N$;

(9)

$$\limsup_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F_1(t, x) dt < -\frac{3T^2 C_0^2}{8\pi^2} \int_0^T f^2(t) dt,$$

$$\limsup_{|x| \rightarrow \infty} |x|^{-\beta} \int_0^T F_2(t, x) dt \leq -8\mu \max_{|s| \leq 1} G(s) \int_0^T k(t) dt,$$

where $\beta = \log_{2\lambda}(2\mu)$.

Then problem (1.1) has at least one periodic solution which minimizes φ on H_T^1 .

Remark 1.5 Theorems 1.1-1.4 extend some existing results: (i) [22], Theorems 1.1-1.4, are special cases of Theorems 1.1-1.4 with control function $h(t) = t^\alpha, \alpha \in [0, 1), t \in [0, +\infty)$; (ii) if $F_2 = 0$, [15], Theorems 1 and 2, are special cases of Theorem 1.1 and Theorem 1.2, respectively; (iii) If $F_2 = 0$, Theorem 1.1 and Theorem 1.2 extend [21], Theorems 1.1 and 1.2, since we weaken the so-called Ahmad-Lazer-Paul type conditions with the control function $h(t)$.

2 Proof of theorems

For $u \in H_T^1$, let $\bar{u} = \frac{1}{T} \int_0^T |\dot{u}(t)| dt$ and $\tilde{u}(t) = u(t) - \bar{u}$. Then one has

$$\|\tilde{u}\|_\infty^2 \leq \frac{T}{12} \int_0^T |\dot{u}(t)|^2 dt \quad (\text{Sobolev's inequality}),$$

$$\|\tilde{u}\|_{L^2}^2 \leq \frac{T^2}{4\pi^2} \int_0^T |\dot{u}(t)|^2 dt \quad (\text{Wirtinger's inequality}).$$

For the sake of convenience, we denote $M_1 = (\int_0^T f^2(t) dt)^{1/2}$, $M_2 = \int_0^T f(t) dt$, $M_3 = \int_0^T g(t) dt$.

Proof of Theorem 1.1 Due to (3), we can choose an $a_1 > T^2/(4\pi^2)$ such that

$$\liminf_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt > \frac{a_1 C_0^2}{2} M_1^2. \tag{2.1}$$

For (B) and the Sobolev inequality, for any $u \in H_T^1$ we have

$$\begin{aligned} & \left| \int_0^T [F_1(t, u(t)) - F_1(t, \bar{u})] dt \right| \\ &= \left| \int_0^T \int_0^1 (\nabla F_1(t, \bar{u} + s\tilde{u}(t)), \tilde{u}(t)) ds dt \right| \\ &\leq \int_0^T \int_0^1 f(t)h(|\bar{u} + s\tilde{u}(t)|) |\tilde{u}(t)| ds dt + \int_0^T \int_0^1 g(t) |\tilde{u}(t)| ds dt \\ &\leq \int_0^T \int_0^1 C_0 f(t)(h(|\bar{u}|) + h(|\tilde{u}(t)|)) |\tilde{u}(t)| ds dt + M_3 \|\tilde{u}\|_\infty \\ &\leq C_0 h(|\bar{u}|) \left(\int_0^T f^2(t) dt \right)^{1/2} \left(\int_0^T |\tilde{u}(t)|^2 dt \right)^{1/2} \\ &\quad + C_0 \int_0^T f(t)h(|\tilde{u}(t)|) |\tilde{u}(t)| dt + M_3 \|\tilde{u}\|_\infty \end{aligned}$$

$$\begin{aligned}
 &\leq C_0M_1h(|\bar{u}|)\|\tilde{u}\|_{L^2} + C_0\int_0^T f(t)(K_1|\tilde{u}(t)|^\alpha + K_2)|\tilde{u}(t)| dt + M_3\|\tilde{u}\|_\infty \\
 &\leq C_0M_1h(|\bar{u}|)\|\tilde{u}\|_{L^2} + C_0M_2K_1\|\tilde{u}\|_\infty^{1+\alpha} + C_0M_2K_2\|\tilde{u}(t)\|_\infty + M_3\|\tilde{u}(t)\|_\infty \\
 &\leq \frac{1}{2a_1}\|\tilde{u}\|_{L^2}^2 + \frac{a_1(C_0M_1)^2}{2}h^2(|\bar{u}|) + C_0M_2K_1\|\tilde{u}\|_\infty^{1+\alpha} \\
 &\quad + C_0M_2K_2\|\tilde{u}(t)\|_\infty + M_3\|\tilde{u}(t)\|_\infty \\
 &\leq \frac{T^2}{8\pi^2a_1}\|\dot{u}\|_{L^2}^2 + \frac{a_1(C_0M_1)^2}{2}h^2(|\bar{u}|) + \left(\frac{T}{12}\right)^{(1+\alpha)/2}C_0M_2K_1\|\dot{u}\|_{L^2}^{1+\alpha} \\
 &\quad + \left(\frac{T}{12}\right)^{1/2}C_0M_2K_2\|\dot{u}\|_{L^2} + \left(\frac{T}{12}\right)^{1/2}M_3\|\dot{u}\|_{L^2}.
 \end{aligned} \tag{2.2}$$

Similarly, from (2) and the Sobolev inequality, for any $u \in H_T^1$ we get

$$\begin{aligned}
 &\int_0^T [F_2(u(t)) - F_2(\bar{u})] dt \\
 &= \int_0^T \int_0^1 \frac{1}{s}(\nabla F_2(\bar{u} + s\tilde{u}(t)) - \nabla F_2(\bar{u}), s\tilde{u}(t)) ds dt \\
 &\geq -\int_0^T \int_0^1 rs^{\gamma-1}|\tilde{u}(t)|^\gamma ds dt \\
 &\geq -\frac{rT}{\gamma}\|\tilde{u}\|_\infty^\gamma \\
 &\geq -\frac{rT}{\gamma}\left(\frac{T}{12}\right)^{\gamma/2}\|\dot{u}\|_{L^2}^\gamma.
 \end{aligned} \tag{2.3}$$

From (2.2) and (2.3) we have

$$\begin{aligned}
 \varphi(u) &= \frac{1}{2}\|\dot{u}\|_{L^2}^2 + \int_0^T [F_1(t, u(t)) - F_1(t, \bar{u})] dt \\
 &\quad + \int_0^T [F_2(u(t)) - F_2(\bar{u})] dt + \int_0^T F(t, \bar{u}) dt \\
 &\geq \left(\frac{1}{2} - \frac{T^2}{8\pi^2a_1}\right)\|\dot{u}\|_{L^2}^2 - \frac{a_1(C_0M_1)^2}{2}h^2(|\bar{u}|) - \left(\frac{T}{12}\right)^{\frac{1+\alpha}{2}}C_0M_2K_1\|\dot{u}\|_{L^2}^{1+\alpha} \\
 &\quad - \left(\frac{T}{12}\right)^{1/2}C_0M_2K_2\|\dot{u}\|_{L^2} - \left(\frac{T}{12}\right)^{1/2}M_3\|\dot{u}\|_{L^2} \\
 &\quad - \frac{rT}{\gamma}\left(\frac{T}{12}\right)^{\alpha/2}\|\dot{u}\|_{L^2}^\gamma + \int_0^T F(t, \bar{u}) dt \\
 &\geq \left(\frac{1}{2} - \frac{T^2}{8\pi^2a_1}\right)\|\dot{u}\|_{L^2}^2 + h^2(|\bar{u}|)\left(h^{-2}(|\bar{u}|)\int_0^T F(t, \bar{u}) dt - \frac{a_1(C_0M_1)^2}{2}\right) \\
 &\quad - \left(\frac{T}{12}\right)^{1/2}(C_0M_2K_2 + M_3)\|\dot{u}\|_{L^2} \\
 &\quad - \left(\frac{T}{12}\right)^{\frac{1+\alpha}{2}}C_0M_2K_1\|\dot{u}\|_{L^2}^{1+\alpha} - \frac{rT}{\gamma}\left(\frac{T}{12}\right)^{\gamma/2}\|\dot{u}\|_{L^2}^\gamma,
 \end{aligned}$$

for all $u \in H_T^1$. So, by (2.1) we get $\varphi(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$.

Hence, applying the least action principle (see [6], Theorem 1.1 and Corollary 1.1), the proof is complete. \square

Proof of Theorem 1.2 Step 1. First, we assert that φ satisfies the (PS) condition. Suppose that $\{u_n\}$ is a (PS) sequence, that is, $\varphi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ and $\{\varphi(u_n)\}$ is bounded. For (5), we can choose an $a_2 > T^2/(4\pi^2)$ such that

$$\limsup_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt < -\left(\frac{a_2}{2} + \frac{\sqrt{a_2}T}{2\pi}\right) C_0^2 \int_0^T f^2(t) dt. \tag{2.4}$$

Similar to the proof of Theorem 1.1, we have

$$\begin{aligned} & \left| \int_0^T (\nabla F_1(t, u_n(t)), \tilde{u}_n(t)) dt \right| \\ & \leq \frac{T^2}{8\pi^2 a_2} \|\dot{u}_n\|_{L^2}^2 + \frac{a_2(C_0 M_1)^2}{2} h^2(|\tilde{u}_n|) + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 \|\dot{u}_n\|_{L^2}^{1+\alpha} \\ & \quad + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) \|\dot{u}_n\|_{L^2} \end{aligned} \tag{2.5}$$

and

$$\int_0^T (\nabla F_2(u_n(t)), \tilde{u}_n(t)) dt \geq -\frac{rT}{\gamma} \left(\frac{T}{12}\right)^{\gamma/2} \|\dot{u}_n\|_{L^2}^\gamma,$$

for all n . Hence we have

$$\begin{aligned} \|\tilde{u}_n\| & \geq \langle \varphi'(u_n), \tilde{u}_n \rangle \\ & = \|\dot{u}_n\|_{L^2}^2 + \int_0^T (\nabla F(t, u_n(t)), \tilde{u}_n(t)) dt \\ & \geq \left(1 - \frac{T^2}{8\pi^2 a_2}\right) \|\dot{u}_n\|_{L^2}^2 - \frac{a_2(C_0 M_1)^2}{2} h^2(|\tilde{u}_n|) \\ & \quad - \left(\frac{T}{12}\right)^{\frac{1+\alpha}{2}} C_0 M_2 K_1 \|\dot{u}_n\|_{L^2}^{1+\alpha} \\ & \quad - \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) \|\dot{u}_n\|_{L^2} - \frac{rT}{\gamma} \left(\frac{T}{12}\right)^{\gamma/2} \|\dot{u}_n\|_{L^2}^\gamma, \end{aligned} \tag{2.6}$$

for large n . So, by Wirtinger's inequality we get

$$\|(\tilde{u}_n)\| \leq \frac{(T^2 + 4\pi^2)^{1/2}}{2\pi} \|\dot{u}_n\|_{L^2}. \tag{2.7}$$

From (2.6) and (2.7),

$$\begin{aligned} & \frac{a_2(C_0 M_1)^2}{2} h^2(|\tilde{u}_n|) \\ & \geq \left(1 - \frac{T^2}{8\pi^2 a_2}\right) \|\dot{u}_n\|_{L^2}^2 - C_0 M_2 K_1 \left(\frac{T}{12}\right)^{\frac{1+\alpha}{2}} \|\dot{u}_n\|_{L^2}^{1+\alpha} \end{aligned}$$

$$\begin{aligned}
 & - \left(\frac{T}{12}\right)^{1/2} (C_0M_2K_2 + M_3)\|\dot{u}_n\|_{L^2} \\
 & - \|\tilde{u}_n\| - \frac{rT}{\gamma} \left(\frac{T}{12}\right)^{\gamma/2} \|\dot{u}_n\|_{L^2}^\gamma \geq \frac{1}{2} \|\dot{u}_n\|_{L^2}^2 + C_1,
 \end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
 C_1 = \min_{s \in [0, +\infty]} & \left\{ \frac{4\pi^2 a_2 - T^2}{8\pi^2 a_2} s^2 - \left(\frac{T}{12}\right)^{\frac{1+\alpha}{2}} C_0M_2K_1s^{1+\alpha} \right. \\
 & \left. - \left[\frac{(T^2 + 4\pi^2)^{1/2}}{2\pi} + C_0M_2K_2 \left(\frac{T}{12}\right)^{1/2} + \left(\frac{T}{12}\right)^{1/2} M_3 \right] s - \frac{rT}{\gamma} \left(\frac{T}{12}\right)^{\gamma/2} s^\gamma \right\}.
 \end{aligned}$$

Note that $a_2 > T^2/(4\pi^2)$ implies $-\infty < C_1 < 0$. Hence, it follows from (2.8) that

$$\|\dot{u}_n\|_{L^2}^2 \leq a_2 C_0^2 M_1^2 h^2(|\bar{u}_n|) - 2C_1, \tag{2.9}$$

and then

$$\|\dot{u}_n\|_{L^2} \leq \sqrt{a_2} C_0 M_1 h(|\bar{u}_n|) + C_2, \tag{2.10}$$

where $0 < C_2 < +\infty$. Similar to the proof of Theorem 1.1, we have

$$\begin{aligned}
 & \left| \int_0^T [F_1(t, u_n(t)) - F_1(t, \tilde{u}_n)] dt \right| \\
 & \leq C_0M_1h(|\bar{u}_n|)\|\tilde{u}_n\|_{L^2} + C_0M_2K_1\|\tilde{u}_n\|_\infty^{1+\alpha} + (C_0M_2K_2 + M_3)\|\tilde{u}_n\|_\infty \\
 & \leq \frac{\pi}{\sqrt{a_2}T}\|\tilde{u}_n\|_{L^2}^2 + \frac{\sqrt{a_2}TC_0^2}{4\pi}M_1^2h^2(|\bar{u}_n|) \\
 & \quad + C_0M_2K_1\|\tilde{u}_n\|_\infty^{1+\alpha} + (C_0M_2K_2 + M_3)\|\tilde{u}_n\|_\infty \\
 & \leq \frac{T}{4\pi\sqrt{a_2}}\|\dot{u}_n\|_{L^2}^2 + \frac{\sqrt{a_2}TC_0^2}{4\pi}M_1^2h^2(|\bar{u}_n|) + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0M_2K_1\|\dot{u}_n\|_{L^2}^{1+\alpha} \\
 & \quad + \left(\frac{T}{12}\right)^{1/2} (C_0M_2K_2 + M_3)\|\dot{u}_n\|_{L^2}.
 \end{aligned} \tag{2.11}$$

By (4), we obtain

$$\begin{aligned}
 & \int_0^T [F_2(u_n(t)) - F_2(\tilde{u}_n)] dt \\
 & = \int_0^T \int_0^1 \frac{1}{s} (\nabla F_2(\tilde{u}_n + s\tilde{u}_n(t)) - \nabla F_2(\tilde{u}_n), s\tilde{u}_n(t)) ds dt \\
 & \leq \int_0^T \int_0^1 \mu s^{\delta-1} |\tilde{u}_n(t)|^\delta ds dt \leq \frac{\mu T}{\delta} \|\tilde{u}_n\|_\infty^\delta \\
 & \leq \frac{\mu T}{\delta} \left(\frac{T}{12}\right)^{\delta/2} \|\dot{u}_n\|_{L^2}^\delta.
 \end{aligned}$$

From the boundedness of $\varphi(u_n)$ and (2.9)-(2.11), we have

$$\begin{aligned}
 C_3 &\leq \varphi(u_n) \\
 &= \frac{1}{2} \|\dot{u}_n\|_{L^2}^2 + \int_0^T [F_1(t, u_n(t)) - F_1(t, \bar{u}_n)] dt + \int_0^T [F_2(u_n(t)) - F_2(\bar{u}_n)] dt \\
 &\quad + \int_0^T F(t, \bar{u}_n(t)) dt \\
 &\leq \left(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_2}}\right) \|\dot{u}\|_{L^2}^2 + \frac{\sqrt{a_2}TC_0^2}{4\pi} M_1^2 h^2 (|\bar{u}|) + C_0 M_2 K_1 \left(\frac{T}{12}\right)^{(1+\alpha)/2} \|\dot{u}_n\|_{L^2}^{1+\alpha} \\
 &\quad + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) \|\dot{u}_n\|_{L^2} + \int_0^T F(t, \bar{u}_n) dt + \frac{\mu T}{\delta} \left(\frac{T}{12}\right)^{\delta/2} \|\dot{u}_n\|_{L^2}^\delta \\
 &\leq \left(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_2}}\right) (a_2 C_0^2 M_1^2 h^2 (|\bar{u}_n|) - 2C_1) + \frac{\sqrt{a_2}TC_0^2}{4\pi} M_1^2 h^2 (|\bar{u}|) \\
 &\quad + C_0 M_2 K_1 \left(\frac{T}{12}\right)^{(1+\alpha)/2} (\sqrt{a_2} C_0 M_1 h (|\bar{u}_n|) + C_2)^{1+\alpha} \\
 &\quad + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) (\sqrt{a_2} C_0 M_1 h (|\bar{u}_n|) + C_2) \\
 &\quad + \int_0^T F(t, \bar{u}_n) dt + \frac{\mu T}{\delta} \left(\frac{T}{12}\right)^{\delta/2} \|\dot{u}_n\|_{L^2}^\delta \\
 &\leq \left(\frac{a_2}{2} + \frac{\sqrt{a_2}T}{2\pi}\right) C_0^2 M_1^2 h^2 (|\bar{u}_n|) - \left(1 + \frac{T}{2\pi\sqrt{a_2}}\right) C_1 \\
 &\quad + \left(\frac{T}{12}\right)^{(1+\alpha)/2} 2^\alpha C_0 M_2 K_1 [(\sqrt{a_2} C_0 M_1)^{1+\alpha} h (|\bar{u}_n|)^{1+\alpha} + C_2^{1+\alpha}] \\
 &\quad + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) (\sqrt{a_2} C_0 M_1 h (|\bar{u}_n|) + C_2) \\
 &\quad + \int_0^T F(t, \bar{u}_n) dt + \frac{\mu T}{\delta} \left(\frac{T}{12}\right)^{\delta/2} 2^{\delta-1} ((\sqrt{a_2} M_1)^\delta h^\delta (|\bar{u}_n|) + C_2^\delta) \\
 &= h^2 (|\bar{u}_n|) \left[h^{-2} (|\bar{u}_n|) \int_0^T F(t, \bar{u}_n) dt + \left(\frac{a_2}{2} + \frac{\sqrt{a_2}T}{2\pi}\right) C_0^2 M_1^2 \right. \\
 &\quad + \left(\frac{T}{12}\right)^{(1+\alpha)/2} 2^\alpha C_0^{2+\alpha} M_2 M_1^{1+\alpha} K_1 h^{\alpha-1} (|\bar{u}_n|) + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 \\
 &\quad + M_3) (\sqrt{a_2} C_0 M_1 h^{-1} (|\bar{u}_n|)) + \left. \frac{\mu T}{\delta} \left(\frac{T}{12}\right)^{\delta/2} 2^{\delta-1} (\sqrt{a_2} M_1)^\delta h^{\delta-2} (|\bar{u}_n|) \right] \\
 &\quad + \left(\frac{T}{12}\right)^{(1+\alpha)/2} 2^\alpha C_0 M_2 K_1 C_2^{1+\alpha} + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 \\
 &\quad + M_3) C_2 - \left(1 + \frac{T}{2\pi\sqrt{a_2}}\right) C_1 + \frac{\mu T}{\delta} \left(\frac{T}{12}\right)^{\delta/2} 2^{\delta-1} C_2^\delta,
 \end{aligned}$$

for large n . So, by (2.4) we see that $|\bar{u}|$ is bounded. Hence $\{u_n\}$ is bounded by (2.9). Arguing as in the proof of Proposition 4.1 of [6], we conclude that the (PS) condition is satisfied.

Step 2. Let $\tilde{H}_T^1 = \{u \in H_T^1 : \bar{u} = 0\}$. We assert that for $u \in \tilde{H}_T^1$,

$$\varphi(u) \rightarrow +\infty, \quad \|u\| \rightarrow \infty. \tag{2.12}$$

In fact, from (1) and Sobolev’s inequality, we get

$$\begin{aligned} & \left| \int_0^T [F_1(t, u(t)) - F_1(t, 0)] dt \right| \\ &= \left| \int_0^T \int_0^1 (\nabla F(t, su(t)), u(t)) ds dt \right| \\ &\leq \int_0^T f(t)h(|u(t)|)|u(t)| dt + \int_0^T g(t)|u(t)| dt \\ &\leq \int_0^T f(t)(K_1|u(t)|^\alpha + K_2)|u(t)| dt + M_3\|u\|_\infty \\ &\leq M_2K_1\|u\|_\infty^{1+\alpha} + M_2K_2\|u\|_\infty + M_3\|u\|_\infty \\ &\leq \left(\frac{T}{12}\right)^{\frac{1+\alpha}{2}} M_2K_1\|\dot{u}\|_{L^2}^{1+\alpha} + \left(\frac{T}{12}\right)^{1/2} (M_2K_2 + M_3)\|\dot{u}\|_{L^2}, \end{aligned}$$

for all $u \in \tilde{H}_T^1$. It follows from (2) that

$$\begin{aligned} & \int_0^T [F_2(u(t)) - F_2(0)] dt \\ &= \int_0^T \int_0^1 \frac{1}{s} (\nabla F_2(s\bar{u}(t)) - \nabla F_2(0), su(t)) ds dt \\ &\geq - \int_0^T \int_0^1 rs^{\gamma-1}|u|^\gamma ds dt \\ &\geq -\frac{rT}{\gamma} \|\dot{u}\|_\infty^\gamma \\ &\geq -\frac{rT}{\gamma} \left(\frac{T}{12}\right)^{\gamma/2} \|\dot{u}\|_{L^2}^\gamma. \end{aligned}$$

So, we get

$$\begin{aligned} \varphi(u) &= \frac{1}{2} \|\dot{u}\|_{L^2}^2 + \int_0^T [F(t, u(t)) - F(t, 0)] dt + \int_0^T F(t, 0) dt \\ &\geq \frac{1}{2} \|\dot{u}\|_{L^2}^2 - \left(\frac{T}{12}\right)^{\frac{1+\alpha}{2}} M_2K_1\|\dot{u}\|_{L^2}^{1+\alpha} - \left(\frac{T}{12}\right)^{1/2} (M_2K_2 + M_3)\|\dot{u}\|_{L^2} \\ &\quad - \frac{rT}{\gamma} \left(\frac{T}{12}\right)^{\gamma/2} \|\dot{u}\|_{L^2}^\gamma + \int_0^T F(t, 0) dt. \end{aligned}$$

By Wirtinger’s inequality, $\|u\| \rightarrow \infty$ if and only if $\|\dot{u}\|_{L^2} \rightarrow \infty$ in \tilde{H}_T^1 . Hence (2.12) holds.

Step 3. By (5), we can easily see that $\int_0^T F(t, x) dt \rightarrow -\infty$ as $|x| \rightarrow \infty$ for all $x \in \mathbb{R}^N$. Thus, for all $u \in (\tilde{H}_T^1)^\perp = \mathbb{R}^N$,

$$\varphi(u) = \int_0^T F(t, u) dt \rightarrow -\infty \quad \text{as } |u| \rightarrow \infty.$$

Now, by saddle point theorem (see, [10], Theorem 4.6), the proof is completed. □

Proof of Theorem 1.3 By (7), we can choose an $a_3 > \frac{T^2}{4\pi^2 - rT^2}$ such that

$$\liminf_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F(t, x) dt > \frac{a_3}{2} M_1^2 C_0^2. \tag{2.13}$$

By (6) and the Sobolev inequality, we have

$$\begin{aligned} & \int_0^T [F_2(u(t)) - F_2(\bar{u})] dt \\ &= \int_0^T \int_0^1 \frac{1}{s} (\nabla F_2(\bar{u} + s\tilde{u}(t)) - \nabla F_2(\bar{u}), s\tilde{u}(t)) ds dt \\ &\geq - \int_0^T \int_0^1 rs |\tilde{u}(t)|^2 ds dt \geq - \frac{rT^2}{8\pi^2} \|\dot{u}\|_{L^2}^2. \end{aligned}$$

By a similar method to that of the proof of Theorem 1.1, we get

$$\begin{aligned} \varphi(u) &= \frac{1}{2} \|\dot{u}\|_{L^2}^2 + \int_0^T F(t, u(t)) dt \\ &= \frac{1}{2} \|\dot{u}\|_{L^2}^2 + \int_0^T [F_1(t, u(t)) - F_1(t, \bar{u})] dt \\ &\quad + \int_0^T [F_2(u(t)) - F_2(\bar{u})] dt + \int_0^T F(t, \bar{u}) dt \\ &\geq \left(\frac{1}{2} - \frac{T^2}{8\pi^2 a_3} - \frac{rT^2}{8\pi^2}\right) \|\dot{u}\|_{L^2}^2 - \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 \|\dot{u}\|_{L^2}^{1+\alpha} \\ &\quad - \left(\frac{T}{12}\right)^{1/2} \left(M_3 + \frac{C_0 M_2 K_2}{2}\right) \|\dot{u}\|_{L^2} - \frac{a_3 C_0^2 M_1^2}{2} h^2(|\bar{u}|) + \int_0^T F(t, \bar{u}) dt \\ &= \left(\frac{1}{2} - \frac{T^2}{8\pi^2 a_3} - \frac{rT^2}{8\pi^2}\right) \|\dot{u}\|_{L^2}^2 - \left(\frac{T}{12}\right)^{1/2} \left(M_3 + \frac{C_0 M_2 K_2}{2}\right) \|\dot{u}\|_{L^2} \\ &\quad - \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 \|\dot{u}\|_{L^2}^{1+\alpha} + h^2(|\bar{u}|) \left(h^{-2}(|\bar{u}|) \int_0^T F(t, \bar{u}) dt - \frac{a_3 C_0^2 M_1^2}{2}\right), \end{aligned}$$

for all $u \in H_T^1$, which implies that $\varphi(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ by (2.13), due to the facts that $r < \frac{4\pi^2}{T^2}$ and $\|u\| \rightarrow \infty$ if and only if $(|\bar{u}|^2 + \|\dot{u}\|_{L^2}^2)^{1/2} \rightarrow \infty$. So, applying the least action principle, Theorem 1.3 holds. □

Proof of Theorem 1.4 First, we assert that φ satisfies the (PS) condition. Suppose that $\{u_n\}$ satisfies $\varphi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ and $\{\varphi(u_n)\}$ is bounded. By (9), we can choose an $a_4 > \frac{T^2}{4\pi^2}$ such that

$$\limsup_{|x| \rightarrow \infty} h^{-2}(|x|) \int_0^T F_1(t, x) dt < - \left(\frac{a_4}{2} + \frac{\sqrt{a_4 T}}{2\pi}\right) C_0^2 M_1^2. \tag{2.14}$$

By the (λ, μ) -subconvexity of $G(x)$, we have

$$G(x) \leq (2\mu|x|^\beta + 1)G_0 \tag{2.15}$$

for all $x \in \mathbb{R}^N$, and a.e. $t \in [0, T]$, where $G_0 = \max_{|s| \leq 1} G(s)$, $\beta = \log_{2\lambda}(2\mu) < 2$. Then

$$\begin{aligned} & \int_0^T (\nabla F_2(t, u_n(t)), \tilde{u}_n(t)) \, dt \\ & \geq - \int_0^T k(t)G(\bar{u}_n) \, dt \\ & \geq - \int_0^T k(t)(2\mu|\bar{u}_n|^\beta + 1)G_0 \, dt \\ & = -2\mu M_4|\bar{u}_n|^\beta - M_4, \end{aligned} \tag{2.16}$$

where $M_4 = G_0 \int_0^T k(t) \, dt$. From (2.5) and (2.16), for large n , we have

$$\begin{aligned} \|\bar{u}_n\| & \geq \langle \varphi(u_n), \tilde{u}_n \rangle \\ & = \|\dot{u}_n\|_{L^2}^2 + \int_0^T (\nabla F(t, u_n(t)), \tilde{u}_n(t)) \\ & \geq \left(1 - \frac{T^2}{8\pi^2 a_4}\right) \|\dot{u}_n\|_{L^2}^2 - \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 \|\dot{u}_n\|_{L^2}^{1+\alpha} - \frac{(C_0 M_1)^2 a_4}{2} h^2(|\bar{u}_n|) \\ & \quad - \left(\frac{T}{12}\right)^{1/2} \left(M_3 + \frac{C_0 M_2 K_2}{2}\right) \|\dot{u}_n\|_{L^2} - 2\mu M_4 |\bar{u}_n|^\beta - M_4. \end{aligned} \tag{2.17}$$

So, from (2.7) and (2.17) we have

$$\begin{aligned} & \frac{(C_0 M_1)^2 a_4}{2} h^2(|\bar{u}_n|) + 2\mu M_4 |\bar{u}_n|^\beta \\ & \geq \left(1 - \frac{T^2}{8\pi^2 a_4}\right) \|\dot{u}_n\|_{L^2}^2 - \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 \|\dot{u}_n\|_{L^2}^{1+\alpha} \\ & \quad - \left(\frac{T}{12}\right)^{1/2} \left(M_3 + \frac{C_0 M_2 K_2}{2}\right) \|\dot{u}_n\|_{L^2} - \frac{(T^2 + 4\pi^2)^{1/2}}{2\pi} \|\dot{u}_n\|_{L^2} - M_4 \\ & \geq \frac{1}{2} \|\dot{u}_n\|_{L^2}^2 + C_4, \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} C_4 = \min \left\{ \left(\frac{1}{2} - \frac{T^2}{8\pi^2 a_4}\right) s^2 - \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 s^{1+\alpha} - \left[\frac{(T^2 + 4\pi^2)^{1/2}}{2\pi}\right] s \right. \\ \left. + \left(\frac{T}{12}\right)^{1/2} \left(M_3 + \frac{C_0 M_2 K_2}{2}\right) \right] s - M_4 \right\}. \end{aligned}$$

Note that $-\infty < C_4 < 0$ due to $a_4 > \frac{T^2}{4\pi^2}$, by (2.18), one has

$$\|\dot{u}_n\|_{L^2}^2 \leq a_4 (C_0 M_1)^2 h^2(|\bar{u}_n|) + 4\mu M_4 |\bar{u}_n|^\beta - 2C_4, \tag{2.19}$$

and then

$$\|\dot{u}_n\|_{L^2} \leq \sqrt{a_4} C_0 M_1 h(|\bar{u}_n|) + 2\sqrt{\mu M_4} |\bar{u}_n|^{\beta/2} + C_5, \tag{2.20}$$

where $C_5 > 0$. From (8) and (2.15), we have

$$\begin{aligned}
 & \left| \int_0^T [F_2(t, u(t)) - F_2(t, \bar{u})] dt \right| \\
 &= \int_0^T \int_0^1 (\nabla F_2(t, \bar{u}_n + s\tilde{u}_n(t)), \tilde{u}_n(t)) ds dt \\
 &\leq \int_0^T \int_0^1 k(t)G(\bar{u}_n + (s+1)\tilde{u}_n) ds dt \\
 &\leq \int_0^T \int_0^1 k(t)(2\mu|\bar{u}_n + (s+1)\tilde{u}_n(t)|^\beta + 1) \\
 &\leq 4\mu \int_0^T k(t)(|\bar{u}_n|^\beta + 2^\beta|\tilde{u}_n|^\beta)G_0 \int_0^T k(t) dt \\
 &\leq \left(\frac{T}{12}\right)^{\beta/2} 2^{\beta+2}\mu M_4 \|\dot{u}_n\|_{L^2}^\beta + 4\mu M_4 |\bar{u}_n|^\beta + M_4, \tag{2.21}
 \end{aligned}$$

for all $u \in H_T^1$. By the boundedness of $\{\varphi(u_n)\}$ and the inequalities (2.19)-(2.21), we get

$$\begin{aligned}
 C_6 &\leq \varphi(u_n) \\
 &= \frac{1}{2} \|\dot{u}_n\|_{L^2}^2 + \int_0^T [F_1(t, u_n(t)) - F_1(t, \bar{u}_n)] dt \\
 &\quad + \int_0^T [F_2(t, u_n(t)) - F_2(t, \bar{u}_n)] dt + \int_0^T F(t, \bar{u}_n) dt \\
 &\leq \left(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_4}}\right) \|\dot{u}_n\|_{L^2}^2 + \frac{\sqrt{a_4}TC_0^2}{4\pi} M_1^2 h^2(|\bar{u}_n|) + \left(\frac{T}{12}\right)^{1/2} (C_0M_2K_2 + M_3) \|\dot{u}_n\|_{L^2} \\
 &\quad + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0M_2K_1 \|\dot{u}_n\|_{L^2}^{1+\alpha} + \left(\frac{T}{12}\right)^{\beta/2} 2^{\beta+2}\mu M_4 \|\dot{u}_n\|_{L^2}^\beta \\
 &\quad + 4\mu M_4 |\bar{u}_n|^\beta + M_4 + \int_0^T F(t, \bar{u}_n) dt \\
 &\leq \left(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_4}}\right) (a_4(C_0M_1)^2 h^2(|\bar{u}_n|) + 4\mu M_4 |\bar{u}_n|^\beta - 2C_4) \\
 &\quad + \frac{\sqrt{a_4}TC_0^2}{4\pi} M_1^2 h^2(|\bar{u}_n|) \\
 &\quad + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0M_2K_1 (\sqrt{a_4}C_0M_1 h(|\bar{u}_n|) + 2\sqrt{\mu M_4} |\bar{u}_n|^{\beta/2} + C_5)^{1+\alpha} \\
 &\quad + \left(\frac{T}{12}\right)^{1/2} (C_0M_2K_2 + M_3) (\sqrt{a_4}C_0M_1 h(|\bar{u}_n|) + 2\sqrt{\mu M_4} |\bar{u}_n|^{\beta/2} + C_5) \\
 &\quad + \left(\frac{T}{12}\right)^{\beta/2} 2^{\beta+2}\mu M_4 (\sqrt{a_4}C_0M_1 h(|\bar{u}_n|) + 2\sqrt{\mu M_4} |\bar{u}_n|^{\beta/2} + C_5)^\beta \\
 &\quad + 4\mu M_4 |\bar{u}_n|^\beta + M_4 + \int_0^T F(t, \bar{u}_n) dt \\
 &\leq \left(\frac{a_4}{2} + \frac{\sqrt{a_4}T}{2\pi}\right) ((C_0M_1)^2 h^2(|\bar{u}_n|))
 \end{aligned}$$

$$\begin{aligned}
 & + \left(6 + \frac{T}{\pi\sqrt{a_4}}\right)\mu M_4 |\bar{u}_n|^\beta - \left(1 + \frac{T}{2\pi\sqrt{a_4}}\right)C_4 \\
 & + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 (2^\alpha a_4^{(1+\alpha)/2} (C_0 M_1)^{1+\alpha} h^{1+\alpha} (|\bar{u}_n|) \\
 & + 2^{3\alpha+1} \mu^{\frac{1+\alpha}{2}} M_4^{\frac{1+\alpha}{2}} |\bar{u}_n|^{\beta(1+\alpha)} + 2^{2\alpha} C_5^{1+\alpha}) \\
 & + \left(\frac{T}{12}\right)^{\beta/2} 2^{2+\beta} \mu M_4 (2^{\beta-1} a_4^{\beta/2} (C_0 M_1 h (|\bar{u}_n|))^\beta \\
 & + 2^{3\beta-2} \mu^{\beta/2} M_4^{\beta/2} |\bar{u}_n|^{\beta^2/2} + 2^{2(\beta-1)} C_5^\beta) \\
 & + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) (\sqrt{a_4} C_0 M_1 h (|\bar{u}_n|) + 2\sqrt{\mu M_4} |\bar{u}_n|^{\beta/2} + C_5) \\
 & + M_4 + \int_0^T F_1(t, \bar{u}_n) dt + \int_0^T F_2(\bar{u}_n) dt \\
 = & h^2 (|\bar{u}_n|) \left[h^{-2} (|\bar{u}_n|) \int_0^T F_1(t, \bar{u}_n) dt + \left(\frac{a_4}{2} + \frac{\sqrt{a_4} T}{2\pi}\right) (C_0 M_1)^2 \right. \\
 & + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0^{2+\alpha} M_1^{1+\alpha} M_2 K_1 2^\alpha a_4^{(1+\alpha)/2} h^{\alpha-1} (|\bar{u}_n|) \\
 & + \left(\frac{T}{12}\right)^{\beta/2} 2^{1+2\beta} \mu (C_0 M_1)^\beta M_4 a_4^{\beta/2} h^{\beta-2} (|\bar{u}_n|) \\
 & \left. + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) \sqrt{a_4} C_0 M_1 h^{-1} (|\bar{u}_n|) \right] \\
 & + |\bar{u}_n|^\beta \left[|\bar{u}_n|^{-\beta} \int_0^T F_2(\bar{u}_n) dt + \left(6 + \frac{T}{\pi\sqrt{a_4}}\right) \mu M_4 \right. \\
 & + \left(\frac{T}{12}\right)^{(1+\alpha)/2} 2^{3\alpha+1} C_0 M_2 K_1 \mu^{(1+\alpha)/2} M_4^{(1+\alpha)/2} |\bar{u}_n|^{\alpha\beta} \\
 & + \left(\frac{T}{12}\right)^{\beta/2} 2^{4\beta} \mu^{(\beta+2)/2} M_4^{(\beta+2)/2} |\bar{u}_n|^{\frac{\beta^2}{2}-\beta} \\
 & \left. + \left(\frac{T}{12}\right)^{1/2} 2(C_0 M_2 K_2 + M_3) \sqrt{\mu M_4} |\bar{u}_n|^{-\beta/2} \right] \\
 & - \left(1 + \frac{T}{2\pi\sqrt{a_4}}\right) C_4 + \left(\frac{T}{12}\right)^{(1+\alpha)/2} 2^{2\alpha} C_0 M_2 K_1 C_5^{1+\alpha} \\
 & + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) C_5 + \left(\frac{T}{12}\right)^{\beta/2} 2^{3\beta} \mu M_4 C_5^\beta + M_4,
 \end{aligned}$$

for large n . The above inequality and (2.14) imply that $\{|\bar{u}|\}$ is bounded. Hence $\{u_n\}$ is bounded by (2.19). By using the standard method, the (PS) condition holds.

Since the rest of the proof is similar to that of Theorem 1.2, we omit the details here. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The main idea of this paper was proposed by D-BW, D-BW prepared the manuscript initially, and KY performed a part of the steps of the proofs in this research. All authors read and approved the final manuscript.

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