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The boundary value condition of an evolutionary $p(x)$-Laplacian equation

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Abstract

Consider an evolutionary equation related to the $p(x)$-Laplacian: $u_t = \operatorname{div}(\rho^\alpha |\nabla u|^{p(x)-2}\nabla u) + \frac{\partial b_i(u,x,t)}{\partial x_i}$, $(x,t) \in Q_T = \Omega \times (0,T)$, which arises from electrorheological fluid mechanics. Since $\rho(x) = \operatorname{dist}(x, \partial\Omega)$, the equation is degenerate on the boundary, one may expect that there is not flux across the boundary. The paper shows that the facts may be unexpected. The paper reviews Fichera-Oleinik theory, then uses the theory to discuss the boundary value condition related to the equation. If $p^- > 2$, the existence and the uniqueness of the solutions are researched. Finally, if $b_i \equiv 0$, the behavior of the solutions near the boundary is studied by the comparison theorem.

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Keywords: evolutionary $p(x)$-Laplacian equation; electrorheological fluid; Fichera-Oleinik theory; boundary degeneracy; boundary value condition

1 Introduction

Consider the equation

$$u_t = \operatorname{div}\left(\rho^\alpha |\nabla u|^{p(x)-2}\nabla u\right) + \sum_{i=1}^{N} \frac{\partial b_i(u,x,t)}{\partial x_i}, \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

where $\Omega \subset R^N$ is a bounded domain with suitably smooth boundary $\partial\Omega$, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ is the distance function from the boundary, $p(x)$ is a measurable function.

If $\alpha = 0$, $b_i(s,x,t) \equiv 0$, (1.1) becomes the evolutionary $p(x)$-Laplacian equation

$$u_t = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.2}$$

which has been researched widely in recent years. The equation emerges in the so-called electrorheological fluid theory, in which $p(x)$ is as a function of the external electromagnetic field (see [1, 2] and the references therein). Certainly, if $p(x) \equiv p$ is a constant in (1.2), it is called the evolutionary p-Laplacian equation and emerges in the non-Newtonian fluid theory. It has been studied by very many papers, we only quote some basic references [3–8] here. By the way, the author also has researched (1.2) for a long time, *cf.* [9–19].

Throughout the paper we denote

$$p^+ = \operatorname*{ess\,sup}_{\overline{\Omega}} p(x), \qquad p^- = \operatorname*{ess\,inf}_{\overline{\Omega}} p(x).$$

To consider the posedness of the solutions to (1.1) and (1.2), a nature basic functional space is $W_0^{1,p(x)}(\Omega)$. Let us introduce some basic definitions and properties of the function spaces with variable exponents; for more details, see [20–23].

1. $L^{p(x)}(\Omega)$ space,

$$L^{p(x)}(\Omega) = \left\{u : u \text{ is a measurable real-valued function, } \int_\Omega \left|u(x)\right|^{p(x)} \mathrm{d}x < \infty\right\}$$

is equipped with the following Luxemburg norm:

$$|u|_{L^{p(x)}}(\Omega) = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leq 1\right\}.$$

The space $(L^{p(x)}(\Omega), |\cdot|_{L^{p(x)}(\Omega)})$ is a separable, uniformly convex Banach space.

2. $W^{1,p(x)}(\Omega)$ space,

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

is endowed with the following norm:

$$|u|_{W^{1,p(x)}} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W^{1,p(x)}(\Omega). \tag{1.3}$$

We use $W_0^{1,p(x)}(\Omega)$ to denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}$.

A very important property of the function spaces with variable exponents was found by Zhikov in [24]. He showed that

$$W_0^{1,p(x)}(\Omega) \neq \left\{v \in W_0^{1,p(x)}(\Omega) : v|_{\partial\Omega} = 0\right\} = \mathring{W}^{1,p(x)}(\Omega).$$

Hence, the property of the space $W_0^{1,p(x)}(\Omega)$ is different from that of the case when p is a constant. The following lemma gives some basic properties of $W^{1,p(x)}(\Omega)$.

Lemma 1.1

(i) *The spaces* $(L^{p(x)}(\Omega), |\cdot|_{L^{p(x)}(\Omega)})$, $(W^{1,p(x)}(\Omega), |\cdot|_{W^{1,p(x)}(\Omega)})$ *and* $W_0^{1,p(x)}(\Omega)$ *are reflexive Banach spaces.*

(ii) *We have* $p(x)$*-Hölder's inequality. Let* $q_1(x)$ *and* $q_2(x)$ *be real functions with* $\frac{1}{q_1(x)} + \frac{1}{q_2(x)} = 1$ *and* $q_1(x) > 1$. *Then the conjugate space of* $L^{q_1(x)}(\Omega)$ *is* $L^{q_2(x)}(\Omega)$. *For any* $u \in L^{q_1(x)}(\Omega)$ *and* $v \in L^{q_2(x)}(\Omega)$, *we have*

$$\left|\int_\Omega uv \,\mathrm{d}x\right| \leq 2|u|_{L^{q_1(x)}(\Omega)}|v|_{L^{q_2(x)}(\Omega)}.$$

(iii) *If* $|u|_{L^{p(x)}(\Omega)} = 1$, *then* $\int_\Omega |u|^{p(x)}\,\mathrm{d}x = 1$.
If $|u|_{L^{p(x)}(\Omega)} > 1$, *then* $|u|_{L^{p(x)}}^{p^-} \leq \int_\Omega |u|^{p(x)}\,\mathrm{d}x \leq |u|_{L^{p(x)}}^{p^+}$.
If $|u|_{L^{p(x)}(\Omega)} < 1$, *then* $|u|_{L^{p(x)}}^{p^+} \leq \int_\Omega |u|^{p(x)}\,\mathrm{d}x \leq |u|_{L^{p(x)}}^{p^-}$.

(iv) *If* $p_1(x) \leq p_2(x)$, *then*

$$L^{p_1(x)}(\Omega) \supset L^{p_2(x)}(\Omega).$$

(v) *If* $p_1(x) \leq p_2(x)$, *then*

$$W^{1,p_2(x)}(\Omega) \hookrightarrow W^{1,p_1(x)}(\Omega).$$

(vi) *We have the* $p(x)$*-Poincaré's inequality. If* $p(x) \in C(\Omega)$, *then there is a constant* $C > 0$, *such that*

$$|u|_{L^{p(x)}}(\Omega) \leq C|\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega). \tag{1.4}$$

This implies that $|\nabla u|_{L^{p(x)}}(\Omega)$ *and* $|u|_{W^{1,p(x)}(\Omega)}$ *are equivalent norms of* $W_0^{1,p(x)}$.

However, if the exponent $p(x)$ is required to satisfy a logarithmic Hölder continuity condition

$$\bigl|p(x) - p(y)\bigr| \leq \omega\bigl(|x - y|\bigr), \tag{1.5}$$

$\forall x, y \in Q_T$, $|x - y| < \frac{1}{2}$ with

$$\overline{\lim_{s\to 0^+}}\, \omega(s) \ln\left(\frac{1}{s}\right) = C < \infty,$$

then (see [25])

$$W_0^{1,p(x)}(\Omega) = \mathring{W}^{1,p(x)}(\Omega). \tag{1.6}$$

By (1.5) and (1.6), Antontsev-Shmarev [26] established the existence and uniqueness results of (1.2). Since then, using the logarithmic Hölder continuity condition, there were many papers in studying the solvability and the regularity of the equation related to (1.2); for examples, see [27, 28] *etc.* When $p^- > 2$, Peng [29] had studied the existence of the solutions of the equation

$$\operatorname{div}\bigl(|\nabla u|^{p(x)-2}\nabla u\bigr) + f(x, u) = 0, \quad x \in \Omega, \tag{1.7}$$

without the condition (1.3). By adopting a time difference method, Lian *et al.* [30] generalized the method of [29] to study

$$u_t = \operatorname{div}\bigl(|\nabla u|^{p(x)-2}\nabla u\bigr) + f(x, t, u), \quad (x, t) \in Q_T = \Omega \times (0, T), \tag{1.8}$$

provided that f satisfied some restrictions.

In our paper, we want to consider the initial boundary value problem of (1.1). By the paper of Yin and Wang [31], which studied the diffusion equation

$$u_t = \operatorname{div}\bigl(\rho^{\alpha}|\nabla u|^{p-2}\nabla u\bigr), \quad (x, t) \in Q_T = \Omega \times (0, T), \tag{1.9}$$

we know that the initial value condition

$$u|_{t=0} = u_0(x), \quad x \in \Omega, \tag{1.10}$$

is always required. But due to the degeneracy of the diffusion ρ^α on the boundary, whether we can require the usual boundary value condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.11}$$

is uncertain. From the point of physics, if we regard (1.9) as a heat transfer equation, since the diffusion coefficient vanishes on the boundary, it seems that there is not heat flux across the boundary. However, Yin and Wang [31] proved that, if $\alpha \geq p-1$, the existence and uniqueness of solutions can be obtained without any boundary value condition. In other words, the solution of the equation is completely controlled by the initial value condition. Thus, whether there is heat flux across the boundary is unknown actually.

The first aim of our paper is to probe how to give a suitable boundary value condition of (1.1). We first review Fichera-Oleinik theory, and then we use it to discuss the suitable boundary value condition related to (1.1). The main point is that, to assure the posedness of the solutions to (1.1), instead of the whole boundary value condition (1.11), we can require only a partial boundary value condition,

$$u(x,t) = 0, \quad (x,t) \in \Sigma_p \times (0,T), \tag{1.12}$$

where Σ_p is a subset of $\partial\Omega$. In some cases, Σ_p can be expressed clearly, whereas in some other cases, it is difficult to write out its explicit formulas.

We assume the following.

(A) We call a bounded domain Ω has the integral non-singularity, if there are constants $\alpha > 0$, $p^- > 2$, such that

$$\int_\Omega \rho^{-\frac{2\alpha}{p^- -2}}\,\mathrm{d}x \leq c. \tag{1.13}$$

(B) For any $i \in \{1,2,\dots,N\}$, $b_i(s,x,t)$ is a C^1 function on $\mathbb{R} \times \overline{\Omega} \times [0,T]$, and there are constants β, c such that

$$\bigl|b_i(s,x,t)\bigr| \leq c|s|^{1+\beta}, \qquad \bigl|b_{is}(s,x,t)\bigr| \leq c|s|^{\beta}, \qquad \bigl|b_{ix_i}(s,x,t)\bigr| \leq c, \tag{1.14}$$

where $b_{is} = \frac{\partial b_i}{\partial s}$, $b_{ix_i} = \frac{\partial b_i}{\partial x_i}$ as usual.

The main results in our paper are the following theorems.

Theorem 1.2 *If* $p^- > 2$ *and* $\alpha < \frac{p^- -2}{2}$, *the bounded domain* Ω *is with the integral non-singularity* (A), $b_i(s,x,t)$ *and its partial derivatives satisfy the condition* (B), *and* u_0 *satisfies*

$$u_0 \in L^\infty(\Omega), \qquad \rho^\alpha |\nabla u_0|^{p^+} \in L^1(\Omega), \tag{1.15}$$

then (1.1) *with initial boundary values* (1.10)-(1.12) *has a solution. In particular, when* $\Sigma_p = \partial\Omega$, *then the solution is unique.*

Remark 1.3 The explicit formula of Σ_p of (1.12) used in the theorem is listed in Section 4.

Theorem 1.4 *If $p^- > 2$ and $\alpha > 1$, let u be a viscous solution of* (1.1), *then there are constants c_1, c_2 such that*

$$c_1\rho(x) \le u(x,t) \le c_2\rho(x), \tag{1.16}$$

when x is near the boundary.

2 The stage of formal operation

Consider the second order equation with the form

$$L(u) = a^{rs}(x)u_{x_r x_s} + b^r(x)u_{x_r} + c(x)u = f(x). \tag{2.1}$$

If for any real vector $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ and any point $x \in \Omega$,

$$a^{rs}\xi_r\xi_s \ge 0 \tag{2.2}$$

is true, then it is called the second order equation with nonnegative characteristic form in Ω. Obviously, it entails an elliptic equation, a parabolic equation, a first-order equation (the case $a^{rs}\xi_r\xi_s \equiv 0$), ultra parabolic equation, Brown motion equation, Tricomi equation on the half-plane and so on.

Consider the first boundary value problem of (2.1) in Ω, Fichera [32] first made thorough research in this problem. In what follows, we use the notations in [33, 34], especially, the pairs of the indices imply summation. Suppose on $\overline{\Omega} = \Omega \cup \Sigma$, all the points x and all $\xi \in R^n$ satisfy the condition (2.2), Ω is appropriately smooth, $a^{rs} \in C^2(\Omega)$, $b^r \in C^1(\Omega)$, $c \in C^0(\Omega)$. Let $\{n_s\}$ be the unit inner normal vector of $\partial\widetilde{\Omega}$ and denote that

$$\Sigma^0 = \{x \in \Sigma : a^{rs}n_r n_s = 0\}.$$

In Σ^0, let us consider the Fichera function

$$b(x) \equiv \left(b^r - a^{rs}_{x_s}\right)n_r. \tag{2.3}$$

We denote

$$\Sigma_1 = \{x \in \Sigma^0 : \left(b_r - a^{rs}_{x_s}\right)n_r > 0\},$$
$$\Sigma_2 = \{x \in \Sigma^0 : \left(b_r - a^{rs}_{x_s}\right)n_r < 0\}$$

and

$$\Sigma_0 = \{x \in \Sigma^0 : \left(b_r - a^{rs}_{x_s}\right)n_r = 0\},$$

$\Sigma \setminus \Sigma^0$ is denoted as Σ_3.

The first boundary value problem of (2.1) is quoted as follows: in $\overline{\Omega} = \Omega \cup \Sigma$, to find a function u such that

$$L(u) = f(x), \quad x \in \Omega, \tag{2.4}$$

$$u = g, \quad x \in \Sigma_2 \cup \Sigma_3, \tag{2.5}$$

where f is a given function, and g is a given function on $\Sigma_2 \cup \Sigma_3$. Clearly, if (2.1) is an elliptic equation, then (2.4)-(2.5) is the usual Dirichlet problem. For the cylindrical region, (2.4)-(2.5) consists of the mixed problem, also known as parabolic equations with the initial boundary values.

Now, if we consider (1.1) in our paper,

$$u_t = \operatorname{div}\big(\rho^\alpha |\nabla u|^{p(x)-2}\nabla u\big) + \frac{\partial b_i(u,x,t)}{\partial x_i}, \quad (x,t) \in Q_T, \tag{2.6}$$

then we can rewrite it as

$$u_t = a^{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \beta_i(x,t)\frac{\partial u}{\partial x_i} + \gamma(u,x,t), \tag{2.7}$$

where

$$\begin{aligned}
&a^{ij}(x,t) = \rho^\alpha |\nabla u|^{p(x)-2}\big[\delta_{ij} + \big(p(x)-2\big)|\nabla u|^{-2}u_{x_i}u_{x_j}\big],\\
&\beta_i = \alpha\rho^{\alpha-1}|\nabla u|^{p(x)-2}\rho_{x_i} + \rho^\alpha |\nabla u|^{p(x)-2}\log|\nabla u| p_{x_i} + b_{iu}(u,x,t),\\
&\gamma(u,x,t) = b_{ix_i}(u,x,t).
\end{aligned}$$

If comparing (2.7) with (2.5), on the lateral boundary, when $t = 0$, the initial value condition (1.10) is required, and as we know when $t = T$, no boundary value is necessary. The more interesting phases appear on the bottom boundary. Generally, only a portion of the bottom boundary can be required as the boundary value. Let us explain what happens as follows.

By

$$\begin{aligned}
a^{ij}_{x_j} &= \alpha\rho^{\alpha-1}\rho_{x_j}|\nabla u|^{p(x)-4}\big[\delta_{ij}|\nabla u|^2 + \big(p(x)-2\big)u_{x_i}u_{x_j}\big]\\
&\quad + \rho^\alpha|\nabla u|^{p(x)-4}\big[p_{x_j}|\nabla u|^2\log|\nabla u| + \big(p(x)-2\big)u_{x_k}u_{x_kx_j} + p_{x_j}u_{x_j}u_{x_i}\big]\\
&\quad + \big(p(x)-2\big)\rho^\alpha|\nabla u|^{p(x)-6}u_{x_j}u_{x_i}\big[p_{x_j}|\nabla u|^2\log|\nabla u| + \big(p(x)-4\big)u_{x_k}u_{x_kx_j}\big]\\
&\quad + \big(p(x)-2\big)\rho^\alpha|\nabla u|^{p(x)-4}(u_{x_ix_j}u_{x_j} + u_{x_i}u_{x_jx_j}),
\end{aligned}$$

if we notice that near the boundary $\Sigma = \partial\Omega$, $\rho^\alpha(x) = o(\rho^{\alpha-1})$, then

$$\begin{aligned}
\big(\beta^i - \alpha^{ij}_{x_j}\big)n_i &= \alpha\rho^{\alpha-1}|\nabla u|^{p(x)-4}\big[\rho_{x_i}|\nabla u|^2 - \rho_{x_j}\big(\delta_{ij}|\nabla u|^2 + \big(p(x)-2\big)u_{x_i}u_{x_j}\big)\big]n_i\\
&\quad + b_{iu}(u,x,t)n_i + o\big(\rho^{\alpha-1}\big)\\
&= -\big(p(x)-2\big)\alpha\rho^{\alpha-1}\big(|\nabla u|^{p(x)-4}u_{x_i}u_{x_j}\big)n_in_j + b_{iu}(u)n_i(x) + o\big(\rho^{\alpha-1}\big).
\end{aligned} \tag{2.8}$$

Because the determinant of $U_x = (u_{x_i}u_{x_j})_{N\times N}$,

$$|u_{x_i}u_{x_j}|_{N\times N} = \prod_{i=1}^N u_{x_i}|u_{x_j}|_{N\times N} = 0, \quad x \in \overline{\Omega},$$

and ith order principal minor determinants are all equal to 0, except that $i = 1$. Then according to the characteristic value theory, due to the symmetry of $N \times N$ matrix U_x, there exists an orthogonal matrix P such that $(u_{x_i}u_{x_j})_{N\times N} = PA_{N\times N}P^{-1}$, where A is a diagonal matrix which is just the characteristic matrix of U_x. Let $\lambda_i \geq 0$ be the characteristic values of U_x. By a direct calculation, we get

$$\lambda_1 = |\nabla u|^2, \quad \lambda_i = 0, i = 2,3,\ldots,N. \tag{2.9}$$

We have

$$\begin{aligned}\left(\beta^i - \alpha^{ij}_{x_j}\right)n_i &= -\left(p(x)-2\right)\alpha\rho^{\alpha-1}|\nabla u|^{p(x)-4}\vec{n}PA_{N\times N}(\vec{n}P)^T + b_{iu}(u,x,t)n_i + o\left(\rho^{\alpha-1}\right) \\ &= -\left(p(x)-2\right)\alpha\rho^{\alpha-1}|\nabla u|^{p(x)-2}m_1^2 \\ &\quad + b_{iu}(u,x,t)n_i + o\left(\rho^{\alpha-1}\right), \quad x\in\partial\Omega,\end{aligned} \tag{2.10}$$

where $\vec{m} = \vec{n}P$.

Then it can be divided into the following cases.

1. $\alpha > 1$, then $\rho^\alpha|_{\partial\Omega} = \rho^{\alpha-1}|_{\partial\Omega} = 0$.

1.1. $b_i \equiv 0$. Then $\Sigma_p = \emptyset$, no boundary value is required.

1.2. b_i is not identical to 0,

$$\Sigma_p = \left\{x\in\partial\Omega : b_{iu}(0,x,t)n_i < 0\right\}. \tag{2.11}$$

It shows that (2.6) still needs the partial boundary condition when $\alpha > 1$, this is different from the case of $b_i \equiv 0$, in which no boundary is required even when $\alpha > 1$. For example, considering the case of a one-dimensional space variable, and $p = 2$, $x \in (0,1)$, $b(u,x,t) = b(u)$, (2.6) becomes

$$u_t = \left(\rho^\alpha u_x\right)_x + b'(u)u_x, \tag{2.12}$$

Σ_p in (2.11) means that (2.12) needs to give the boundary condition at $x = 0$ when $b'(0) < 0$ and needs to give the boundary condition at $x = 1$ when $b'(0) > 0$.

In this case, certainly, when $b_{iu}(0,x,t)n_i(x) \geq 0$ is true for all $x \in \partial\Omega$, Σ_p is an empty set, then (2.6) does not require any boundary condition now.

2. $\alpha = 1$, then

$$I = \left(\beta^i - \alpha^{ij}_{x_j}\right)n_i = -\left(p(x)-2\right)|\nabla u|^{p(x)-2}m_1^2 + b_{iu}(0,x,t)n_i + o(\rho).$$

2.1. $b_i \equiv 0$,

$$I = \left(\beta^i - \alpha^{ij}_{x_j}\right)n_i = -\left(p(x)-2\right)|\nabla u|^{p(x)-2}m_1^2 + o(\rho).$$

If $N = 1$, then

$$\Sigma_p = \left\{x\in\partial\Omega : p(x) > 2\right\}.$$

If $N \geq 2$, then

$$\Sigma_p = \left\{x\in\partial\Omega : p(x) > 2, m_1(x) \neq 0\right\}.$$

2.2. b_i is not identical to 0,

$$I = \left(\beta^i - \alpha^{ij}_{x_j}\right)n_i = -\left(p(x) - 2\right)|\nabla u|^{p(x)-2} m_1^2 + b_{iu}(0,x,t)n_i + o(\rho).$$

If $N = 1$, when for all $x \in \partial\Omega$,

$$p(x) > 2, \qquad b_{iu}(0,x,t)n_i(x) \leq 0,$$

then $\Sigma_p = \partial\Omega$. Generally, it is only a subset of $\partial\Omega$ and it is difficult to write out the explicit formula.

If $N \geq 2$, when for all $x \in \partial\Omega$,

$$p(x) > 2, \qquad b_{iu}(0,x,t)n_i(x) \leq 0, \qquad m_1(x) \neq 0,$$

then $\Sigma_p = \partial\Omega$.

If $N \geq 2$, when for all $x \in \partial\Omega$,

$$b_{iu}(0,x,t)n_i(x) \geq 0, \qquad m_1(x) = 0,$$

then $\Sigma_p = \emptyset$. Generally, it is only a subset of $\partial\Omega$ and it is difficult to write out the explicit formula.

3. $\alpha < 1$.

3.1. $b_i \equiv 0$.

When $N = 1$, (2.10) becomes

$$I = -\alpha\left(p(x) - 2\right)\rho^{\alpha-1}\left|u'(x)\right|^{p(x)-2} m_1^2 + O\left(\rho^\alpha\right).$$

If $p(x) \equiv p$, when $p > 2$, $I < 0$, then $\Sigma_p = \partial\Omega$. When $p \leq 2$, $I \geq 0$, then $\Sigma_p = \emptyset$.

If $p(x)$ is just a continuous function, then

$$\Sigma_p = \left\{x \in \partial\Omega : p(x) > 2\right\}.$$

When $N \geq 2$, if $p(x) \equiv p > 2$, then

$$\Sigma_p = \left\{x \in \partial\Omega : m_1(x) \neq 0\right\}.$$

If $p(x) \equiv p \leq 2$, then $\Sigma_p = \emptyset$.

If $p(x)$ is just a function, then

$$\Sigma_p = \left\{x \in \partial\Omega : p(x) > 2, m_1(x) \neq 0\right\}.$$

3.2. b_i is not identical to 0.

When $N = 1$, (2.10) becomes

$$I = -\alpha\left(p(x) - 2\right)\rho^{\alpha-1}\left|u'(x)\right|^{p(x)-2} m_1^2 + b_u(0,x,t)n + O\left(\rho^\alpha\right).$$

If $p(x) \equiv p$, when $p > 2$, $I < 0$, then $\Sigma_p = \partial\Omega$. When $p = 2$,

$$\Sigma_p = \{x \in \partial\Omega : b(u)(0,x,t)n < 0\}.$$

When $p < 2$, $I > 0$, then $\Sigma_p = \emptyset$.

If $p(x)$ is just a function, then

$$\Sigma_p = \{x \in \partial\Omega : p(x) > 2\} \cup \{x \in \partial\Omega : p(x) = 2, b_{iu}(0,x,t)n_i < 0\}.$$

Let $N \geq 2$, if $p(x) \equiv p$, when $p > 2$, then

$$\Sigma_p = \{x \in \partial\Omega : m_1(x) \neq 0\},$$

when $p = 2$,

$$\Sigma_p = \{x \in \partial\Omega : b_{iu}(0,x,t)n_i(x) < 0\},$$

when $p < 2$, when for all $x \in \partial\Omega$, $m_1(x) \neq 0$, then $I > 0$, $\Sigma_p = \emptyset$. In general, it is just a subset of $\partial\Omega$, and it is difficult to write out the explicit formula.

If $p(x)$ is just a continuous function, then

$$\begin{aligned}\Sigma_p = {} & \{x \in \partial\Omega : p(x) > 2\} \cup \{x \in \partial\Omega : p(x) = 2, b_{iu}(0,x,t)n_i < 0\} \\ & \cup \{x \in \partial\Omega : p(x) < 2, m_1(x) \neq 0, b_{iu}(0,x,t)n_i < 0\}.\end{aligned}$$

In other words, the boundary value condition of (1.1) is so complicated; it may depend on whether $\alpha > 1, = 1$, or < 1, whether $N = 1$, or $N \geq 2$, whether $p(x) > 2$ or not, whether $b_i \equiv 0$ or not. In Sections 3 and 4, we only consider the existence and the uniqueness of the solutions when $p^- > 2$. In last section, we only consider the behavior of the solutions near the boundary when $\alpha \geq 1$ and $b_i \equiv 0$.

Certainly, as we already know that a degenerate parabolic equation generally only has a weak solution, the above linearization is only formal. We only give some ideas of how to give the partial boundary value condition to assure the posedness of the weak solutions.

3 The existence of the solution related to the initial value

Let

$$u_0 \in L^\infty(\Omega), \qquad \rho^\alpha |\nabla u_0|^{p_+} \in L^1(\Omega). \tag{3.1}$$

Definition 3.1 A function $u(x,t)$ is said to be a solution of (1.1) with the initial value condition (1.10), if the initial condition is satisfied, in the sense of a trace, and u satisfies

$$u \in L^\infty(Q_T), \qquad \rho^\alpha |\nabla u|^{p(x)} \in L^1(Q_T), \qquad u_t \in L^2(Q_T), \tag{3.2}$$

and for any function $\varphi \in C_0^\infty(Q_T)$, the following integral equivalence holds:

$$\iint_{Q_T} \left(-u\varphi_t + \rho^\alpha |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi + b_i(u,x,t)\varphi_{x_i}\right) \mathrm{d}x\,\mathrm{d}t = 0. \tag{3.3}$$

We consider the following regularized problem:

$$u_{\varepsilon t} - \operatorname{div}\left(\rho_\varepsilon^\alpha \left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p(x)-2}{2}} \nabla u_\varepsilon\right) - \frac{\partial b_i(u,x,t)}{\partial x_i} = 0, \quad (x,t) \in Q_T, \tag{3.4}$$

$$u_\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3.5}$$

$$u_\varepsilon(x,0) = u_{\varepsilon,0}(x), \quad x \in \Omega, \tag{3.6}$$

where $\rho_\varepsilon = \rho * \delta_\varepsilon + \varepsilon$, $\varepsilon > 0$, δ_ε is the usual mollifier. For all $\varepsilon > 0$, selecting $u_{\varepsilon,0}$ such that $\|u_{\varepsilon,0}\|_{L^\infty(\Omega)}$ and $\|\rho_\varepsilon^\alpha |\nabla u_{\varepsilon,0}|^{p^+}\|_{L^1(\Omega)}$ are uniformly bounded, and $u_{\varepsilon,0}$ converges to u_0 in $W^{1,p^+}_{\mathrm{loc}}(\Omega)$. For any $u_{\varepsilon,0} \in C_0^\infty(\Omega)$, $\rho_\varepsilon^\alpha |\nabla u_{\varepsilon,0}|^{p^+} \in L^1(\Omega)$, it is well known that the above problem has a unique classical solution [35]. Hence for any $\varphi \in C_0^\infty(Q_T)$, u_ε satisfies the following integral equivalence:

$$\iint_{Q_T} \left(u_{\varepsilon t}\varphi + \rho_\varepsilon^\alpha |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \cdot \nabla\varphi + b_i(u_\varepsilon,x,t)\varphi_{x_i}\right) \mathrm{d}x\,\mathrm{d}t = 0. \tag{3.7}$$

Lemma 3.2 *If* $p^- > 2$, Ω *is a suitably smooth bounded domain, the assumptions* (A) *and* (B) *are true, then the solution* u_ε *of the initial boundary value problem* (3.4)-(3.6) *is weakly star convergent to* u *and strongly convergent to* $u \in L^2_{\mathrm{loc}}(Q_T)$, *and its limit function* u *satisfies* (3.2) *and is the solution of* (1.1) *with the initial value condition* (1.10).

Proof By the maximum principle, there is a constant c, only dependent on $\|u_0\|_{L^\infty(\Omega)}$ but independent of ε, such that

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \le c. \tag{3.8}$$

Multiplying (3.4) by u_ε and integrating over Q_T, we get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega u_\varepsilon^2\,\mathrm{d}x + \iint_{Q_T} \rho_\varepsilon^\alpha \left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p(x)-2}{2}} |\nabla u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \iint_{Q_T} u_\varepsilon \frac{\partial b_i(u_\varepsilon x,t)}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega u_{\varepsilon,0}^2\,\mathrm{d}x.
\end{aligned}$$

By the fact

$$\begin{aligned}
&\iint_{Q_T} u_\varepsilon \frac{\partial b_i(u_\varepsilon x,t)}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= -\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial x_i} b_i(u_\varepsilon,x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= -\iint_{Q_T} \frac{\partial}{\partial x_i}\int_0^{u_\varepsilon} b_i(s,x,t)\,\mathrm{d}s\,\mathrm{d}x + \iint_{Q_T}\int_0^{u_\varepsilon} b_{ix_i}(s,x,t)\,\mathrm{d}s\,\mathrm{d}x \\
&\quad= \iint_{Q_T}\int_0^{u_\varepsilon} b_{ix_i}(s,x,t)\,\mathrm{d}s\,\mathrm{d}x,
\end{aligned} \tag{3.9}$$

and by (B),

$$\frac{1}{2}\int_\Omega u_\varepsilon^2\,\mathrm{d}x + \iint_{Q_T} \rho_\varepsilon^\alpha \left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p(x)-2}{2}} |\nabla u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \le c.$$

Let $\Omega_\lambda = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \lambda\}$. Since $p^+ \geq p^- > 2$, we have

$$\int_0^T \int_{\Omega_\lambda} |\nabla u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \leq c\left(\int_0^T \int_{\Omega_\lambda} |\nabla u_\varepsilon|^{p^-} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p^-}} \leq c(\lambda). \tag{3.10}$$

Multiplying (3.4) by $u_{\varepsilon t}$, integrating over Q_T,

$$\begin{aligned} &\iint_{Q_T} (u_{\varepsilon t})^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \iint_{Q_T} \operatorname{div}\bigl(\rho_\varepsilon^\alpha |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon\bigr) \cdot u_{\varepsilon t} \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} u_{\varepsilon t} \frac{\partial b_i(u_\varepsilon, x, t)}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.11}$$

We notice that

$$\bigl(|\nabla u_\varepsilon|^2 + \varepsilon\bigr)^{\frac{p(x)-2}{2}} \nabla u_\varepsilon \cdot \nabla u_{\varepsilon t} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^{|\nabla u_\varepsilon(x,t)|^2 + \varepsilon} s^{\frac{p(x)-2}{2}} \,\mathrm{d}s.$$

Thus,

$$\begin{aligned} &\iint_{Q_T} \operatorname{div}\Bigl(\rho_\varepsilon^\alpha \bigl(|\nabla u_\varepsilon|^2 + \varepsilon\bigr)^{\frac{p(x)-2}{2}} \nabla u_\varepsilon\Bigr) \cdot u_{\varepsilon t} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = -\iint_{Q_T} \rho_\varepsilon^\alpha \bigl(|\nabla u_\varepsilon|^2 + \varepsilon\bigr)^{\frac{p(x)-2}{2}} \nabla u_\varepsilon \nabla u_{\varepsilon t} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = -\frac{1}{2} \iint_{Q_T} \rho_\varepsilon^\alpha \frac{\mathrm{d}}{\mathrm{d}t} \int_0^{|\nabla u_\varepsilon(x,t)|^2 + \varepsilon} s^{\frac{p(x)-2}{2}} \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.12}$$

By condition (B),

$$\begin{aligned} &\iint_{Q_T} u_{\varepsilon t} \frac{\partial b_i(u_\varepsilon, x, t)}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq \iint_{Q_T} \bigl|b_{iu}(u_\varepsilon, x, t)\bigr| |u_{\varepsilon x_i}| |u_{\varepsilon t}| \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \bigl|b_{ix_i}(u_\varepsilon, x, t)\bigr| |u_{\varepsilon t}| \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq \frac{1}{4} \iint_{Q_T} (u_{\varepsilon t})^2 \,\mathrm{d}x\,\mathrm{d}t + c \iint_{Q_T} |u_\varepsilon|^{2\beta} |\nabla u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \frac{1}{4} \iint_{Q_T} (u_{\varepsilon t})^2 \,\mathrm{d}x\,\mathrm{d}t + c. \end{aligned} \tag{3.13}$$

Here, we have used the fact that $|u_\varepsilon|$ is bounded, $b_i(s, x, t) \in C^1(\mathbb{R} \times \overline{\Omega} \times [0, T])$.

By Hölder's inequality and $\alpha \leq \frac{p^- - 2}{2}$,

$$\begin{aligned} &\iint_{Q_T} |u_\varepsilon|^{2\beta} |\nabla u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq c \iint_{Q_T} |\nabla u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t = c \iint_{Q_T} \rho_\varepsilon^{-\frac{2\alpha}{p^-}} \cdot \rho_\varepsilon^{\frac{2\alpha}{p^-}} |\nabla u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq c\left(\iint_{Q_T} \rho_\varepsilon^{-\frac{2\alpha}{p^- - 2}} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p^- - 2}{p^-}} \cdot \left(\iint_{Q_T} \rho_\varepsilon^{a} |\nabla u_\varepsilon|^{p^-} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{p^-}} \leq c. \end{aligned} \tag{3.14}$$

Combining (3.11)-(3.14), we have

$$\iint_{Q_T} (u_{\varepsilon t})^2 \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \rho_\varepsilon^\alpha \frac{\mathrm{d}}{\mathrm{d}t} \int_0^{|\nabla u_\varepsilon(x,t)|^2} s^{\frac{p-2}{2}} \,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \le c,$$

by the inequality, we have

$$\iint_{Q_T} (u_{\varepsilon t})^2 \,\mathrm{d}x\,\mathrm{d}t \le c + c \int_\Omega \rho_\varepsilon^\alpha |\nabla u_{\varepsilon,0}|^{p(x)} \,\mathrm{d}x \le c. \tag{3.15}$$

By (3.10), (3.15), we know that

$$\int_0^T \int_{\Omega_\lambda} |\nabla u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \le c, \qquad \int_0^T \int_\Omega |u_{\varepsilon t}|^2 \,\mathrm{d}x\,\mathrm{d}t \le c. \tag{3.16}$$

By (3.10), (3.15), and (3.16), we know that there exists a subsequence (still denoted u_ε) of u_ε, which is weakly star convergent to u, and strongly convergent to $u \in L^2_{\mathrm{loc}}(Q_T)$, and it satisfies (3.2). In particular, $u_\varepsilon \to u$ a.e. in Q_T, and there exists an n-dimensional vector function $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$,

$$|\vec{\zeta}| \in L^{\frac{p(x)}{p(x)-1}}(Q_T),$$

such that

$$u_\varepsilon \rightharpoonup^* u \quad \text{in } L^\infty(Q_T), \qquad u_\varepsilon \to u \quad \text{in } L^2_{\mathrm{loc}}(Q_T),$$
$$\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^{p(x)}_{\mathrm{loc}}(Q_T),$$
$$\rho_\varepsilon^\alpha |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \rightharpoonup \vec{\zeta} \quad \text{in } L^{\frac{p(x)}{p(x)-1}}(Q_T).$$

So u satisfies (1.10) in the sense of a trace. In order to prove that u satisfies equivalence (3.1), we notice that, for any function $\varphi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} \left(-u_\varepsilon \varphi_t + \rho_\varepsilon^\alpha \left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p(x)-2}{2}} \nabla u_\varepsilon \cdot \nabla \varphi + b_i(u_\varepsilon, x, t) \varphi_{x_i}\right) \mathrm{d}x\,\mathrm{d}t = 0. \tag{3.17}$$

By $u_\varepsilon \to u$ a.e. in Q_T, then $b_i(u_\varepsilon, x, t) \to b_i(u, x, t)$, and so

$$\iint_{Q_T} \left(\frac{\partial u}{\partial t} \varphi + \vec{\zeta} \cdot \nabla \varphi + b_i(u, x, t) \varphi_{x_i}\right) \mathrm{d}x\,\mathrm{d}t = 0. \tag{3.18}$$

Now, it is not difficult to prove that (*cf.* [36, 37])

$$\iint_{Q_T} \rho^\alpha |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T} \vec{\zeta} \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t, \tag{3.19}$$

for any function $\varphi \in C_0^\infty(Q_T)$, then u satisfies (3.1) and it is the solution of (1.1) with the initial value (1.10). Thus, we have proved Lemma 3.2. □

4 The existence and the uniqueness of solutions

Definition 4.1 Let $\alpha < p^- - 1$, $p^- > 2$. The function $u(x,t)$ is said to be the weak solution of (1.1) with the initial value (1.10) and with the boundary value condition

$$u|_{\Sigma_p \times (0,T)} = 0, \tag{4.1}$$

if u satisfies Definition 3.1, and for any function

$$\phi(x,t) \in C^2(\overline{Q}_T), \qquad \operatorname{supp}\phi(x,t) \subset \overline{\Omega} \times (0,T),$$

$\phi = 0$ near $\Sigma_p' = \partial\Omega \setminus \Sigma_p$, and the following integral equivalence holds:

$$\begin{aligned} &\iint_{Q_T} u_t \phi \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \rho^\alpha |\nabla u|^{p(x)-2} \nabla u \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \iint_{Q_T} b_i(u,x,t)\phi_{x_i} \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T \int_{\Sigma_p} b_i(0,x,t) n_i \phi \,\mathrm{d}\sigma\,\mathrm{d}t, \end{aligned} \tag{4.2}$$

where Σ_p is defined in Section 2 in detail. We quote it as follows.

(i) If $\alpha > 1$, then for any given $t \in (0,T)$,

$$\Sigma_p = \{x \in \partial\Omega : b_{iu}(0,x,t) n_i(x) < 0\}.$$

(ii) If $\alpha \leq 1$, then for any given $t \in (0,T)$,

$$\Sigma_p = \partial\Omega.$$

(iii) If $\alpha = 1$, then for any given $t \in (0,T)$, Σ_p is just a subset of $\partial\Omega$, and it is difficult to write out its explicit formula, except some special cases.

For examples, we have the following special cases.

Case 1. When $N = 1$, for any given $t \in (0,T)$, if for all $x \in \partial\Omega$, we have

$$b_{iu}(0,x,t) n_i \leq 0,$$

then

$$\Sigma_p = \partial\Omega.$$

Case 2. When $N \geq 2$, for any given $t \in (0,T)$, if for all $x \in \partial\Omega$, we have

$$b_{iu}(0,x,t) n_i \leq 0, \qquad m_1(x) \neq 0,$$

then

$$\Sigma_p = \partial\Omega.$$

Theorem 4.2 *Let* $\alpha < \frac{p^- - 2}{2}$, *conditions* (A) *and* (B) *be true. Suppose*

$$u_0 \in L^\infty(\Omega), \qquad \rho^\alpha |\nabla u_0|^{p^+} \in L^1(\Omega),$$

then there is a solution of (1.1) *with the initial value condition* (1.10) *and with the partial boundary value condition* (4.1).

Proof For all $\varepsilon > 0$, selecting $u_{\varepsilon,0}$ such that $\|u_{\varepsilon,0}\|_{L^\infty(\Omega)}$ and $\|\rho_\varepsilon^\alpha |\nabla u_{\varepsilon,0}|^{p^+}\|_{L^1(\Omega)}$ are uniformly bounded, and $u_{\varepsilon,0}$ converges to u_0 in $W^{1,p^+}_{\mathrm{loc}}(\Omega)$. Let u_ε be the solution of the initial boundary value problem (3.4)-(3.6). By the condition $\alpha < \frac{p^- - 2}{2}$, we have Lemma 3.2, and u_ε converges to u in $L^2_{\mathrm{loc}}(Q_T)$, and the limit function u is a weak solution of (1.1) with the initial condition (1.10). Now, just as in [31], we can prove that there is $\gamma \in (1, p^- - \frac{\alpha}{\beta})$ such that

$$\iint_{Q_T} |\nabla u_\varepsilon|^\gamma \,\mathrm{d}x\,\mathrm{d}t \le c.$$

Here c is independent of ε. So ∇u_ε is uniformly bounded in $L^\gamma(Q_T)$, and u has a trace on the boundary.

Let $\phi(x,t) \in C^2(\overline{Q}_T)$, $\mathrm{supp}\phi(x,t) \subset \overline{\Omega} \times (0,T)$, $\varphi = 0$ near Σ_p'. Equation (1.1) is multiplied by $\varphi(x,t)$ on both sides, integrated over Q_T, then

$$\begin{aligned}
&\iint_{Q_T} u_t \phi \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \rho^\alpha |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \phi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \iint_{Q_T} b_i(u,x,t) \phi_{x_i} \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T \int_{\Sigma_p} b_i(0,x,t) n_i \phi \,\mathrm{d}\sigma\,\mathrm{d}t
\end{aligned}$$

That means (4.2) is true. $\square$

Theorem 4.3 *Let conditions* (A) *and* (B) *be true and* $\alpha < p^- - 1$,

$$u_0 \in L^\infty(\Omega), \qquad \rho^\alpha |\nabla u_0|^{p^+} \in L^1(\Omega).$$

If $\Sigma_p = \partial\Omega$, *the solution of the problem* (1.1)-(1.10)-(4.1) *is unique.*

Proof Let u and v be two weak solutions, $u(x,0) = v(x,0)$. We have $\rho^\alpha |\nabla u|^{p(x)}, \rho^\alpha |\nabla v|^{p(x)} \in L^1(Q)$, and for all $\varphi \in C_0^\infty(Q_T)$,

$$\begin{aligned}
\iint_{Q_T} \varphi \frac{\partial(u-v)}{\partial t} \,\mathrm{d}x\,\mathrm{d}t = &-\iint_{Q_T} \rho^\alpha \bigl(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\bigr) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&- \iint_{Q_T} b_i(u,x,t) \cdot \varphi_{x_i} \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

For any given positive integer n, let $g_n(s)$ be an odd function. When $s > 0$ it is defined as

$$g_n(s) = \begin{cases} 1, & s > \frac{1}{n}, \\ n^2 s^2 e^{1-n^2 s^2}, & s \le \frac{1}{n}. \end{cases}$$

Choosing $g_n(u-v)$ as the test function, then

$$\begin{aligned}&\iint_{Q_T} g_n(u-v)\frac{\partial(u-v)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\iint_{Q_T}\rho^\alpha\bigl(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\bigr)\cdot\nabla(u-v)g_n'\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\iint_{Q_T}\bigl(b_i(u,x,t)-b_i(v,x,t)\bigr)\cdot(u-v)_{x_i}g_n'\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}\tag{4.3}$$

Since for any given $s>0$, $g_n(s)$ is a monotone increasing sequence of n, and clearly

$$\lim_{n\to\infty} g_n(s)=1,\qquad s>0$$

and

$$\lim_{n\to\infty} g_n(s)=\operatorname{sgn}(x),\qquad s\in\mathbb{R},$$

where $\operatorname{sgn}(x)$ is the sign function. Thus, we have

$$\lim_{n\to\infty}\int_\Omega g_n(u-v)\frac{\partial(u-v)}{\partial t}\,\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\|u-v\|_1.\tag{4.4}$$

At the same time, it is clear that

$$\iint_{Q_T}\rho^\alpha\bigl(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\bigr)\cdot\nabla(u-v)g_n'\,\mathrm{d}x\,\mathrm{d}t\geq 0.\tag{4.5}$$

Now, according to the definition of $g_n(s)$,

$$\bigl|g_n'(s)\bigr|\leq\frac{c}{s},\qquad |s|\leq\frac{1}{n}.$$

We use the following facts:

$$\begin{aligned}&\left|\iint_{Q_T\cap\{|u-v|<\frac{1}{n}\}}\bigl[b_i(u,x,t)-b_i(v,x,t)\bigr]g_n(u-v)_{x_i}\,\mathrm{d}x\,\mathrm{d}t\right|\\&\quad=\left|\iint_{Q\cap\{|u-v|<\frac{1}{n}\}}\bigl[b_i(u,x,t)-b_i(v,x,t)\bigr]g_n'(u-v)(u-v)_{x_i}\,\mathrm{d}x\,\mathrm{d}t\right|\\&\quad\leq c\iint_{Q\cap\{|u-v|<\frac{1}{n}\}}\left|\frac{b_i(u,x,t)-b_i(v,x,t)}{u-v}\right|\bigl|(u-v)_{x_i}\bigr|\,\mathrm{d}x\,\mathrm{d}t\\&\quad=c\iint_{Q\cap\{|u-v|<\frac{1}{n}\}}\left|\rho^{-\frac{\alpha}{p^-}}\frac{b_i(u,x,t)-b_i(v,x,t)}{u-v}\right|\bigl|\rho^{\frac{\alpha}{p^-}}(u-v)_{x_i}\bigr|\,\mathrm{d}x\,\mathrm{d}t\\&\quad\leq c\left[\iint_{Q\cap\{|u-v|<\frac{1}{n}\}}\left(\left|\rho^{-\frac{\alpha}{p^-}}\frac{b_i(u,x,t)-b_i(v,x,t)}{u-v}\right|\right)^{\frac{p^-}{p^--1}}\mathrm{d}x\,\mathrm{d}t\right]^{\frac{p^--1}{p^-}}\\&\qquad\cdot\left(\iint_{Q\cap\{|u-v|<\frac{1}{n}\}}\bigl|\rho^\alpha\nabla(u-v)\bigr|^{p^-}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p^-}}.\end{aligned}\tag{4.6}$$

Since $\alpha < p^- - 1$,

$$\iint_{Q\cap\{|u-v|<\frac{1}{n}\}} \left(\left|\rho^{-\frac{\alpha}{p^-}} \frac{b_i(u,x,t) - b_i(v,x,t)}{u-v}\right|\right)^{\frac{p}{p^- -1}} \,dx\,dt$$
$$\leq \iint_Q \rho^{-\frac{\alpha}{p^- -1}} \left|b_i'(\xi,x,t)\right|^{\frac{p}{p^- -1}} \,dx\,dt \leq c \iint_Q \rho^{-\frac{\alpha}{p^- -1}} \,dx\,dt \leq c, \tag{4.7}$$

where $b_i'(\xi,x,t) = \frac{\partial b_i(s,x,t)}{\partial s}|_{s=\xi}$, which is bounded by the assumption (B).

In (4.6), let $n \to \infty$. If $\{x \in \Omega : |u - v| = 0\}$ is a set with 0 measure, then

$$\lim_{n\to\infty} \iint_{Q\cap\{|u-v|<\frac{1}{n}\}} \left|\rho^{\frac{\alpha}{p^- -1}} b_i'(\xi,x,t)\right| \,dx\,dt$$
$$= \iint_{Q\cap\{|u-v|=0\}} \left|\rho^{\frac{\alpha}{p^- -1}} b_i'(\xi,x,t)\right| \,dx\,dt = 0. \tag{4.8}$$

If the set $\{x \in \Omega : |u - v| = 0\}$ has a positive measure, then

$$\lim_{n\to\infty} \iint_{Q\cap\{|u-v|<\frac{1}{n}\}} \rho^\alpha \left|\nabla(u-v)\right|^{p^-} \,dx\,dt$$
$$= \iint_{Q\cap\{|u-v|=0\}} \rho^\alpha \left|\nabla(u-v)\right|^{p^-} \,dx\,dt = 0. \tag{4.9}$$

Therefore, in both cases, (4.6) tends to 0 as $n \to \infty$.

Thus we have

$$\lim_{n\to\infty} \iint_{Q_T} \big(b_i(u,x,t) - b_i(v,x,t)\big) g_n'(u-v)(u-v)_{x_i} \,dx\,dt = 0. \tag{4.10}$$

Now, let $n \to \infty$ in (4.2). Then, by (4.3)-(4.10), we have

$$\frac{d}{dt}\|u - v\|_1 \leq 0.$$

It implies that

$$\int_\Omega \left|u(x,t) - v(x,t)\right| \,dx \leq \int_\Omega |u_0 - v_0| \,dx = 0, \quad \forall t \in [0,T).$$

By the arbitrariness of t,

$$u(x,t) = v(x,t) \quad \text{a.e. } (x,t) \in Q_T.$$

Theorem 4.3 is proved. □

5 The behavior of solutions near the boundary

Without loss the generality, we also assume that the boundary $\partial\Omega$ is of class C^2. That is, there exists a number $\rho_0 \in (0,1)$ such that for all $x_0 \in \partial\Omega$ the portion of $\partial\Omega$ within the ball $B_{\rho_0}(x_0)$ can be represented, in a local system of coordinates, as the graph of a C^2 function $\varphi^{(x_0)}$ such that $\varphi^{(x_0)}(x_0) = 0$, and for $x \in B_{\rho_0}(x_0) \cap \Omega = \{x = (x_1, x_2, \ldots, x_{N-1}, x_N) : x_N > 0\}$, $x \in$

$B_{\rho_0}(x_0) \cap \partial\Omega = \{x = (x_1, x_2, \ldots, x_{N-1}, x_N) : x_N = 0\}$. We call this local coordinate transform a planarization technique. In this section, we use some ideas of [38, 39].

Definition 5.1 If $u_0(x)$ satisfies (3.1), u is the limit of the solutions $\{u_n\}$ of the following equations:

$$\frac{\partial u}{\partial t} = \operatorname{div}\left[\left(\rho^\alpha + \frac{1}{n}\right)\left(|\nabla u|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}} \nabla u\right], \quad (x,t) \in Q_T, \tag{5.1}$$

$$u(x,0) = u_{0,n}(x), \quad x \in \Omega, \tag{5.2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{5.3}$$

where $u_{0,n}(x)$ is the smoothly mollified functions of $u_0(x)$. Then we say u is a viscous solution of (1.1).

We shall get *estimates above and near* $\partial\Omega$.

Theorem 5.2 *Let u, $0 \le u \le M$, be a nonnegative bounded viscous solution of* (1.1) *in the sense of Definition* 5.1. *If $\alpha \ge 1$, then for any given $s \in (0,T)$, we have*

$$u(x,t) \le kC\rho(x), \quad (x,t) \in \Omega \times (s,T), \tag{5.4}$$

where the constant C depending upon M, N, p, s, and the constant k is a constant independent of s, M.

Proof Fix $(x_0, t_0) \in \partial\Omega \times (s,T)$. By the planarization technique, we may assume that $(x_0, t_0) \equiv (0,0)$ and in the vicinity of $(0,0)$, after flattening of $\partial\Omega$ near x_0, without loss of generality, let us assume that $\partial\Omega$ coincides with the portion of hyperplane $\{x_N = 0\}$, and the inclusion $\Omega \cap \{|x| < \rho_0\} \subset \{x_N > 0\}$ is true. Let $y = (0, \ldots, 0, -1)$, and define the set

$$\aleph_k = \left\{(x,t) : x_N > 0, 1 < |x - y| < 1 + \frac{1}{k}, -s_n \le t \le 0\right\}. \tag{5.5}$$

We assume k is so large that $\aleph_k \subset B^+_{\rho_0} \times (s, 0]$. Consider the following problem:

$$\frac{\partial v}{\partial t} = \operatorname{div}\left[\left(\rho^\alpha + \frac{1}{n}\right)\left(|\nabla v|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}} \nabla v\right], \quad (x,t) \in \aleph_k, \tag{5.6}$$

$$v(x, -s_n) = u_n(x, -s_n), \quad x \in B^+_k, \tag{5.7}$$

$$v(x,t) = u_n(x,t) - \frac{1}{n}, \quad (x,t) \in \partial B^+_k \times [-s_n, 0], \tag{5.8}$$

where u_n is the solution of the problem (5.1)-(5.3), $0 < s_n < s < T$, $s_n n$ is small enough, and

$$B^+_k = \left\{x : x_N > 0, 1 < |x - y| < 1 + \frac{1}{k}\right\} \cap B^+_{\rho_0}.$$

By the comparison theorem ([40], p.119), we have

$$v \le u_n. \tag{5.9}$$

Let us construct a barrier for u in $\aleph_k$. Consider the function

$$\eta_k(x,t) = e^{-k(|x-y|-1)}e^t,$$

and the barrier is given by

$$\Psi_k = CM\bigl(1-\eta_k(x,t)\bigr) + \gamma t, \quad (x,t) \in \aleph_k,$$

where the constants γ, C are to be chosen later so large that $v \leq \Psi_k$ on the parabolic boundary of $\aleph_k$. This holds true on the portion of such a boundary lying on the hyperplane $x_N = 0$,

$$\Psi_k = CM\bigl(1-\eta_k(x,t)\bigr) + \gamma t \geq CM\bigl(1-e^{-k(|x-y|-1)}\bigr) - \gamma s_n \geq -\frac{1}{n}$$

provided that $\gamma \leq \frac{1}{ns_n}$. On the portion $\{|x-y| = 1+\frac{1}{k}\} \cap \{x_N \geq 0\}$, we have

$$\Psi_k \geq CM\bigl(1-e^{-1+t}\bigr) - \gamma s_n \geq 2M \geq v,$$

if $C \geq 3(1-e^{-1})^{-1}$. On the bottom of $\aleph_k$ we have

$$\Psi_k \geq CM\bigl(1-e^{-s}\bigr) - \gamma s_n \geq 2M \geq v,$$

provided that

$$C \geq \frac{2M+\gamma s}{M(1-e^{-s})}.$$

By direct calculation,

$$\begin{aligned}
\Psi_{k,x_j} &= kCM\eta_k \frac{x_j-y_j}{|x-y|},\\
\Psi_{k,x_lx_j} &= -k^2CM\eta_k \frac{(x_j-y_j)(x_l-y_l)}{|x-y|^2} + kCM\eta_k \frac{1}{|x-y|^2}\left[\delta_{jl}|x-y| - \frac{(x_j-y_j)(x_l-y_l)}{|x-y|}\right],\\
\Psi_{k,x_ix_i} &= -k^2CM\eta_k + kCM\eta_k \frac{1}{|x-y|^2}\bigl(N|x-y| - |x-y|\bigr)\\
&= kCM\eta_k\left(-k + \frac{N-1}{|x-y|}\right),\\
&\left[\left(|\nabla\Psi_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}} \Psi_{k,x_i}\right]_{x_i}\\
&= \frac{p(x)-2}{2}\left(|\nabla\Psi_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-4}{2}} 2\Psi_{k,x_l}\Psi_{k,x_lx_i}\Psi_{k,x_i}\\
&\quad + \left(|\nabla\Psi_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}} \Psi_{k,x_ix_i} + \ln\left(|\nabla\Psi_k|^2 + \frac{1}{n}\right)p_{x_i}(x)\cdot \frac{x_i-y_i}{|x-y|}KCM\eta_k\\
&= \bigl(p(x)-2\bigr)\left[(kCM\eta_k)^2 + \frac{1}{n}\right]^{\frac{p(x)-4}{2}} (kCM\eta_k)^2 \frac{(x_j-y_j)(x_l-y_l)}{|x-y|^2}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \left\{-k^2 CM\eta_k \frac{(x_j-y_j)(x_l-y_l)}{|x-y|^2} + \frac{kCM\eta_k}{|x-y|^2}\left[\delta_{jl}|x-y| - \frac{(x_j-y_j)(x_l-y_l)}{|x-y|}\right]\right\} \\
&+ \left[(kCM\eta_k)^2 + \frac{1}{n}\right]^{\frac{p(x)-2}{2}} kCM\eta_k\left(-k + \frac{N-1}{|x-y|}\right) \\
&+ \ln\left[(KCM\eta_k)^2 + \frac{1}{n}\right] p_{x_i}(x) \cdot \frac{x_i-y_i}{|x-y|} KCM\eta_k \\
= {}& G_n(x) kCM\eta_k\left[-k + \frac{1}{|x-y|^2}\bigl(|x-y| - |x-y|\bigr)\right] \\
&+ \left[(kCM\eta_k)^2 + \frac{1}{n}\right]^{\frac{p(x)-2}{2}} kCM\eta_k\left(-k + \frac{N-1}{|x-y|}\right) \\
&+ \ln\left[(KCM\eta_k)^2 + \frac{1}{n}\right] p_{x_i}(x) \cdot \frac{x_i-y_i}{|x-y|} KCM\eta_k \\
= {}& -G_n(x) k^2 CM\eta_k + \left[(kCM\eta_k)^2 + \frac{1}{n}\right]^{\frac{p(x)-2}{2}} kCM\eta_k\left(-k + \frac{N-1}{|x-y|}\right) \\
&+ \ln\left[(KCM\eta_k)^2 + \frac{1}{n}\right] p_{x_i}(x) \cdot \frac{x_i-y_i}{|x-y|} KCM\eta_k,
\end{aligned}
$$

where $G_n(x) = (p(x)-2)[(kCM\eta_k)^2 + \frac{1}{n}]^{\frac{p(x)-4}{2}}(kCM\eta_k)^2$,

$$
\begin{aligned}
&\Psi_{k,t} - \operatorname{div}\left[\left(\rho^\alpha + \frac{1}{n}\right)\left(|\nabla\Psi_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}} \nabla\Psi_k\right] \\
&= -CMe^{-k(|x-y|-1)}e^t + \gamma - \alpha\rho^{\alpha-1}\rho_{x_i}\left[\left(|\nabla\Psi_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}} \Psi_{k,x_i}\right] \\
&\quad - \left(\rho^\alpha + \frac{1}{n}\right)\left[\left(|\nabla\Psi_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}} \Psi_{k,x_i}\right]_{x_i} \\
&= -CMe^{-k(|x-y|-1)}e^t + \gamma - \alpha\rho^{\alpha-1}\rho_{x_i}\left[\left(|kCM\eta_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}} kCM\eta_k\frac{x_i-y_i}{|x-y|}\right] \\
&\quad - \left(\rho^\alpha + \frac{1}{n}\right)\left\{-G_n(x)k^2CM\eta_k + \left[(kCM\eta_k)^2 + \frac{1}{n}\right]^{\frac{p(x)-2}{2}} kCM\eta_k\left(-k + \frac{N-1}{|x-y|}\right)\right\} \\
&\quad - \ln\left[(KCM\eta_k)^2 + \frac{1}{n}\right] p_{x_i}(x) \cdot \frac{x_i-y_i}{|x-y|} KCM\eta_k \\
&\geq -CMe^{-k(|x-y|-1)}e^t + \gamma - \alpha\rho^{\alpha-1}\left(|kCM\eta_k|^2 + \frac{1}{n}\right)^{\frac{p(x)}{2}} \\
&\quad + \left(\rho^\alpha + \frac{1}{n}\right)G_n k^2 CM\eta_k + k\left(\rho^\alpha + \frac{1}{n}\right)\left(|kCM\eta_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}} kCM\eta_k \\
&\quad - (N-1)\left(\rho^\alpha + \frac{1}{n}\right)\left(|kCM\eta_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}} \\
&\quad - \ln\left[(KCM\eta_k)^2 + \frac{1}{n}\right] p_{x_i}(x) \cdot \frac{x_i-y_i}{|x-y|} KCM\eta_k,
\end{aligned}
$$

where we have used the facts that $|\rho_{x_i}| = 1$, $\alpha \geq 1$.

Clearly, if we choose k large enough, then we have

$$\Psi_{k,t} - \operatorname{div}\left[\left(\rho^{\alpha} + \frac{1}{n}\right)\left(|\nabla \Psi_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}} \nabla \Psi_k\right] \geq 0.$$

It follows by the comparison theorem that the solution of the problem (5.6)-(5.8) v, $v \leq \Psi_k$ in $\aleph_k$. In particular, $\forall 0 < x_N < \frac{1}{k}$, we have

$$\begin{aligned} u(0,0,\ldots,0_{N-1},x_N,0) &= \lim_{n\to\infty} v(0,0,\ldots,0_{N-1},x_N,0) \\ &\leq \Psi_k(0,0,\ldots,0_{N-1},x_N,0) \\ &= CM\left(1 - e^{-kx_N}\right) \leq kCMx_N. \end{aligned}$$

Therefore there exists a constant k depending only upon N, such that

$$u(x,t) \leq kM \operatorname{dist}(x, \partial\Omega),$$

for all $x \in \Omega$ such that $\rho(x) \leq \frac{1}{k}$. On the other hand, if $\rho(x) > \frac{1}{k}$, we have

$$u(x,t) \leq M \leq kCM\rho(x).$$

Thus (5.5) holds in both cases. □

Estimates below and near $\partial\Omega$: Let u be a nonnegative bounded viscous solution of (1.1) in the sense of Definition 5.1,

$$u \leq M,$$

for some $M > 0$. For $r > 0$ let

$$\Omega_r \equiv \left\{x \in \Omega \mid d(x, \partial\Omega) \geq r\right\},$$

$$\Omega_{r,t} \equiv \Omega_r \times [s,t], \quad \forall s < t \leq T$$

and

$$\mu(r) \equiv \inf_{(x,\tau)\in\Omega_{r,t}} u(x,\tau).$$

For $0 < s < t < T$, let

$$r(M,s,t) = \rho_0 \min\{1, \sqrt{t-s}\},$$

where the constant ρ_0 makes the inclusion $\Omega \cap \{|x| < \rho_0\} \subset \{x_N > 0\}$ true as before.

Now, we estimate u below, near the boundary $\partial\Omega$.

Theorem 5.3 *If the hypothesis of Theorem* 5.2 *is true, then* $\forall 0 < s < t < T$, $\forall x \in \Omega$, $\rho(x) \leq r(M,s,t)$, *the inequality*

$$u(x,t) \geq \mu\left(r(M,s,t)\right)\rho(x) \tag{5.10}$$

holds.

Proof Fix $(x_0,t_0)\in\partial\Omega\times(s,T)$ and let $\mu_0\equiv\mu(r(M,s,t_0))$. After flattening of $\partial\Omega$ near x_0, we may assume that $(x_0,t_0)\equiv(0,0)$ as before. Introduce the point

$$\overline{y}\equiv\left(0,0,\dots,0_{N-1},1+\frac{1}{k}\right),$$

and the domain

$$\overline{\aleph}_{kn}=\left\{(x,t):0<x_N<\frac{1}{k},1<|x-\overline{y}|<1+\frac{1}{k},-s_n\le t\le 0\right\}\subset B^+_{\rho_0}\times[-s,0],$$

where

$$\frac{1}{k}=r(M,s,t_0)=\rho_0\min\{1,\sqrt{s}\}.$$

Consider $-s<-s_n<t\le 0$,

$$\frac{\partial v}{\partial t}=\operatorname{div}\left[\left(\rho^\alpha+\frac{1}{n}\right)\left(|\nabla v|^2+\frac{1}{n}\right)^{\frac{p(x)-2}{2}}\nabla v\right],\quad (x,t)\in\overline{\aleph}_{kn},\tag{5.11}$$

$$v_n(x,-s_n)=u_n(x,-s_n),\quad (x,s_n)\in\overline{\aleph}_{kn},\tag{5.12}$$

$$v(x,t)=\frac{1}{n}+u_n(x,t),\quad (x,t)\in\partial\overline{B}_k\times[-s_n,0],\tag{5.13}$$

where u_n is the nonnegative solution of the problem (5.1)-(5.3), $s_n n$ is small enough, and $\overline{B}_k=\{x:0<x_N<\frac{1}{k},1<|x-y|<1+\frac{1}{k}\}$. Also by the comparison theorem ([40], p.119), we have

$$v\ge u_n.\tag{5.14}$$

Consider the function

$$\overline{\eta}_k(x,t)=e^{-k(|x-\overline{y}|-1)}e^{\frac{t}{s}}$$

and construct the barrier

$$\overline{\Psi}_k(x,t)=\mu_0\left(\overline{\eta}_k(x,t)-e^{-1}\right)_+-\gamma t,$$

where $\gamma=\gamma(s,\mu_0,k)$ is a large enough constant to be chosen later. Let us show that $v\ge\overline{\Psi}_k$ on the parabolic boundary of $\overline{\aleph}_k$. On the portion $\{|x-\overline{y}|=1+\frac{1}{k}\}\times[-s_n,0]$ we have $\overline{\Psi}_k=-\gamma t\le\gamma s_n\le\frac{1}{n}\le v$. On the portion lying on the hyperplane $\{x_N=\frac{1}{k}\}$ one checks that $\overline{\Psi}_k\le\mu_0\le u\le v$. On the bottom of $\overline{\aleph}_k$, we have

$$\overline{\Psi}_k|_{t=-s_n}\le\mu_0\left(e^{-k(|x-\overline{y}|-1)}-1\right)_++\gamma s_n\le 0.$$

By direct calculation

$$\overline{\Psi}_{k,x_j}=-k\mu_0\overline{\eta}_k\frac{x_j-y_j}{|x-y|},\qquad \frac{\partial\overline{\Psi}}{\partial t}=\mu_0\overline{\eta}_k\frac{1}{s_n}-\gamma$$

and

$$\overline{\Psi}_{k,x_l x_j} = k^2\mu_0\overline{\eta}_k \frac{(x_j-\overline{y}_j)(x_l-\overline{y}_l)}{|x-\overline{y}|^2} - k\mu_0\overline{\eta}_k \frac{1}{|x-\overline{y}|^2}\left[\delta_{jl}|x-\overline{y}| - \frac{(x_j-\overline{y}_j)(x_l-\overline{y}_l)}{|x-\overline{y}|}\right],$$

$$\begin{aligned}
\overline{\Psi}_{k,x_i x_i} &= k^2\mu_0\overline{\eta}_k - k\mu_0\overline{\eta}_k \frac{1}{|x-\overline{y}|^2}\left[N|x-\overline{y}| - |x-\overline{y}|\right] \\
&= k\mu_0\overline{\eta}_k\left(k - \frac{N-1}{|x-\overline{y}|}\right),
\end{aligned}$$

$$\begin{aligned}
&\left[\left(|\nabla\overline{\Psi}_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}}\overline{\Psi}_{k,x_i}\right]_{x_i} \\
&\quad = \frac{p(x)-2}{2}\left(|\nabla\overline{\Psi}_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-4}{2}} 2\overline{\Psi}_{k,x_l}\overline{\Psi}_{k,x_l x_i}\overline{\Psi}_{k,x_i} \\
&\qquad + \left(|\nabla\overline{\Psi}_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}}\overline{\Psi}_{k,x_i x_i} - \ln\left(|\nabla\overline{\Psi}_k|^2 + \frac{1}{n}\right)\nabla p(x)\cdot\nabla\overline{\Psi}_k \\
&\quad = \big(p(x)-2\big)\left[(k\mu_0\overline{\eta}_k)^2 + \frac{1}{n}\right]^{\frac{p(x)-4}{2}}(k\mu_0\overline{\eta}_k)^2\frac{(x_i-\overline{y}_i)(x_l-\overline{y}_l)}{|x-\overline{y}|^2} \\
&\qquad \cdot\left\{k^2\mu_0\overline{\eta}_k\frac{(x_i-\overline{y}_i)(x_l-\overline{y}_l)}{|x-\overline{y}|^2} + \frac{k\mu_0\overline{\eta}_k}{|x-\overline{y}|^2}\left[\delta_{il}|x-\overline{y}| - \frac{(x_j-\overline{y}_j)(x_l-\overline{y}_l)}{|x-\overline{y}|}\right]\right\} \\
&\qquad + \left[(k\mu_0\overline{\eta}_k)^2 + \frac{1}{n}\right]^{\frac{p(x)-2}{2}} k\mu_0\overline{\eta}_k\left(k - \frac{N-1}{|x-\overline{y}|}\right) \\
&\qquad - \ln\left[(k\mu_0\overline{\eta}_k)^2 + \frac{1}{n}\right]p_{x_i}(x)\frac{x_i-y_i}{|x-y|}k\mu_0\overline{\eta}_k \\
&\quad = \overline{G}_n(x)k\mu_0\overline{\eta}_k\left[k - \frac{1}{|x-\overline{y}|^2}\big(|x-\overline{y}| - |x-\overline{y}|\big)\right] \\
&\qquad + \left[(k\mu_0\overline{\eta}_k)^2 + \frac{1}{n}\right]^{\frac{p(x)-2}{2}} k\mu_0\overline{\eta}_k\left(k - \frac{N-1}{|x-\overline{y}|}\right) \\
&\qquad - \ln\left[(k\mu_0\overline{\eta}_k)^2 + \frac{1}{n}\right]p_{x_i}(x)\frac{x_i-y_i}{|x-y|}k\mu_0\overline{\eta}_k \\
&\quad = \overline{G}_n(x)k^2\mu_0\overline{\eta}_k + \left[(k\mu_0\eta_k)^2 + \frac{1}{n}\right]^{\frac{p(x)-2}{2}} k\mu_0\overline{\eta}_k\left(k - \frac{N-1}{|x-y|}\right) \\
&\qquad - \ln\left[(k\mu_0\overline{\eta}_k)^2 + \frac{1}{n}\right]p_{x_i}(x)\frac{x_i-y_i}{|x-y|}k\mu_0\overline{\eta}_k,
\end{aligned}$$

where $\overline{G}_n(x) = (p(x)-2)[(k\mu_0\overline{\eta}_k)^2 + \frac{1}{n}]^{\frac{p(x)-4}{2}}(k\mu_0\overline{\eta}_k)^2$,

$$\begin{aligned}
&\overline{\Psi}_{k,t} - \operatorname{div}\left[\left(\rho^\alpha + \frac{1}{n}\right)\left(|\nabla\overline{\Psi}_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}}\nabla\overline{\Psi}_k\right] \\
&\quad = \mu_0 e^{-k(|x-y|-1)}e^{\frac{t}{s_n}}\frac{1}{s_n} - \gamma - \alpha\rho^{\alpha-1}\rho_{x_i}\left[\left(|\nabla\overline{\Psi}_k|^2 + \frac{1}{n}\right)^{\frac{p-2}{2}}\overline{\Psi}_{k,x_i}\right] \\
&\qquad - \left(\rho^\alpha + \frac{1}{n}\right)\left[\left(|\nabla\overline{\Psi}_k|^2 + \frac{1}{n}\right)^{\frac{p-2}{2}}\overline{\Psi}_{k,x_i}\right]_{x_i}
\end{aligned}$$

$$
\begin{aligned}
&= \mu_0 e^{-k(|x-y|-1)} e^{\frac{t}{s_n}} \frac{1}{s_n} - \gamma - \alpha\rho^{\alpha-1}\rho_{x_i}\left[\left(|k\mu_0\overline{\eta}_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}} k\mu_0\overline{\eta}_k \frac{x_i-\overline{y}_i}{|x-\overline{y}|}\right] \\
&\quad - \left(\rho^\alpha + \frac{1}{n}\right)\left\{\overline{G}_n(x)k^2\mu_0\overline{\eta}_k + \left[(k\mu_0\overline{\eta}_k)^2 + \frac{1}{n}\right]^{\frac{p(x)-2}{2}} k\mu_0\overline{\eta}_k\left(k - \frac{N-1}{|x-\overline{y}|}\right)\right\} \\
&\quad - \ln\left[(k\mu_0\overline{\eta}_k)^2 + \frac{1}{n}\right]p_{x_i}(x)\frac{x_i-y_i}{|x-y|}k\mu_0\overline{\eta}_k \\
&\le \mu_0 e^{-k(|x-y|-1)} e^{\frac{t}{s_n}} \frac{1}{s_n} - \gamma + \alpha\rho^{\alpha-1}\left(|k\mu_0\eta_k|^2 + \frac{1}{n}\right)^{\frac{p(x)}{2}} \\
&\quad - \left(\rho^\alpha + \frac{1}{n}\right)\overline{G}_n k^2\mu_0\overline{\eta}_k - k\left(\rho^\alpha + \frac{1}{n}\right)|k\mu_0\overline{\eta}_k|^{p(x)-1} \\
&\quad + (N-1)\left(\rho^\alpha + \frac{1}{n}\right)\left(|k\mu_0\eta_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}} \\
&\quad - \ln\left[(k\mu_0\overline{\eta}_k)^2 + \frac{1}{n}\right]p_{x_i}(x)\frac{x_i-y_i}{|x-y|}k\mu_0\overline{\eta}_k,
\end{aligned}
$$

where we have used the fact that $|\rho_{x_i}| = 1$ too.

Clearly, if we choose γ large enough, then we have

$$
\Psi_{k,t} - \operatorname{div}\left[\left(\rho^\alpha + \frac{1}{n}\right)\left(|\nabla\Psi_k|^2 + \frac{1}{n}\right)^{\frac{p(x)-2}{2}} \nabla\Psi_k\right] \le 0.
$$

It follows from the comparison theorem that $v \ge \overline{\Psi}_k$ in $\overline{\aleph}_k$. In particular, $\forall 0 < x_N < \frac{1}{k}$,

$$
v(0,0,\ldots,0_{N-1},x_N,0) \ge \frac{\mu_0}{e}\left(e^{kx_N} - 1\right) \ge \frac{\mu_0 k}{e}x_N.
$$

Let $n \to \infty$. We have

$$
u(x,t) \ge \frac{k}{e}\mu\left(r(M,s,t)\right)\rho(x) \ge \mu\left(r(M,s,t)\right)\rho(x).
$$

Theorem 5.3 is proved. □

Competing interests

The author declares to have no competing interests.

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