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Global existence and asymptotic behavior of solutions to a semilinear parabolic equation on Carnot groups

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Abstract

In this paper we consider the semilinear parabolic equation $\sum_{i,j=1}^{m} a_{ij}X_iX_ju - \partial_t u + Vu^p = 0$ with a general class of potentials $V = V(\xi, t)$, where $A = \{a_{ij}\}_{i,j}$ is a positive definite symmetric matrix and the X_i's denotes a system of left-invariant vector fields on a Carnot group G. Based on a fixed point argument and by establishing some new estimates involving the heat kernel, we study the existence and large-time behavior of global positive solutions to the preceding equation.

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Keywords: Carnot group; existence; asymptotic behavior

1 Introduction

Global existence and asymptotic behavior of solutions to nonlinear parabolic equations have been followed with interest over the past years [1–8].

In this paper we are concerned with the existence and asymptotic behavior of global positive solutions for the semilinear parabolic equation

$$\begin{cases} Hu = \sum_{i,j=1}^{m} a_{ij}X_iX_ju - \frac{\partial}{\partial t}u + V(\xi,t)u^p = 0, & (\xi,t) \in G \times (0,+\infty), \\ u(\xi,0) = u_0(\xi), & \xi \in G. \end{cases} \tag{1.1}$$

Here $p > 1$, $X_1, \ldots, X_m$ are left-invariant vector fields on a Carnot group G, and the matrix $A = \{a_{ij}\}_{i,j}$ is symmetric and positive definite, that is,

$$\Lambda^{-1}|Z|^2 \le \langle AZ, Z\rangle \le \Lambda|Z|^2 \tag{1.2}$$

for some constant $\Lambda > 0$, every $Z \in \mathbb{R}^m$. For a fixed $\Lambda \ge 1$, we denote by M_Λ the set of $m \times m$ symmetric matrices A satisfying (1.2).

It is well known that the Euclidean space $\mathbb{R}^n$, with its usual Abelian group structure, is a trivial Carnot group. In the Euclidean case, we first recall that Zhang [4] studied the global existence for a parabolic problem in divergence form analogous to (1.1) when the potential V is in parabolic Kato class at infinity P^∞, the asymptotic behavior of solutions for the problem was studied by Zhang and Zhao [5]. Riahi [6] extended the results in [4] and [5] to a new functional class $P^{\infty}_{\mathrm{loc}}$ more general than the parabolic Kato class P^{∞} and proved that the problem $a\Delta u - \partial_t u + V(x,t)u^p = 0$ has a global continuous solution. The author in [6] also gave the asymptotic behavior of the global solutions when $V = V(x)$ is independent of time. The proofs in [4–6] rely on some heat kernel estimates. We see that the fundamental solutions to the parabolic operators on the Euclidean space have explicit expression. A natural question to ask is whether the results in [4–6] can be generalized to general degenerate parabolic operators whose fundamental solutions are not known explicitly.

One of the most important degenerate parabolic operators is the heat operator associated with the subelliptic operator on a Carnot group. These classes of operators naturally arise in many different settings: geometry in several complex variables, curvature problem for CR-manifolds, sub-Riemannian geometry, diffusion processes, control theory, human vision (see [9, 10] and the references therein). In recent years many authors have undertaken the research on degenerate heat equation on Carnot groups; see for example [7, 8] and [11–18].

In the present paper we will generalize the global existence and asymptotic results of [5] and [6] to the degenerate heat equation on the Carnot group G. Let us briefly discuss the method we are going to adopt. We will use a fixed point argument to achieve existence. This requires us to obtain a number of new estimates involving the heat kernel, which are based on the Gaussian bounds for fundamental solutions for the operator $L_A = \sum_{i,j=1}^{m} a_{ij}X_iX_j - \partial_t$ (see [11]). The asymptotic behavior of the global solutions are obtained by establishing global Gaussian upper bounds for the fundamental solution of certain linear degenerate parabolic operators on Carnot groups. The result will serves as a bridge between the degenerate parabolic problem and the corresponding subelliptic stationary problem.

The paper is organized as follows. In the next section, we first present the necessary background material concerning homogeneous structures on Carnot groups and introduce some basic definitions. Then we summarize our results in Theorem 2.4 and Theorem 2.5. Section 3 is devoted to the proof of some heat kernel estimates which will be used in the following sections. Theorem 2.4 and Theorem 2.5 will be proved in Section 4 and Section 5, respectively. In the Appendix, we present two results as regards the class of potentials $P^{\infty}_{\mathrm{loc}}$.

2 Main results

We start by giving the definition of a Carnot group. We will consider $G = (\mathbb{R}^N, \cdot)$ as a Carnot group with a group operation $\cdot$ and a family of dilations, compatible with the Lie structure. A Carnot group G of step $r \geq 1$ is a simply connected nilpotent Lie group whose Lie algebra g admits a stratification $\mathrm{g} = \bigoplus_{j=1}^{r} V_j$, with $[V_1, V_j] = V_{j+1}$, for $1 \leq j < r$, $[V_1, V_r] = \{0\}$. We assume that a scalar product $\langle\cdot,\cdot\rangle$ is given on g for which the V_j's are mutually orthogonal.

Via the exponential map, it is possible to induce on G a family of non-isotropic dilations defined by

$$\delta_\lambda\bigl(x^{(1)}, x^{(2)}, \ldots, x^{(r)}\bigr) = \bigl(\lambda x^{(1)}, \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)}\bigr).$$

Here $x^{(i)} \in \mathbb{R}^{N_i}$ for $i = 1, \ldots, r$ and $N_1 + \cdots + N_r = N$. The topological dimension of G is N, whereas the homogeneous dimension of G, attached to the dilations $\{\delta_\lambda\}_{\lambda>0}$, is given by $Q = \sum_{j=1}^{r} jN_j$. Let $m = N_1$ and $X = \{X_1, \dots, X_m\}$ be the dimension and a basis of V_1, respectively. Let $Xu = (X_1 u, \dots, X_m u)$ denote the horizontal gradient for a function u and $|Xu| = [\sum_{i=1}^{m} (X_i u)^2]^{\frac{1}{2}}$.

Let e be the identity on G. For $\xi \in G$, we denote by ξ^{-1} the inverse of ξ with respect to the group operation. In the sequel, ρ will denote the Carnot-Carathéodory control distance generated on G by the X_i's (see [15]). There is a remarkable link between the control distance ρ and the homogeneous Lie group structure on G. Indeed, we have

$$\rho(\xi \cdot \eta, \xi \cdot \zeta) = \rho(\eta, \zeta), \quad \xi, \eta, \zeta \in G,$$
$$\rho\bigl(e, \delta_\lambda(\xi)\bigr) = \lambda \rho(e, \xi), \quad \xi \in G, \lambda > 0.$$

By denoting $\rho(\xi, e)$ simply by $\rho(\xi)$, we define a norm function $\rho(\xi) \in C^\infty(G \backslash \{e\}) \cap C(G)$ such that

(1) $\rho(\xi) = 0$ if and only if $\xi = e$;
(2) $\rho(\xi) = \rho(\xi^{-1})$;
(3) $\rho(\delta_\lambda(\xi)) = \lambda \rho(\xi)$, $\lambda > 0$.

Moreover, $\rho(\cdot)$ satisfies the triangle inequality

$$\rho(\xi \cdot \eta) \le \rho(\xi) + \rho(\eta), \quad \xi, \eta \in G. \tag{2.1}$$

We recall that this Carnot-Carathéodory distance is equivalent to any quasi-distance induced by a homogeneous norm on G.

We denote by

$$B_\rho(\xi, R) = \bigl\{\eta \in G : \rho(\xi, \eta) < R\bigr\} \tag{2.2}$$

the open ball of center ξ and radius R. Since the Lebesgue measure is a Haar measure on G, we have $|B_\rho(\xi, R)| = |B_\rho(e, 1)| R^Q$. The following polar coordinates formula holds:

$$\int_{B_\rho(e,R)} f\bigl(\rho(\xi)\bigr)\, d\xi = Q\bigl|B_\rho(e,1)\bigr| \int_0^R f(\rho)\rho^{Q-1}\, d\rho \tag{2.3}$$

for every measurable function f.

Following [11], we next briefly recall some well-known results on the fundamental solution for the operator $L_A = \sum_{i,j=1}^{m} a_{ij} X_i X_j - \partial_t$ in (1.1). There exists a positive function Γ_A in $G \times (0, +\infty)$ such that the fundamental solution for L_A is given by $\Gamma_A(\xi, t; \eta, s) := \Gamma_A(\eta^{-1} \cdot \xi, t - s)$. It is $\Gamma_A(\xi, t) = 0$ for $t \le 0$, $\Gamma_A(\xi, t) = \Gamma_A(\xi^{-1}, t)$ and $\Gamma_A(\delta_\lambda(\xi), \lambda^2 t) = \lambda^{-Q}\Gamma_A(\xi, t)$. In particular, Γ_A vanishes at infinity. For every $t > 0$, $\int_G \Gamma_A(\xi, t)\, d\xi = 1$. For every $\xi \in G$, $t > 0$ and $\tau > 0$, the following reproduction property holds (see [12]):

$$\Gamma_A(\xi, t + \tau) = \int_G \Gamma_A\bigl(\eta^{-1} \cdot \xi, t\bigr)\Gamma_A(\eta, \tau)\, d\eta. \tag{2.4}$$

One of the main tools we shall use in the paper is the following remarkable uniform Gaussian estimates (see [11]): there exist positive constants C_Λ, $C_{\Lambda 1}$, $C_{\Lambda 2}$ such that, for every $i, j = 1, \dots, m$ and for every $A \in M_\Lambda$, we have

$$C_\Lambda^{-1} t^{-\frac{Q}{2}} \exp\biggl(-\frac{C_\Lambda \rho^2(\xi)}{t}\biggr) \le \Gamma_A(\xi, t) \le C_\Lambda t^{-\frac{Q}{2}} \exp\biggl(-\frac{\rho^2(\xi)}{C_\Lambda t}\biggr), \tag{2.5}$$

$$\left|X_i\Gamma_A(\xi,t)\right| \le C_{\Lambda 1}t^{-\frac{Q+1}{2}}\exp\left(-\frac{\rho^2(\xi)}{C_\Lambda t}\right) \tag{2.6}$$

and

$$\left|X_iX_j\Gamma_A(\xi,t)\right| + \left|\partial_t\Gamma_A(\xi,t)\right| \le C_{\Lambda 2}t^{-(\frac{Q}{2}+1)}\exp\left(-\frac{\rho^2(\xi)}{C_\Lambda t}\right) \tag{2.7}$$

for every $\xi \in G$, $t > 0$.

Let us introduce the class P^∞_{loc} on the Carnot group G.

Definition 2.1 A measurable function $V = V(\xi,t)$ on $G \times \mathbb{R}$ is said to be in the class P^∞_{loc} if it satisfies for all $c > 0$,

$$\begin{aligned} N_c(V) = {}& \sup_{(\xi,t)\in G\times\mathbb{R}} \int_{-\infty}^{t}\int_G \Gamma_c(\xi,t;\eta,s)\left|V(\eta,s)\right| d\eta\, ds \\ &+ \sup_{(\eta,s)\in G\times\mathbb{R}} \int_s^{+\infty}\int_G \Gamma_c(\xi,t;\eta,s)\left|V(\xi,t)\right| d\xi\, dt < +\infty, \end{aligned}$$

and, for any compact subset $K \subset G \times \mathbb{R}$,

$$\begin{aligned} \lim_{r\to 0}\Bigg\{ & \sup_{(\xi,t)\in K} \int_{t-r}^{t}\int_{\rho(\xi,\eta)<\sqrt{r}} \Gamma_c(\xi,t;\eta,s)\left|V(\eta,s)\right| d\eta\, ds \\ &+ \sup_{(\eta,s)\in K} \int_s^{s+r}\int_{\rho(\xi,\eta)<\sqrt{r}} \Gamma_c(\xi,t;\eta,s)\left|V(\xi,t)\right| d\xi\, dt \Bigg\} = 0, \end{aligned}$$

where $\Gamma_c(\xi,t;\eta,s) = (t-s)^{-\frac{Q}{2}}e^{-c\frac{\rho^2(\xi,\eta)}{t-s}}$, for $t > s$.

Obviously, $P^\infty_{\mathrm{loc}} \subset L^1_{\mathrm{loc}}(G\times\mathbb{R})$ and we have the following.

Remark 2.2 (1) As far as time independent V is concerned, we will show in Proposition A.2 below the fact that $V \in P^\infty_{\mathrm{loc}}$ if $V \in L^1(G)$ and $\int_G \frac{|V(\eta)|}{\rho^{Q-2}(\xi,\eta)}\,d\eta$ is bounded in G. Then it follows that the function $V(\xi) = \frac{1}{1+\rho^\alpha(\xi)}$, $\alpha > Q$, belongs to P^∞_{loc}. In fact all functions in $L^1(G)\cap L^\infty(G)$ belong to P^∞_{loc}.

(2) In the case when V is independent of ξ, the function $V(t) = \frac{1}{1+t^\beta}$, $\beta > 1$, also belongs to P^∞_{loc}. Following a similar argument to the proof of Proposition 2.1 in [4], we have all functions in $L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ belong to P^∞_{loc}.

Let u_0 be a positive function in $L^\infty(G)$ and $c > 0$, we write

$$h_c(\xi,t) = \int_G \Gamma_c(\xi,t;\eta,0)u_0(\eta)\,d\eta. \tag{2.8}$$

Following [7], we introduce the following definition of weak solutions.

Definition 2.3 A function $u = u(\xi,t)$ is called a weak solution of (1.1) if $u, X_1u, \ldots, X_mu \in L^2_{\mathrm{loc}}(G\times(0,+\infty))$, $Vu \in L^1_{\mathrm{loc}}(G\times(0,+\infty))$ and

$$u(\xi,t) = \int_G \Gamma_A(\xi,t;\eta,0)u_0(\eta)\,d\eta + \int_0^t\int_G \Gamma_A(\xi,t;\eta,s)V(\eta,s)u^p(\eta,s)\,d\eta\,ds$$

for all $(\xi,t) \in G\times(0,+\infty)$.

The main results of the paper are the next two theorems.

Theorem 2.4 (Global existence) *Let $V \in P^{\infty}_{\mathrm{loc}}$ be a nonnegative function. For any $M > C_\Lambda > 1$, there is a constant $b_0 > 0$ such that for each nonnegative $u_0 \in C^2(G)$ satisfying $\|u_0\|_\infty \leq b_0$, there exists a positive continuous solution of* (1.1) *such that*

$$M^{-1}h_{C_\Lambda}(\xi,t) \leq u(\xi,t) \leq Mh_{\frac{1}{C_\Lambda}}(\xi,t)$$

for all $(\xi,t) \in G \times (0,+\infty)$.

In the next theorem we present a result about the large-time behavior.

Theorem 2.5 *Let $V(\xi,t) = V(\xi) \in P^{\infty}_{\mathrm{loc}}$ be a nonnegative function. If $u_0 \in C^2(G)$, $\sum_{i,j=1}^m a_{ij} \times X_iX_ju_0 \in P^{\infty}_{\mathrm{loc}}$, and $0 < \alpha_1 \leq u_0 \leq \alpha_2$ for some positive constants α_1, α_2, then the problem* (1.1) *has a global positive solution which converges pointwise to a positive solution of the subelliptic problem*

$$\sum_{i,j=1}^{m} a_{ij}X_iX_ju + V(\xi)u^p = 0. \tag{2.9}$$

Throughout this paper, the letter C denotes a positive constant which may vary from line to line but is independent of the terms which will take part in any limit process.

3 Preliminaries and auxiliary estimates

For $\alpha, \beta \geq 0$ we use $\alpha \vee \beta$ and $\alpha \wedge \beta$ to mean $\max\{\alpha,\beta\}$ and $\min\{\alpha,\beta\}$, respectively. We also need to use the inequality

$$e^{-\theta} \leq \frac{1 \vee (\frac{m}{e})^m}{(1 \vee \theta)^m} \quad \text{for all } \theta \geq 0 \text{ and } m > 0. \tag{3.1}$$

For $(\xi,t), (\eta,s) \in G \times \mathbb{R}$, we can define on $G \times \mathbb{R}$ the parabolic distance corresponding to ρ as

$$d\big((\xi,t),(\eta,s)\big) \equiv \rho(\xi,\eta) \vee |t-s|^{\frac{1}{2}}, \tag{3.2}$$

where $|\cdot|$ denotes the Euclidean distance on $\mathbb{R}$. In particular d satisfies the triangle inequality via (2.1). For $r > 0$, we can also define the ball with center (ξ,t) and radius r with respect to the parabolic distance and its complement as $B((\xi,t),r)$ and $B^c((\xi,t),r)$, respectively.

For any $\beta \leq 1$, the parabolic CC-Hölder space $\Gamma^\beta(G \times \mathbb{R})$ related to d is defined by

$$\Gamma^\beta(G \times \mathbb{R}) := \left\{ f \in L^\infty \cap C^0 : \sup_{(\xi,t) \neq (\eta,s)} \frac{|f(\xi,t) - f(\eta,s)|}{d((\xi,t),(\eta,s))^\beta} < \infty \right\}.$$

We refer the readers to [16] for more information.

The following two lemmas concern the continuity of the potentials $\iint \Gamma_A|V|\,d\eta\,ds$ when $V \in P^{\infty}_{\mathrm{loc}}$, which will be used in the proof of Theorem 2.4. The proof of the results in the Euclidean case was given in [6].

Lemma 3.1 *Let $A \in M_\Lambda$. Then there exist constants $C = C(\Lambda) > 0$ and $c = c(\Lambda) > 0$ such that for all $r \in (0,1)$, $(\xi_0, t_0) \in G \times \mathbb{R}$, $(\xi, t) \in B((\xi_0, t_0), \frac{\sqrt{r}}{8})$, $t \geq t_0$ and $(\eta, s) \in B^c((\xi_0, t_0), \frac{\sqrt{r}}{2})$, we have*

$$\left|\Gamma_A(\xi,t;\eta,s) - \Gamma_A(\xi_0,t_0;\eta,s)\right| \leq C\frac{d((\xi,t),(\xi_0,t_0))}{r^{\frac{1}{2}}}\Gamma_c(\xi,t;\eta,s). \tag{3.3}$$

Proof Case 1: $s \geq t$. The left-hand side term is equal to zero and so the inequality is trivial.

Case 2: $t > s \geq t_0$. From the assumptions we have $0 < t - s \leq t - t_0$ and $\rho(\xi,\eta) \vee |t-s|^{\frac{1}{2}} \geq \frac{3}{8}\sqrt{r}$. By noting that $\Gamma_A(\xi_0, t_0; \eta, s) = 0$ and applying (2.5) and (3.1) with $m = \frac{1}{2}$, we have

$$\begin{aligned}\left|\Gamma_A(\xi,t;\eta,s) - \Gamma_A(\xi_0,t_0;\eta,s)\right| &\leq C_\Lambda (t-s)^{-\frac{Q}{2}} e^{-\frac{\rho^2(\xi,\eta)}{C_\Lambda (t-s)}} \\ &\leq \frac{C(t-s)^{\frac{1}{2}}}{(t-s)^{\frac{1}{2}} \vee \rho(\xi,\eta)}(t-s)^{-\frac{Q}{2}} e^{-\frac{\rho^2(\xi,\eta)}{2C_\Lambda (t-s)}} \\ &\leq C\frac{(t-t_0)^{\frac{1}{2}}}{r^{\frac{1}{2}}}\Gamma_{\frac{1}{2C_\Lambda}}(\xi,t;\eta,s).\end{aligned}$$

Case 3: $t_0 > s$. We get

$$\begin{aligned}&\left|\Gamma_A(\xi,t;\eta,s) - \Gamma_A(\xi_0,t_0;\eta,s)\right| \\ &\quad\leq \left|\Gamma_A(\xi,t;\eta,s) - \Gamma_A(\xi_0,t;\eta,s)\right| + \left|\Gamma_A(\xi_0,t;\eta,s) - \Gamma_A(\xi_0,t_0;\eta,s)\right|.\end{aligned} \tag{3.4}$$

Thus, from (2.6) and (3.1) with $m = \frac{1}{2}$, we have

$$\begin{aligned}&\left|\Gamma_A(\xi,t;\eta,s) - \Gamma_A(\xi_0,t;\eta,s)\right| \\ &\quad\leq \rho(\xi,\xi_0) \sup_{\zeta:\rho(\zeta,\xi_0)\leq\rho(\xi,\xi_0)} \left|X\Gamma_A(\zeta,t;\eta,s)\right| \\ &\quad\leq \rho(\xi,\xi_0) \sup_{\zeta:\rho(\zeta,\xi_0)\leq\rho(\xi,\xi_0)} \frac{1}{\rho(\zeta,\eta)\vee(t-s)^{\frac{1}{2}}}\frac{C}{(t-s)^{\frac{Q}{2}}}e^{-\frac{\rho^2(\zeta,\eta)}{2C_\Lambda(t-s)}},\end{aligned} \tag{3.5}$$

where we have used the Lagrange mean value theorem on G (see Theorem 20.3.1 in [15]). On the other hand, for the previous ζ we have

$$\begin{aligned}\rho(\zeta,\eta)\vee(t-s)^{\frac{1}{2}} &\geq \rho(\xi,\eta)\vee(t-s)^{\frac{1}{2}} - \rho(\xi,\zeta) \\ &\geq \frac{1}{3}\rho(\xi,\eta)\vee(t-s)^{\frac{1}{2}} \geq \frac{\sqrt{r}}{8},\end{aligned} \tag{3.6}$$

which yields $\rho^2(\zeta,\eta) + t - s \geq \frac{1}{9}(\rho^2(\xi,\eta) + t - s)$ and therefore

$$\frac{\rho^2(\zeta,\eta)}{t-s} \geq \frac{1}{9}\frac{\rho^2(\xi,\eta)}{t-s} - \frac{8}{9}. \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5) leads to

$$\left|\Gamma_A(\xi,t;\eta,s) - \Gamma_A(\xi_0,t;\eta,s)\right| \leq C\frac{\rho(\xi,\xi_0)}{r^{\frac{1}{2}}}\Gamma_{\frac{1}{18C_\Lambda}}(\xi,t;\eta,s). \tag{3.8}$$

Analogously, there exists a $\tau \in (t_0, t)$ such that

$$\left|\Gamma_A(\xi_0, t; \eta, s) - \Gamma_A(\xi_0, t_0; \eta, s)\right| \le |t - t_0| \cdot \left|\frac{\partial}{\partial \tau}\Gamma_A(\xi_0, \tau; \eta, s)\right|. \tag{3.9}$$

Moreover, by applying (2.7) and (3.1) with $m = \frac{Q}{2} + 1$, and using the inequality $0 < \tau - s < t - s$, we obtain

$$\left|\frac{\partial}{\partial \tau}\Gamma_A(\xi_0, \tau; \eta, s)\right| \le \frac{C}{(\rho(\xi_0, \eta) \vee (\tau - s)^{\frac{1}{2}})^{Q+2}} e^{-\frac{\rho^2(\xi_0, \eta)}{2C_\Lambda (t-s)}}.$$

Since $\rho(\xi_0, \eta) \vee (\tau - s)^{\frac{1}{2}} \ge \rho(\xi, \eta) \vee (t - s)^{\frac{1}{2}} - \rho(\xi, \xi_0) \vee (t - \tau)^{\frac{1}{2}} \ge \frac{2}{3}\rho(\xi, \eta) \vee (t - s)^{\frac{1}{2}} \ge \frac{\sqrt{r}}{4}$, and $\frac{\rho^2(\xi_0, \eta)}{t-s} \ge \frac{2}{9}\frac{\rho^2(\xi, \eta)}{t-s} - \frac{7}{9}$, we find

$$\left|\frac{\partial}{\partial \tau}\Gamma_A(\xi_0, \tau; \eta, s)\right| \le \frac{C}{r}\Gamma_{\frac{1}{18C_\Lambda}}(\xi, t; \eta, s). \tag{3.10}$$

Substituting (3.10) into (3.9) gives

$$\left|\Gamma_A(\xi_0, t; \eta, s) - \Gamma_A(\xi_0, t_0; \eta, s)\right| \le C\left(\frac{t - t_0}{r}\right)^{\frac{1}{2}} \Gamma_{\frac{1}{18C_\Lambda}}(\xi, t; \eta, s). \tag{3.11}$$

Combining (3.4), (3.8), and (3.11), we obtain the inequality stated in the lemma. □

Remark 3.2 If we replace $t \ge t_0$ by $t \le t_0$, we obtain the same inequality provided that $\Gamma_c(\xi, t; \eta, s)$ is replaced by $\Gamma_c(\xi_0, t_0; \eta, s)$.

We now set, for every $(\xi, t), (\xi_0, t_0) \in G \times \mathbb{R}$,

$$K_A(\xi, t; \xi_0, t_0) = \int_{-\infty}^{t \vee t_0} \int_G \left|\Gamma_A(\xi, t; \eta, s) - \Gamma_A(\xi_0, t_0; \eta, s)\right| \cdot \left|V(\eta, s)\right| d\eta\, ds.$$

We can prove the following lemma.

Lemma 3.3 *Let $V \in P^\infty_{\mathrm{loc}}$. Then for every $A \in M_\Lambda$ we have*

$$\lim_{\rho(\xi, \xi_0) \vee |t - t_0|^{\frac{1}{2}} \to 0} K_A(\xi, t; \xi_0, t_0) = 0.$$

Proof Let $(\xi_0, t_0) \in G \times \mathbb{R}$ be fixed. Set $K = \bar{B}((\xi_0, t_0), 1)$. Since $V \in P^\infty_{\mathrm{loc}}$ for $\varepsilon > 0$, there exists $r > 0$ sufficiently small such that

$$\begin{aligned} 0 &< \sup_{(\xi, t) \in K} \int_{t-r}^{t} \int_{\rho(\xi, \eta) < \sqrt{r}} \Gamma_A(\xi, t; \eta, s) \left|V(\eta, s)\right| d\eta\, ds \\ &\le C_\Lambda \sup_{(\xi, t) \in K} \int_{t-r}^{t} \int_{\rho(\xi, \eta) < \sqrt{r}} (t - s)^{-\frac{Q}{2}} e^{-\frac{\rho^2(\xi, \eta)}{C_\Lambda (t-s)}} \left|V(\eta, s)\right| d\eta\, ds < \varepsilon. \end{aligned} \tag{3.12}$$

For $\rho(\xi,\xi_0)\vee|t-t_0|^{\frac{1}{2}}\leq\frac{\sqrt{r}}{8}$, we have

$$\begin{aligned}K_A(\xi,t;\xi_0,t_0)&=\int_{-\infty}^{t\vee t_0}\int_G\left|\Gamma_A(\xi,t;\eta,s)-\Gamma_A(\xi_0,t_0;\eta,s)\right|\cdot\left|V(\eta,s)\right|d\eta\,ds\\&=\iint_{B((\xi_0,t_0),\frac{\sqrt{r}}{2})}\cdots d\eta\,ds+\iint_{B^c((\xi_0,t_0),\frac{\sqrt{r}}{2})}\cdots d\eta\,ds\\&\triangleq I_1(\xi,t;\xi_0,t_0)+I_2(\xi,t;\xi_0,t_0).\end{aligned}$$

When $\rho(\xi,\xi_0)\vee|t-t_0|^{\frac{1}{2}}\leq\frac{\sqrt{r}}{8}$, one gets from (3.12)

$$\begin{aligned}I_1(\xi,t;\xi_0,t_0)\leq&\iint_{B((\xi,t),\sqrt{r})}\Gamma_A(\xi,t;\eta,s)\left|V(\eta,s)\right|d\eta\,ds\\&+\iint_{B((\xi_0,t_0),\sqrt{r})}\Gamma_A(\xi_0,t_0;\eta,s)\left|V(\eta,s)\right|d\eta\,ds<2\varepsilon.\end{aligned}$$

By Lemma 3.1, for $\rho(\xi,\xi_0)\vee|t-t_0|^{\frac{1}{2}}\leq\frac{\sqrt{r}}{8}$, we have

$$I_2(\xi,t;\xi_0,t_0)\leq C\frac{d((\xi,t)(\xi_0,t_0))}{r^{\frac{1}{2}}}N_c(V),$$

which vanishes as $\rho(\xi,\xi_0)\vee|t-t_0|^{\frac{1}{2}}\to 0$. This ends the proof. □

Remark 3.4 Similarly, for $(\eta,s),(\eta_0,s_0)\in G\times\mathbb{R}$ and $A\in M_\Lambda$, let

$$K_A^*(\eta,s;\eta_0,s_0)=\int_{s\wedge s_0}^{+\infty}\int_G\left|\Gamma_A^*(\eta,s;\xi,t)-\Gamma_A^*(\eta_0,s_0;\xi,t)\right|\cdot\left|V(\xi,t)\right|d\xi\,dt.$$

If $V\in P_{\mathrm{loc}}^\infty$, then $\lim_{\rho(\eta,\eta_0)\vee|s-s_0|^{\frac{1}{2}}\to 0}K_A^*(\eta,s;\eta_0,s_0)=0$, where $\Gamma_A^*(\eta,s;\xi,t)=\Gamma_A(\xi,t;\eta,s)$ is the fundamental solution of the formal adjoint operator to L_A.

We will use the following lemma established in Lemma 6.1 of [7].

Proposition 3.5 *Suppose $0<a<b$. There exist positive constants $C_{a,b}$ and c depending only on a and b such that*

$$\text{(i)}\quad\int_s^t\int_G\Gamma_a(\xi,t;\zeta,\tau)\left|V(\zeta,\tau)\right|\Gamma_b(\zeta,\tau;\eta,s)\,d\zeta\,d\tau\leq C_{a,b}N_c(V)\Gamma_a(\xi,t;\eta,s);$$

$$\text{(ii)}\quad\int_s^t\int_G\Gamma_b(\xi,t;\zeta,\tau)\left|V(\zeta,\tau)\right|\Gamma_a(\zeta,\tau;\eta,s)\,d\zeta\,d\tau\leq C_{a,b}N_c(V)\Gamma_a(\xi,t;\eta,s).$$

Applying an analogous proof to that of Lemma 2.1(a) in [7], we obtain the following result, which in the Euclidean setting was first given by Zhang [4].

Lemma 3.6 *Given $a>0$, let $h_a(\xi,t)$ be as in* (2.8), *where u_0 is a bounded nonnegative function. Then for every given $p>1$ and $0<\gamma<1$, there exists a constant $C(p,\gamma)$ such that*

$$h_a^p(\xi,t)\leq C(p,\gamma)\|u_0\|_\infty^{p-1}h_a\left(\xi,\frac{t}{p\gamma}\right)\tag{3.13}$$

for all $t>0$.

Proof Clearly

$$h_a(\xi,t) = \int_G t^{-\frac{Q}{2p}} e^{-\frac{a\gamma\rho^2(\xi,\eta)}{t}} u_0(\eta) t^{-\frac{Q}{2q}} e^{-\frac{a(1-\gamma)\rho^2(\xi,\eta)}{t}}\, d\eta,$$

where q is the conjugate of p. Using the Hölder inequality and the fact that

$$\int_G t^{-\frac{Q}{2}} e^{-\frac{aq(1-\gamma)\rho^2(\xi,\eta)}{t}}\, d\eta \le C,$$

we have

$$\begin{aligned} h_a^p(\xi,t) &\le \int_G t^{-\frac{Q}{2}} e^{-\frac{pa\gamma\rho^2(\xi,\eta)}{t}} u_0^p(\eta)\, d\eta \left[\int_G t^{-\frac{Q}{2}} e^{-\frac{aq(1-\gamma)\rho^2(\xi,\eta)}{t}}\, d\eta\right]^{\frac{p}{q}} \\ &\le C(p,\gamma)\|u_0\|_\infty^{p-1} h_a\left(\xi,\frac{t}{p\gamma}\right). \end{aligned}$$

The last inequality implies (3.13). This proves the claim. □

4 Proof of the existence result

In this section we shall first prove Theorem 2.4.

Proof Suppose that the initial value $u_0 \in C^2(G)$ is nonnegative and satisfies $\|u_0\|_\infty \le b_0$ for some constant $b_0 > 0$. Let

$$h(\xi,t) = \int_G \Gamma_A(\xi,t;\eta,0)u_0(\eta)\, d\eta.$$

Then the function h is continuous on $G \times [0,+\infty)$ and $0 \le h \le \|u_0\|_\infty$. In fact, for any $\varepsilon > 0$, there exists a constant $r > 0$ sufficiently small such that

$$\int_{B_\rho(\xi,\sqrt{r})} \Gamma_A(\xi,t;\eta,0)u_0(\eta)\, d\eta < \varepsilon. \tag{4.1}$$

For any $(\xi,t),(\xi_0,t_0) \in G \times [0,+\infty)$ satisfying $\rho(\xi,\xi_0) \vee |t-t_0|^{\frac{1}{2}} \le \frac{\sqrt{r}}{8}$, we have

$$\begin{aligned} \bigl|h(\xi,t) - h(\xi_0,t_0)\bigr| &\le \int_G \bigl|\Gamma_A(\xi,t;\eta,0) - \Gamma_A(\xi_0,t_0;\eta,0)\bigr|u_0(\eta)\, d\eta \\ &= \int_{B_\rho(\xi_0,\frac{\sqrt{r}}{2})} \cdots\, d\eta + \int_{B^c_\rho(\xi_0,\frac{\sqrt{r}}{2})} \cdots\, d\eta \\ &\triangleq \bar{I}_1(\xi,t;\xi_0,t_0) + \bar{I}_2(\xi,t;\xi_0,t_0). \end{aligned}$$

By (4.1) we obtain

$$\bar{I}_1 \le \int_{B_\rho(\xi,\sqrt{r})} \Gamma_A(\xi,t;\eta,0)u_0(\eta)\, d\eta + \int_{B_\rho(\xi_0,\sqrt{r})} \Gamma_A(\xi_0,t_0;\eta,0)u_0(\eta)\, d\eta < 2\varepsilon.$$

Further, using Lemma 3.1 we have clearly that $\bar{I}_2(\xi,t;\xi_0,t_0) \to 0$ as $\rho(\xi,\xi_0) \vee |t-t_0|^{\frac{1}{2}} \to 0$. It follows that h is continuous on $G \times [0,+\infty)$.

We denote by $C_b(G \times [0,+\infty))$ the set of all bounded continuous functions on $G \times [0,+\infty)$ and note that $(C_b(G \times [0,+\infty)), \|\cdot\|_\infty)$ is a Banach space. For $M > C_\Lambda > 1$, let us define the set

$$S = \{u \in C_b(G \times [0,+\infty)) : M^{-1}h_{C_\Lambda} \le u \le M h_{\frac{1}{C_\Lambda}}\}.$$

Obviously S is a nonempty closed subset of $C_b(G \times [0,+\infty))$. We define an integral operator T on $C_b(G \times [0,+\infty))$ by

$$Tu(\xi,t) = h(\xi,t) + \int_0^t \int_G \Gamma_A(\xi,t;\eta,s) V(\eta,s) u^p(\eta,s)\, d\eta\, ds. \tag{4.2}$$

Since u is bounded, it follows from Lemma 3.3 that $Tu \in C_b(G \times [0,+\infty))$. Moreover, since $u \le M h_{\frac{1}{C_\Lambda}}(\xi,t)$, according to Lemma 3.6 one has

$$\begin{aligned} u^p(\eta,s) &\le M^p \big[h_{\frac{1}{C_\Lambda}}(\eta,s)\big]^p \le C(p,\gamma) M^p \|u_0\|_\infty^{p-1} h_{\frac{1}{C_\Lambda}}\left(\eta, \frac{s}{p\gamma}\right) \\ &\le C(p,\gamma) M^p b_0^{p-1} h_{\frac{1}{C_\Lambda}}\left(\eta, \frac{s}{p\gamma}\right) \end{aligned} \tag{4.3}$$

for every $\gamma \in (0,1)$.

Now taking $\gamma < 1$ such that $p\gamma > 1$, we obtain

$$u^p(\eta,s) \le C(p,\gamma) M^p b_0^{p-1} \int_G \Gamma_{\frac{1}{C_\Lambda}}\left(\eta, \frac{s}{p\gamma}; \zeta, 0\right) u_0(\zeta)\, d\zeta. \tag{4.4}$$

Substituting (4.4) into (4.2) and using Proposition 3.5(i), we have

$$\begin{aligned} &Tu(\xi,t) - h(\xi,t) \\ &\quad\le C(p,\gamma) M^p b_0^{p-1} \int_G \int_0^t \int_G \Gamma_A(\xi,t;\eta,s) V(\eta,s) \Gamma_{\frac{1}{C_\Lambda}}\left(\eta, \frac{s}{p\gamma}; \zeta, 0\right) d\eta\, ds\, u_0(\zeta)\, d\zeta \\ &\quad\le C(p,\gamma) C_\Lambda (p\gamma)^{\frac{Q}{2}} M^p b_0^{p-1} C_{\frac{1}{C_\Lambda}, \frac{p\gamma}{C_\Lambda}} N_c(V) h_{\frac{1}{C_\Lambda}}(\xi,t). \end{aligned} \tag{4.5}$$

It follows that, for b_0 sufficiently small,

$$Tu(\xi,t) \le M h_{\frac{1}{C_\Lambda}}(\xi,t). \tag{4.6}$$

In addition,

$$Tu(\xi,t) \ge h(\xi,t) \ge M^{-1} h_{C_\Lambda}(\xi,t) \tag{4.7}$$

since $V \ge 0$. Equations (4.6) and (4.7) show that $Tu \in S$ and so $TS \subset S$.

Moreover, for all $u, v \in S$, we have

$$Tu - Tv = \int_0^t \int_G \Gamma_A(\xi,t;\eta,s) V(\eta,s) \big[u^p(\eta,s) - v^p(\eta,s)\big]\, d\eta\, ds.$$

By a straightforward computation using (4.3),

$$\begin{aligned}\left|u^p(\eta,s)-v^p(\eta,s)\right| &\le p\left|u(\eta,s)-v(\eta,s)\right|\cdot\left|u^{p-1}(\eta,s)+v^{p-1}(\eta,s)\right|\\ &\le 2pM^{p-1}\left|u(\eta,s)-v(\eta,s)\right|\left[h_{\frac{1}{C_\Lambda}}(\eta,s)\right]^{p-1}\\ &\le 2CpM^{p-1}\|u_0\|_\infty^{p-1}\left|u(\eta,s)-v(\eta,s)\right|,\end{aligned}$$

and hence

$$\begin{aligned}\|Tu-Tv\|_\infty &\le 2CpM^{p-1}b_0^{p-1}\|u-v\|_\infty\int_0^t\int_G\Gamma_A(\xi,t;\eta,s)V(\eta,s)\,d\eta\,ds\\ &\le 2Cp(Mb_0)^{p-1}N_{\frac{1}{C_\Lambda}}(V)\|u-v\|_\infty.\end{aligned}$$

In particular for b_0 small enough, we obtain $\|Tu-Tv\|_\infty\le\frac{1}{2}\|u-v\|_\infty$, which means that T is a $\frac{1}{2}$-Lipschitz mapping form S into itself. Therefore, according to the fixed point theorem there exists $u\in S$ such that $Tu=u$. This completes the proof. □

The next theorem shows that when $V\ge 0$, the condition $N_c(V)<+\infty$ in the definition of class P^∞_{loc} is optimal for Theorem 2.4 to hold.

Theorem 4.1 *Assume that V is defined on $G\times(0,+\infty)$, $V\ge 0$, and the result of Theorem* 2.4 *holds for all $A\in M_\Lambda$. Then we have, for all $c\ge C_\Lambda$,*

$$\sup_{(\xi,t)\in G\times(0,+\infty)}\int_0^t\int_G\Gamma_c(\xi,t;\eta,s)\left|V(\eta,s)\right|d\eta\,ds<+\infty.$$

Proof According to the assumptions, for all $M>C_\Lambda>1$ there is a constant b_0 such that for each nonnegative $u_0\in C^2(G)$ with $\|u_0\|_\infty\le b_0$, there exists a solution u of the integral equation

$$u(\xi,t)=\int_G\Gamma_A(\xi,t;\eta,0)u_0(\eta)\,d\eta+\int_0^t\int_G\Gamma_A(\xi,t;\eta,s)V(\eta,s)u^p(\eta,s)\,d\eta\,ds$$

satisfying

$$M^{-1}\int_G h_{C_\Lambda}(\xi,t;\eta,0)u_0(\eta)\,d\eta\le u(\xi,t)\le M\int_G h_{\frac{1}{C_\Lambda}}(\xi,t;\eta,0)u_0(\eta)\,d\eta.$$

For $u_0\equiv b_0$, we obtain $u(\xi,t)=b_0+\int_0^t\int_G\Gamma_A(\xi,t;\eta,s)V(\eta,s)u^p(\eta,s)\,d\eta\,ds$ and $M^{-1}b_0C_1\le u(\xi,t)\le Mb_0C_2$, which implies

$$\begin{aligned}&\left(M^{-1}b_0C_1\right)^p\int_0^t\int_G\Gamma_A(\xi,t;\eta,s)V(\eta,s)\,d\eta\,ds\\ &\quad\le\int_0^t\int_G\Gamma_A(\xi,t;\eta,s)V(\eta,s)u^p(\eta,s)\,d\eta\,ds\le(MC_2-1)b_0.\end{aligned}$$

Therefore

$$\sup_{(\xi,t)\in G\times(0,\infty)}\int_0^t\int_G\Gamma_A(\xi,t;\eta,s)V(\eta,s)\,d\eta\,ds\le(MC_2-1)b_0^{1-p}M^pC_1^{-p}<+\infty.$$

Combining this with (2.5), we deduce the result. □

5 Proof of the asymptotic behavior

This section is divided into two parts. In the first part we establish the global Gaussian upper bounds for fundamental solutions of certain linear degenerate parabolic equations. We emphasize that the parameters in the bounds are independent of time. In the next part we will prove Theorem 2.5 by means of the newly obtained Gaussian estimates.

Lemma 5.1 *Let $A \in M_\Lambda$, $V(\xi,t) \in \Gamma^\beta(G \times \mathbb{R})$ $(0 < \beta < 1)$, and Γ be the fundamental solution of the degenerate parabolic operator*

$$\sum_{i,j=1}^{m} a_{ij} X_i X_j u(\xi,t) - \frac{\partial}{\partial t} u(\xi,t) + V(\xi,t) u(\xi,t).$$

Suppose that $N_{c'}(V)$ is sufficiently small for a suitable $c' > 0$. Then there exist positive constants a and C such that

$$\Gamma(\xi,t;\eta,0) \leq C t^{-\frac{Q}{2}} e^{-a\frac{\rho^2(\xi,\eta)}{t}}$$

for all $t > 0$ and $\xi,\eta \in G$.

Proof Without loss of generality, we assume that V is bounded and supported in $G \times [0,T]$, where T is a positive number. The general case can be covered by a limiting argument. What is important is to make sure all constants are independent of T.

According to the result in Bramanti *et al.* [13], there are positive constants $a < \frac{1}{C_\Lambda}$ and $B = B(T)$ such that

$$\Gamma(\xi,t;\eta,s) \leq B(t-s)^{-\frac{Q}{2}} e^{-a\frac{\rho^2(\xi,\eta)}{t-s}} \tag{5.1}$$

for all $\xi,\eta \in G$ and $s < t$. We suppose that B is the smallest positive number satisfying (5.1). We claim that such a B does exist by our extra assumption that $V(\xi,t) = 0$ and thus $\Gamma = \Gamma_A$ if $t > T$. For $t \leq T$, the claim can be checked by showing that B depends on V only in the form of $N_{c'}(V)$.

By the Duhamel principle, (2.5) and (5.1), we have, for all $\xi,\eta \in G$ and $s < t$,

$$\begin{aligned}\Gamma(\xi,t;\eta,s) &= \Gamma_A(\xi,t;\eta,s) + \int_s^t \int_G \Gamma(\xi,t;\zeta,\tau)\big|V(\zeta,\tau)\big|\Gamma_A(\zeta,\tau;\eta,s)\,d\zeta\,d\tau \\ &\leq \Gamma_A(\xi,t;\eta,s) + \int_s^t \int_G \frac{B}{(t-\tau)^{\frac{Q}{2}}} e^{-a\frac{\rho^2(\xi,\zeta)}{t-\tau}} \big|V(\zeta,\tau)\big| \\ &\quad \times \frac{C_\Lambda}{(\tau-s)^{\frac{Q}{2}}} e^{-\frac{\rho^2(\zeta,\eta)}{C_\Lambda(\tau-s)}}\,d\zeta\,d\tau.\end{aligned}$$

We then derive from Proposition 3.5

$$\begin{aligned}\Gamma(\xi,t;\eta,s) &\leq \frac{C_\Lambda}{(t-s)^{\frac{Q}{2}}} e^{-\frac{\rho^2(\xi,\eta)}{C_\Lambda(t-s)}} + BC_{a,\frac{1}{C_\Lambda}} N_{c'}(V) \frac{1}{(t-s)^{\frac{Q}{2}}} e^{-a\frac{\rho^2(\xi,\eta)}{t-s}} \\ &\leq \big[C_\Lambda + BC_{a,\frac{1}{C_\Lambda}} N_{c'}(V)\big] \frac{1}{(t-s)^{\frac{Q}{2}}} e^{-a\frac{\rho^2(\xi,\eta)}{t-s}}.\end{aligned}$$

Hence, by the definition of B, we obtain

$$B \leq C_\Lambda + BC_{a,\frac{1}{C_\Lambda}} N_{c'}(V).$$

When $C_{a,\frac{1}{C_\Lambda}} N_{c'}(V) < \frac{1}{2}$ we have $B \leq 2C_\Lambda$. This finishes the proof. □

Now we are ready to give the proof of Theorem 2.5.

Proof First we recall the assumptions imposed on the initial function u_0:

$$u_0 \in C^2(G), \qquad 0 < \alpha_1 \leq u_0 \leq \alpha_2,$$

where α_2 is a small number so that (1.1) has global positive solutions by Theorem 2.4. The reason to impose a positive lower bound α_1 for u_0 is to guarantee that the equilibrium solution is non-trivial.

Since $V, \sum_{i,j=1}^m a_{ij} X_i X_j u_0 \in P_{\mathrm{loc}}^\infty$ by assumption, we have

$$\sup_\xi \int_G \frac{|\sum_{i,j=1}^m a_{ij} X_i X_j u_0(\eta)| + |V(\eta)|}{\rho^{Q-2}(\xi,\eta)}\, d\eta < +\infty, \tag{5.2}$$

which will be proved in Proposition A.2 in the Appendix.

We temporarily assume that V is smooth. Hence the solution $u(\xi,t)$ of (1.1) is smooth. Let us write $w = u_t$. Differentiating (1.1), we find that w solves

$$\begin{cases} \sum_{i,j=1}^m a_{ij} X_i X_j w - \frac{\partial}{\partial t} w + pV(\xi) u^{p-1}(\xi,t) w(\xi,t) = 0, & (\xi,t) \in G \times (0,+\infty), \\ w(\xi,0) = \sum_{i,j=1}^m a_{ij} X_i X_j u_0(\xi) + V(\xi) u_0^p(\xi), & \xi \in G, Q \geq 3. \end{cases} \tag{5.3}$$

Let $V_1(\xi,t) = pV(\xi)u^{p-1}(\xi,t)$ and let $\bar{\Gamma}$ be the fundamental solution of the operator $\sum_{i,j=1}^m a_{ij} X_i X_j - \partial_t + V_1$. Then

$$w(\xi,t) = \int_G \bar{\Gamma}(\xi,t;\eta,0) \left[\sum_{i,j=1}^m a_{ij} X_i X_j u_0(\eta) + V(\eta) u_0^p(\eta) \right] d\eta. \tag{5.4}$$

When u_0 is small, we know by Theorem 2.4 that u is small, and so $N_c(V_1) \leq p \sup |u|^{p-1} \times N_c(V)$ is small. From Lemma 5.1, there exist positive constants a, C independent of t, ξ, and η such that

$$\bar{\Gamma}(\xi,t;\eta,0) \leq Ct^{-\frac{Q}{2}} e^{-a\frac{\rho^2(\xi,\eta)}{t}}. \tag{5.5}$$

Substituting (5.5) into (5.4), we obtain

$$\left|u_t(\xi,t)\right| \leq \int_G Ct^{-\frac{Q}{2}} e^{-a\frac{\rho^2(\xi,\eta)}{t}} \left[\sum_{i,j=1}^m a_{ij} X_i X_j u_0(\eta) + V(\eta) u_0^p(\eta) \right] d\eta. \tag{5.6}$$

For any $t > 0$, by integrating the above inequality from t to $+\infty$ we obtain, via the Fubini theorem and (5.2),

$$\begin{aligned}&\int_t^{+\infty}\left|u_s(\xi,s)\right|ds\\&\quad\le C\int_G\frac{1}{\rho(\xi,\eta)^{Q-2}}\left[\left|\sum_{i,j=1}^m a_{ij}X_iX_ju_0(\eta)\right|+\left|V(\eta)u_0^p(\eta)\right|\right]d\eta<+\infty,\end{aligned}\tag{5.7}$$

where we have used the inequality $\int_t^{+\infty}s^{-\frac{Q}{2}}e^{-a\frac{\rho^2(\xi,\eta)}{s}}\,ds\le\frac{C}{\rho^{Q-2}(\xi,\eta)}$.

Now we define the function

$$u_\infty(\xi)=\lim_{t\to+\infty}u(\xi,t).\tag{5.8}$$

We claim that the rate of convergence in (5.8) depends only on ξ and the rate of convergence of the following limit:

$$\lim_{M\to+\infty}\sup_\xi\int_{\rho(\eta)\ge M}\frac{|\sum_{i,j=1}^m a_{ij}X_iX_ju_0(\eta)|+|V(\eta)|}{\rho^{Q-2}(\xi,\eta)}\,d\eta=0.$$

Here is a proof of the claim. Obviously, for a fixed ξ and any $\varepsilon>0$, there exists a constant $M>0$ such that

$$\int_{\rho(\xi,\eta)\ge M}\frac{|\sum_{i,j=1}^m a_{ij}X_iX_ju_0(\eta)|+|V(\eta)|}{\rho(\xi,\eta)^{Q-2}}\,d\eta<\frac{\varepsilon}{2}.$$

From (5.6), we have

$$\begin{aligned}\left|u(\xi,t)-u_\infty(\xi)\right|&\le\int_t^{+\infty}\left|u_s(\xi,s)\right|ds\\&\le C\int_t^{+\infty}\int_G s^{-\frac{Q}{2}}e^{-a\frac{\rho^2(\xi,\eta)}{s}}\left[\left|\sum_{i,j=1}^m a_{ij}X_iX_ju_0(\eta)\right|+\left|V(\eta)\right|\right]d\eta\,ds\\&\le C\int_t^{+\infty}\int_{\rho(\xi,\eta)\le M}\cdots\,d\eta\,ds+C\int_t^{+\infty}\int_{\rho(\xi,\eta)\ge M}\cdots\,d\eta\,ds\\&\le C\int_t^{+\infty}\int_{\rho(\xi,\eta)\le M}s^{-\frac{Q}{2}}\frac{M^{Q-2}}{\rho^{Q-2}(\xi,\eta)}\left[\left|\sum_{i,j=1}^m a_{ij}X_iX_ju_0(\eta)\right|+\left|V(\eta)\right|\right]d\eta\,ds\\&\quad+C\int_{\rho(\xi,\eta)\ge M}\frac{1}{\rho^{Q-2}(\xi,\eta)}\left[\left|\sum_{i,j=1}^m a_{ij}X_iX_ju_0(\eta)\right|+\left|V(\eta)\right|\right]d\eta\,ds\\&\le Ct^{-\frac{Q-2}{2}}M^{Q-2}+\frac{C\varepsilon}{2}<C\varepsilon,\end{aligned}\tag{5.9}$$

when t is sufficiently large. This proves the claim.

From (5.6) we derive a pointwise estimate on $|u_t|$:

$$\left|u_t(\xi,t)\right|\le\frac{C}{t}\int_G\frac{1}{\rho^{Q-2}(\xi,\eta)}\left[\left|\sum_{i,j=1}^m a_{ij}X_iX_ju_0(\eta)\right|+\left|V(\eta)u_0^p(\eta)\right|\right]d\eta,\tag{5.10}$$

by using the obvious inequality $t^{-\frac{Q}{2}}e^{-a\frac{\rho^2(\xi,\eta)}{t}}\le\frac{C}{t\rho^{Q-2}(\xi,\eta)}$.

It remains to prove that $u = u_\infty(\xi)$ is a non-trivial positive solution of the subelliptic equation. Given any $\phi \in C_0^\infty(G)$, we have

$$\int_G u(\xi,t)\sum_{i,j=1}^m a_{ij}X_iX_j\phi(\xi)\,d\xi - \int_G u_t(\xi,t)\phi(\xi)\,d\xi + \int_G V(\xi)u^p(\xi,t)\phi(\xi)\,d\xi = 0.$$

Since u is bounded, we obtain by (5.8), (5.10), and the dominated convergence theorem

$$\int_G u_\infty(\xi)\sum_{i,j=1}^m a_{ij}X_iX_j\phi(\xi)\,d\xi + \int_G V(\xi)u_\infty^p(\xi)\phi(\xi)\,d\xi = 0.$$

According to Theorem 2.4 we find that $u = u(\xi,t)$ is bounded away from zero when $u_0 \geq \alpha_1 > 0$. Therefore, u_∞ is a positive solution of the subelliptic equation.

Now we set $V \in P_{\mathrm{loc}}^\infty$. We claim that there is a sequence of smooth functions $\{V_n\}$ such that $V_n \to V$ a.e. as $n \to \infty$, and for any domain $D \subset G$,

$$\sup_{\xi\in G}\int_D \frac{|V_n(\eta)|}{\rho^{Q-2}(\xi,\eta)}\,d\eta \leq \sup_{\xi\in G}\int_D \frac{|V(\eta)|}{\rho^{Q-2}(\xi,\eta)}\,d\eta. \tag{5.11}$$

We prove the claim as follows. Let J be the standard mollifier. Define

$$V_n(\eta) = \int_{\rho(\zeta)\leq 1} J(\zeta)V\bigl(\bigl(\delta_{\frac{1}{n}}(\zeta)\bigr)^{-1}\cdot\eta\bigr)\,d\zeta.$$

Then we only need to check (5.11). Clearly

$$\begin{aligned}\int_D \frac{|V_n(\eta)|}{\rho^{Q-2}(\xi,\eta)}\,d\eta &\leq \int_D \frac{1}{\rho^{Q-2}(\xi,\eta)}\int_{\rho(\zeta)\leq 1} J(\zeta)\bigl|V\bigl(\bigl(\delta_{\frac{1}{n}}(\zeta)\bigr)^{-1}\cdot\eta\bigr)\bigr|\,d\zeta\,d\eta\\ &= \int_{\rho(\zeta)\leq 1} J(\zeta)\int_D \frac{|V(\eta)|}{\rho^{Q-2}(\xi,\delta_{\frac{1}{n}}(\zeta)\cdot\eta)}\,d\eta\,d\zeta.\end{aligned}$$

Hence

$$\sup_{\xi\in G}\int_D \frac{|V_n(\eta)|}{\rho^{Q-2}(\xi,\eta)}\,d\eta \leq \sup_{\xi\in G}\int_D \frac{|V(\eta)|}{\rho^{Q-2}(\xi,\eta)}\,d\eta\int_{\rho(\zeta)\leq 1} J(\zeta)\,d\zeta.$$

This proves the claim.

The previous argument implies that, for each n, there is a global solution u_n of (1.1) when V is replaced by the smooth function V_n. Moreover, $\lim_{t\to\infty} u_n(\xi,t) = u_{n,\infty}(\xi)$ pointwise. The claim about the rate of convergence of (5.8) and (5.11) show that the convergence is uniform with respect to n. Therefore, a subsequence, still called $\{u_n\}$, converges uniformly to a function $u(\xi,t)$ in any compact subset of $G\times(0,+\infty)$. Following the previous argument, we find that u is a positive solution of (1.1) and $u = u(\xi,t)$ converges pointwise to a $u_\infty(\xi)$ as $t \to +\infty$, and $u_\infty(\xi)$ is a positive solution of (2.9). □

Appendix

The objective of the section is to give two propositions about the class P_{loc}^∞, among which Proposition A.2 was used to obtain (5.2) in Section 5. The corresponding results in the Euclidean case were first given in [6].

For a measurable function V on $G \times \mathbb{R}$, we put

$$p_A^V(\xi,t) = \int_{-\infty}^{t} \int_G \Gamma_A(\xi,t;\eta,s) \big| V(\eta,s) \big| \, d\eta \, ds \tag{A.1}$$

and

$$p_A^{*V}(\eta,s) = \int_{s}^{+\infty} \int_G \Gamma_A(\xi,t;\eta,s) \big| V(\xi,t) \big| \, d\xi \, dt. \tag{A.2}$$

By Lemma 3.3 and Remark 3.4 we obviously have the following result.

Proposition A.1 *Let $V \in L^1(G \times \mathbb{R}) \cap P^\infty_{\mathrm{loc}}$. Then for all $A \in M_\Lambda$, the potentials $p_A^V, p_A^{*V} \in C_b(G \times \mathbb{R})$.*

Proposition A.2 *Let $V(\xi,t) = V(\xi) \in L^1(G)$. Then $V \in P^\infty_{\mathrm{loc}}$ if and only if the potential $p^V(\xi) := \int_G \frac{|V(\eta)|}{\rho^{Q-2}(\xi,\eta)} d\eta \in C_b(G)$.*

Proof We first give the proof of the sufficiency. Assume that $V(\xi) \in L^1(G)$ and $p^V(\xi) \in C_b(G)$. We then obtain

$$N_c(V) \le C \sup_{\xi \in G} \int_G \frac{|V(\eta)|}{\rho^{Q-2}(\xi,\eta)} \, d\eta + C \sup_{\eta \in G} \int_G \frac{|V(\xi)|}{\rho^{Q-2}(\xi,\eta)} \, d\xi < C_0$$

for some constant $C_0 > 0$.

For simplicity we write, for $r \in (0,1)$,

$$p_r^V(\xi) = \int_{B_\rho(\xi,\sqrt{r})} \frac{|V(\eta)|}{\rho^{Q-2}(\xi,\eta)} \, d\eta, \qquad q_r^V(\xi) = \int_{B^c_\rho(\xi,\sqrt{r})} \frac{|V(\eta)|}{\rho^{Q-2}(\xi,\eta)} \, d\eta.$$

We will prove that $q_r^V(\xi)$ is continuous. Indeed, for $\xi_0 \in G$, when $\rho(\xi,\xi_0) < \frac{\sqrt{r}}{8}$,

$$\big| q_r^V(\xi) - q_r^V(\xi_0) \big| \le J_1(\xi,\xi_0) + J_2(\xi,\xi_0),$$

where

$$J_1(\xi,\xi_0) = \int_{B^c_\rho(\xi,\sqrt{r})} \left| \frac{1}{\rho^{Q-2}(\xi,\eta)} - \frac{1}{\rho^{Q-2}(\xi_0,\eta)} \right| \cdot \big| V(\eta) \big| \, d\eta$$

and

$$J_2(\xi,\xi_0) = \int |1_{B^c_\rho(\xi,\sqrt{r})} - 1_{B^c_\rho(\xi_0,\sqrt{r})}| \frac{|V(\eta)|}{\rho^{Q-2}(\xi_0,\eta)} \, d\eta.$$

Recall that $Q \ge 3$, we can calculate via the obvious inequality

$$\begin{aligned} &\left| \frac{1}{\rho^{Q-2}(\xi,\eta)} - \frac{1}{\rho^{Q-2}(\xi_0,\eta)} \right| \\ &\quad \le (Q-2)\rho(\xi,\xi_0) \left[\frac{1}{\rho^{Q-2}(\xi,\eta)\rho(\xi_0,\eta)} + \frac{1}{\rho(\xi,\eta)\rho^{Q-2}(\xi_0,\eta)} \right]. \end{aligned}$$

Therefore

$$0 < J_1(\xi,\xi_0) \le C(Q-2)\frac{\rho(\xi,\xi_0)}{\sqrt{r}}\int_{B_\rho^c(\xi,\sqrt{r})}\left|\frac{1}{\rho^{Q-2}(\xi,\eta)}+\frac{1}{\rho^{Q-2}(\xi_0,\eta)}\right|\cdot\left|V(\eta)\right|d\eta.$$

On the other hand, by the dominated convergence theorem, we also have

$$J_2(\xi,\xi_0)\to 0 \quad \text{as } \rho(\xi,\xi_0)\to 0.$$

Hence $q_r^V(\xi)$ is continuous. We then have $p_r^V(\xi) = p^V(\xi) - q_r^V(\xi)$ is continuous and $\lim_{r\to 0} p_r^V(\xi) = 0$. So, by the Dini theorem, $\lim_{r\to 0}\sup_{\xi\in K'} p_r^V(\xi) = 0$ for any compact subset $K'\subset G$. It assists us to deduce that

$$\begin{aligned}
&\lim_{r\to 0}\left\{\sup_{(\xi,t)\in K}\int_{t-r}^{t}\int_{\rho(\xi,\eta)<\sqrt{r}}\Gamma_c(\xi,t;\eta,s)\left|V(\eta)\right|d\eta\,ds\right.\\
&\qquad\left.+\sup_{(\eta,s)\in K}\int_{s}^{s+r}\int_{\rho(\xi,\eta)<\sqrt{r}}\Gamma_c(\xi,t;\eta,s)\left|V(\xi)\right|d\xi\,dt\right\}\\
&\quad\le\lim_{r\to 0}\left\{C\sup_{\xi\in K'}\int_{\rho(\xi,\eta)<\sqrt{r}}\frac{|V(\eta)|}{\rho^{Q-2}(\xi,\eta)}\,d\eta\right.\\
&\qquad\left.+C\sup_{\eta\in K'}\int_{\rho(\xi,\eta)<\sqrt{r}}\frac{|V(\xi)|}{\rho^{Q-2}(\xi,\eta)}\,d\xi\right\}=0
\end{aligned}$$

for any compact $K\subset G\times\mathbb{R}$. Thus $V\in P^\infty_{\mathrm{loc}}$.

We can now proceed to the proof of the necessity. Let $V\in L^1(G)\cap P^\infty_{\mathrm{loc}}$. By (2.3) and the general properties of Lebesgue integral, there exists a constant $C>0$ such that

$$p^V(\xi)\le\int_{\rho(\xi,\eta)\le 1}\frac{|V(\eta)|}{\rho^{Q-2}(\xi,\eta)}\,d\eta+\int_{\rho(\xi,\eta)\ge 1}\left|V(\eta)\right|d\eta<C.$$

Let $\varepsilon>0$. Then there exists $r>0$ such that

$$\int_{B_\rho(\xi,2\sqrt{r})}\frac{|V(\eta)|}{\rho^{Q-2}(\xi,\eta)}\,d\eta<\varepsilon.$$

Therefore, when $\rho(\xi,\xi_0)<\frac{\sqrt{r}}{8}$, it follows that

$$\begin{aligned}
\left|p^V(\xi)-p^V(\xi_0)\right| &\le\int_G\left|\frac{1}{\rho^{Q-2}(\xi,\eta)}-\frac{1}{\rho^{Q-2}(\xi_0,\eta)}\right|\cdot\left|V(\eta)\right|d\eta\\
&\le\int_{B_\rho(\xi,\sqrt{r})}\cdots\,d\eta+\int_{B_\rho^c(\xi_0,\frac{\sqrt{r}}{2})}\cdots\,d\eta\\
&\triangleq I_1(\xi,\xi_0)+I_2(\xi,\xi_0),
\end{aligned}$$

where

$$I_1(\xi,\xi_0)\le\int_{B_\rho(\xi,\sqrt{r})}\frac{|V(\eta)|}{\rho^{Q-2}(\xi,\eta)}\,d\eta+\int_{B_\rho(\xi_0,2\sqrt{r})}\frac{|V(\eta)|}{\rho^{Q-2}(\xi_0,\eta)}\,d\eta<2\varepsilon$$

and

$$I_2(\xi,\xi_0) \le C(Q-2)\frac{\rho(\xi,\xi_0)}{\sqrt{r}}\int_{B^c_\rho(\xi_0,\frac{\sqrt{r}}{2})}\left[\frac{1}{\rho^{Q-2}(\xi,\eta)}+\frac{1}{\rho^{Q-2}(\xi_0,\eta)}\right]\cdot\left|V(\eta)\right|d\eta \to 0$$

as $\rho(\xi,\xi_0) \to 0$. This finishes the proof. □

Competing interests
The author declares to have no competing interests.

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